



This is a digital copy of a book that was preserved for generations on library shelves before it was carefully scanned by Google as part of a project to make the world's books discoverable online.

It has survived long enough for the copyright to expire and the book to enter the public domain. A public domain book is one that was never subject to copyright or whose legal copyright term has expired. Whether a book is in the public domain may vary country to country. Public domain books are our gateways to the past, representing a wealth of history, culture and knowledge that's often difficult to discover.

Marks, notations and other marginalia present in the original volume will appear in this file - a reminder of this book's long journey from the publisher to a library and finally to you.

Usage guidelines

Google is proud to partner with libraries to digitize public domain materials and make them widely accessible. Public domain books belong to the public and we are merely their custodians. Nevertheless, this work is expensive, so in order to keep providing this resource, we have taken steps to prevent abuse by commercial parties, including placing technical restrictions on automated querying.

We also ask that you:

- + *Make non-commercial use of the files* We designed Google Book Search for use by individuals, and we request that you use these files for personal, non-commercial purposes.
- + *Refrain from automated querying* Do not send automated queries of any sort to Google's system: If you are conducting research on machine translation, optical character recognition or other areas where access to a large amount of text is helpful, please contact us. We encourage the use of public domain materials for these purposes and may be able to help.
- + *Maintain attribution* The Google "watermark" you see on each file is essential for informing people about this project and helping them find additional materials through Google Book Search. Please do not remove it.
- + *Keep it legal* Whatever your use, remember that you are responsible for ensuring that what you are doing is legal. Do not assume that just because we believe a book is in the public domain for users in the United States, that the work is also in the public domain for users in other countries. Whether a book is still in copyright varies from country to country, and we can't offer guidance on whether any specific use of any specific book is allowed. Please do not assume that a book's appearance in Google Book Search means it can be used in any manner anywhere in the world. Copyright infringement liability can be quite severe.

About Google Book Search

Google's mission is to organize the world's information and to make it universally accessible and useful. Google Book Search helps readers discover the world's books while helping authors and publishers reach new audiences. You can search through the full text of this book on the web at <http://books.google.com/>

NYPL RESEARCH LIBRARIES



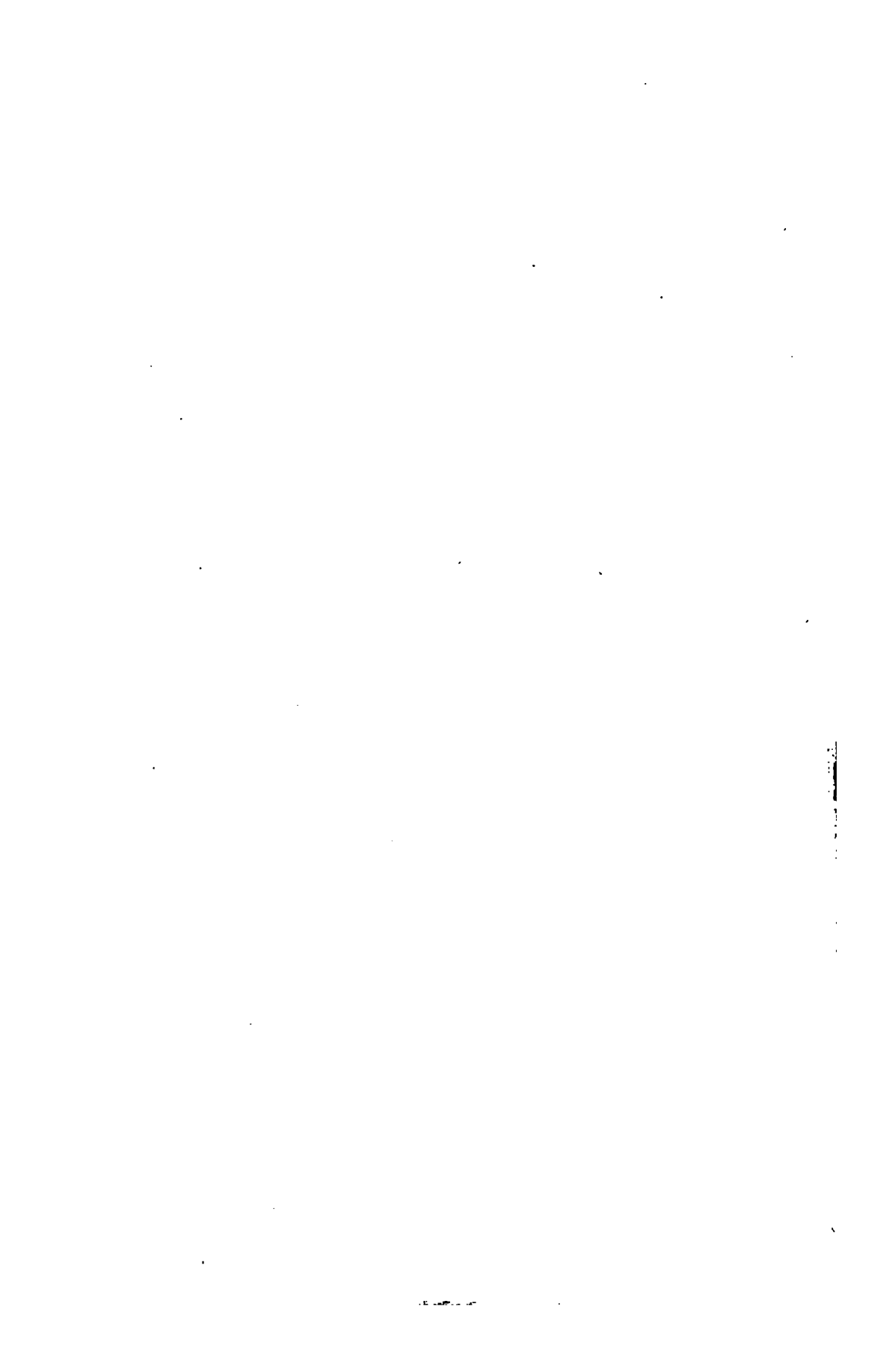
3 3433 06642293 6

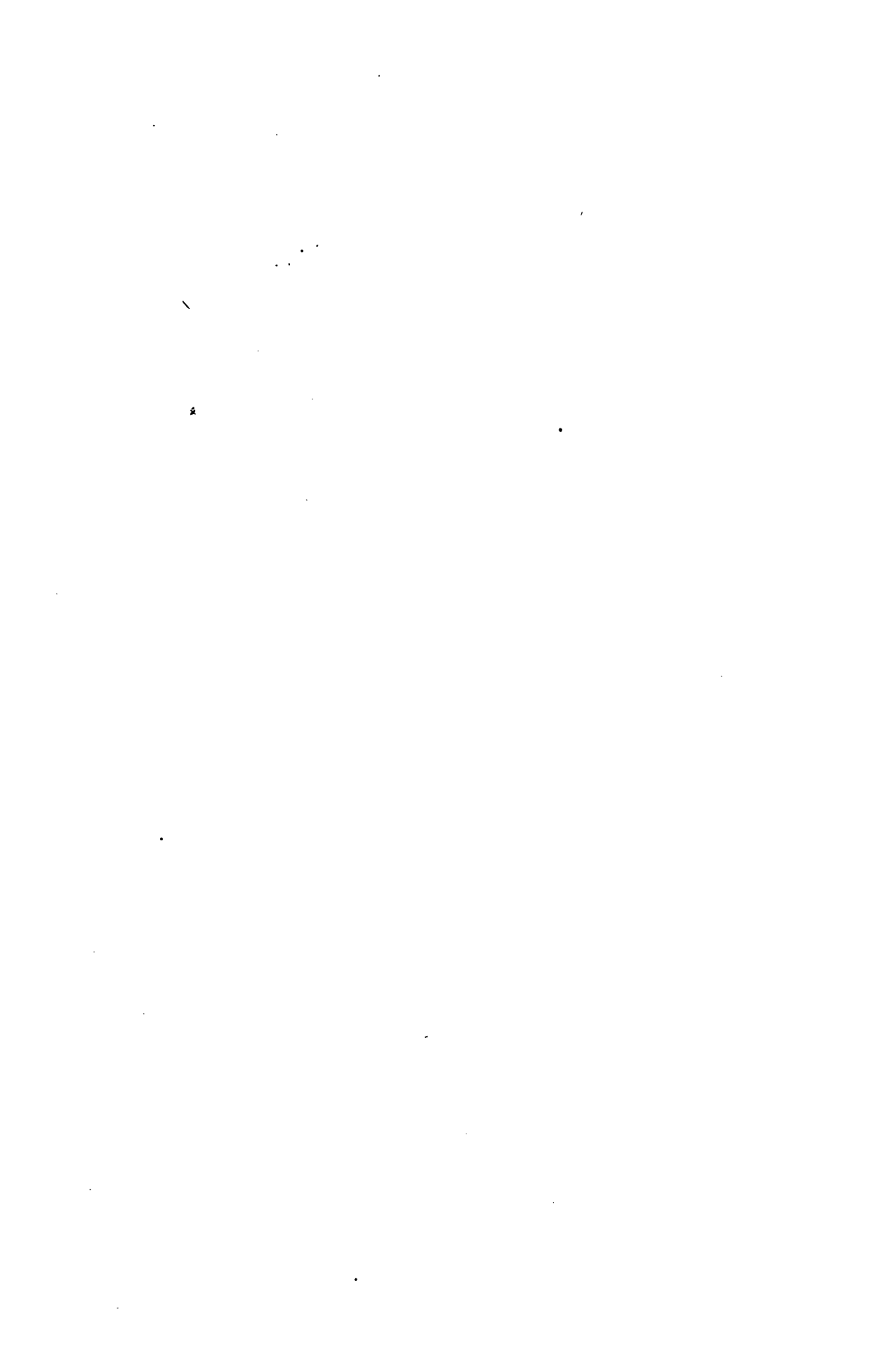


Analyst

DEA







THE ANALYST.

A JOURNAL OF

PURE AND APPLIED MATHEMATICS.

EDITED AND PUBLISHED BY

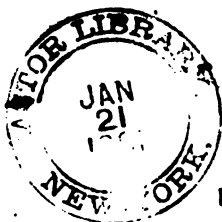
J. E. HENDRICKS, A.M.

VOLUME V.

DES MOINES, IOWA:

MILLS & CO., BOOK AND JOB PRINTERS.

1878.



242

NY 49
208
1971

ALPHABETICAL INDEX.

	PAGE		PAGE
Approximate Multisection of an Angle,	172	Hall, Prof. Asaph,	44
A Problem and its Solution,	181	Hart, Dr. David S.,	118
Announcement,	192	Heal, William E.,	143, 148
Adcock, R. J.,	21, 53, 149	Johnson, Prof. W. W.,	7, 176
A Case of Symbolic vs. Operative Exp'n,	38	Kirkwood, Prof. Daniel,	178
A Problem and its Solution,	45	Kummell, Chas H.,	17, 78, 172
A Question and its Solution,	50	Least Squares,	21
A Problem in Least Squares,	53	Ladd, Christine,	116, 145
Answer to Query,	55	Martin, Artemas,	52, 141
A Collection of Proofs of the Relation		Mc Adam, Prof. D. J.,	87
$r' + r'' + r''' - r = 4R,$	82	Macnie, John,	80, 174
A Problem with Solution,	86	Note,	49, 54, 55
Answer to Query at page 62,	147	Note on the Correspondence of Material	
Barbour, Prof. L. G.,	73, 180	Forms with Mathematical Relations,	140
Baker, Marcus,	24, 82, 143	Note on the Solution of Problem 217,	175
Borden, John,	41, 110	Note on the Polynomial Theorem,	176
Bibliographical Note, and Query,	143	On the Limit of Repeated Adjustments,	129
Correspondence,	174	On the Angle between two Lines given	
Continued Roots,	20	by their Equations,	148
Cubic Equations,	117	On Some Properties of Four Circles	
Casey, Prof. W. P.,	147	Inscribed in One,	116
De Forest, E. L.,	1, 66, 129	On Repeated Adjustments and on Signs	
Dixon, T. S. E.,	20	of Residuals,	65
Discussion of the General Equation		On Equations of the Third Degree,	73
of the Third Degree,	110	On the Motion of the Centre of Gravity	
Demonstration of Proposition on page 8,	87	of the Earth and Moon,	33
Discussion of an Equation,	41	On the Roots of Equations,	51
Enumeration of Primes,	7	On the Grouping of Signs of Residuals,	1
Errata,	32, 64, 96, 128, 160, 192	Proposition,	8
Evaluation of Elliptic Functions of		Problems,	31, 63, 95, 127, 143, 159, 193
the Second and Third Species,	97	Publications Received,	32, 64, 96, 128, 192
Evans, Alexander,	161	Proof of the Theorem that every Eq'n	
Eastwood, George,	50, 170	has a Root,	80
Eggers, Dr. H.,	181	Proposition in Transversals,	113
Ficklin, Prof. Joseph,	177	Quinquesection of the Circumference	
Geometrical Solution of Problem 125,	147	of a Circle,	180
Hill, G. W.,	33, 114	Revised Solution of Problem 218,	170
Hathaway, A. S.,	38, 86	Rectification of the Hyperbola,	52
Hyde, Prof. E. W.,	39, 113	Remarks on Mr. Meech's Article on	
		Elliptic Functions,	17

Short Method of Elliptic Functions,	9	To find the Differential of a Variable	
Successive and Simult's Quadrature,	10	without the Use of Infin's or Limits,	177
Quadrantal Quadrature,	12	The Meteor of August 11, 1878,	178
Integration of Funct's of 2d & 3d Ord.	14	Turrell, Isaac H.,	166
Solution of Problems, 24, 26, 39, 63, 88,		Trowbridge, Prof. David,	168
120, 150, 183		To find the Earth's Dist. from the Sun,	141
Solution of Problem 206,	149	The Polynomial Theorem,	145
Solution of an Indeterminate Problem,	118	The Secular Acceleration of the Moon,	105
Seitz, E. B.,	45	The Centre of Gravity of the Apparent	
Solution of the Eq'n of the Fifth Deg.,	161	Disk of a Planet,	44
Summation of two Series,	168	Ultimate Proposition in Attraction,	22
To Draw a Circle Tangent to Three		Worpitzky, Professor,	51
Given Circles,	166		

THE ANALYST.

VOL. V.

JANUARY, 1878.

No. 1.

ON THE GROUPING OF SIGNS OF RESIDUALS.

BY E. L. DE FOREST, A. M., WATERTOWN, CONN.

WHEN a series of equidistant terms, affected by errors of observation, is adjusted by means of such a formula as

$$u'_0 = l_0 u_0 + l_1(u_1 + u_{-1}) + l_2(u_2 + u_{-2}) + \dots + l_m(u_m + u_{-m}), \quad (1)$$

the extent to which the adjusting process should be carried can often be determined by observing the fortuitous distribution of the signs of the residuals v which result from subtracting each adjusted term from the corresponding observed term. (See the ANALYST for July, 1877, p. 111, and the pamphlet *Interpolation and Adjustment of Series*, p. 31.) This method of testing an adjustment presupposes that the errors of consecutive terms are wholly independent of each other, so that the fact that a particular observed term is, for instance, in excess of its normal value, shall be no reason for presuming that the next term will be also in excess, or the reverse. The method would not apply to such a case as the determination of the normal curve of annual temperature from a series of daily means covering only a single year, because variations from the normal occur in waves of heat or cold, extending over periods of several days each. But it does apply to many kinds of physical and statistical series, as for instance to the experience table of mortality discussed in the pamphlet referred to. The normal or probable distribution of signs was there ascertained, on the assumption that the adjusted series is the true one, and it was inferred that the adjusting process ought to be carried far enough to secure such a distribution, within certain ascertained limits of probable error. This standard of adjustment, however, is in some degree an ideal one, which we ought not to expect fully to attain even with the best adjustment formulas we can use, because the adjusted series can hardly ever be the true one, and the actual residuals are not the true errors. The distribution of signs will depend

somewhat upon the ratio existing between the probable errors ϵ' and ϵ of the adjusted and the observed terms. The case is analogous to that of the other test proposed, where the mean error ϵ_1 of each observed term is supposed to be known, and it was shown that, assuming the adjusted series to be the true one, the arithmetical mean of all the values of $\left(\frac{v}{\epsilon_1}\right)^2$ for the N terms of the series would be

$$1 \pm .6745\sqrt{\frac{2}{N}}. \quad (2)$$

But it was also shown (ANALYST, p. 109, Vol. IV) that, when we use an adjustment formula whose error-ratio is

$$r = \frac{\epsilon'}{\epsilon} = \sqrt{[l_0^2 + 2(l_1^2 + l_2^2 + \dots + l_n^2)]},$$

the mean of $\left(\frac{v}{\epsilon_1}\right)^2$ will most probably be

$$(1 - r^2)\left(1 \pm .6745\sqrt{\frac{2}{N}}\right), \quad (3)$$

which agrees with (2) only in the extreme case, when we suppose the adjusting process to be perfect, so that $\epsilon' = 0$ and $r = 0$. The difference between (2) and (3) is rather too large to be disregarded. This test cannot be used when we have not the data for finding ϵ_1 , but we may use equally well, if the nature of the errors justifies it, the test afforded by the observed grouping of the signs of the residuals. It is the object of the present paper to ascertain, as far as may be, what the probable distribution of these signs will be for any given value of r .

The terms of the adjusted series may be regarded as occupying an intermediate position among the successive observed terms, similar to that which an arithmetical mean occupies with reference to several observed values of a single quantity. (Compare the ANALYST, p. 108.) Let n_1 be a number such that the ratio which the probable error of the mean of n_1 such values bears to that of a single value shall be equal to the error-ratio r , so that we have

$$r = \frac{1}{\sqrt{n_1}}. \quad (4)$$

In a group of n_1 successive values, it is most probable that half of the terms will be in excess of the mean, and the other half in defect, so that if a particular residual is positive, for instance, it is most likely that the number of other positive residuals in the group will be $\frac{1}{2}n_1 - 1$, while the number of negative ones will be $\frac{1}{2}n_1$. Hence, the probability that another residual in the group, that is to say, the residual next to that which we have supposed positive, will be likewise positive, is

$$\frac{\frac{1}{2}n_1 - 1}{n_1 - 1},$$

and giving to n_1 its value from (4), we find that the probability that any two consecutive residuals will have like signs is

$$q = \frac{1 - 2r^2}{2 - 2r^2}. \quad (5)$$

If r equals or exceeds $\frac{1}{2}\sqrt{2} = 0.707$, then q becomes zero or negative, and it appears impossible, in such a case, for any two consecutive residuals to have like signs. It is not really impossible, however, as can be seen by trial with, for example, the formula

$$u'_0 = \frac{1}{16}[10u_0 + 4(u_1 + u_{-1}) - (u_2 + u_{-2})],$$

where $r = 0.723$. Formula (5), therefore, does not give a rigorously true value of q , but it is found to give a good approximate value, for such adjustment formulas as are likely to be used in practice, at least when $r < \frac{1}{2}$. It is strictly true when $r = 0$.

The probability that any group of n consecutive residuals will have like signs is approximately q^{n-1} , provided that n is a small number, and less than n_1 . The probability that such a group of signs will be both alike and isolated, that is, different from the signs next preceding and following it, is

$$p = q^{n-1}(1-q)^2,$$

and consequently

$$p = \frac{(1-2r^2)^{n-1}}{(2-2r^2)^{n+1}}. \quad (6)$$

In a series of N signs in all, the whole number of possible groups of n consecutive signs each will be N if the series is periodic, that is, if its first and last terms are consecutive. Hence by well known principles, the most probable number of isolated groups of n like signs occurring from accidental causes, is for a periodic series approximately

$$\mu = pN \pm .6745\sqrt{[p(1-p)N]}. \quad (7)$$

If we give to n in (6) the values 1 and 2, we have

$$p_1 = \frac{1}{4(1-r^2)^2}, \quad p_2 = \frac{1-2r^2}{8(1-r^2)^3},$$

and the total number of signs included in groups of only one or two signs each is $M = (p_1 + 2p_2)N \pm .6745\sqrt{[p_1(1-p_1) + 4p_2(1-p_2)]N}$.

Assigning to p_1 and p_2 their values as above, and expanding binomials and neglecting powers of r^2 in the probable error, we have with sufficient accuracy when $r < \frac{1}{2}$,

$$M = \frac{(2-3r^2)N}{4(1-r^2)^3} \pm .533(1+\frac{1}{2}r^2)\sqrt{N}. \quad (8)$$

A perfect adjustment, making $r = 0$, would give

$$M = \frac{1}{2}N \pm .533\sqrt{N}, \quad (9)$$

as was found in the ANALYST, p. 111, Vol. IV. Actually, an adjusted series will always have some probable error, and the most probable value of M as given by (8) will always be somewhat larger than $\frac{1}{2}N$. And since q in formula (5) is less than $\frac{1}{2}$, making it more probable that any two consecutive signs will be unlike than that they will be alike, it follows that the largest group likely to occur will contain a rather smaller number of signs than

$$1 + 3.32 \log N,$$

which is the number found in the ANALYST referred to.

Formulas (7) and (8) will also apply to the case of ordinary or non-periodic series, for purposes of rough estimation, provided we treat the first and last residuals as consecutive, so that their signs, if they are alike, belong to the same group. A more precise determination is as follows. The first and last signs of the series, and as many adjacent signs as happen to be like them, cannot properly belong to any isolated group at all, the next signs beyond being unknown. Omitting the first and last sign, the whole number of possible isolated groups of n signs each will be $N-n-1$, and the number of such groups which will most probably occur is

$$\mu = p(N-n-1) \pm .6745\sqrt{[p(1-p)(N-n-1)]}. \quad (10)$$

Proceeding as before, we find for the probable total number of signs included in groups of only one or two signs each

$$M = p_1(N-2) + 2p_2(N-3) \pm .6745\sqrt{[p_1(1-p_1)(N-2) + 4p_2(1-p_2)(N-3)]}.$$

Assigning to p_1 and p_2 their values, expanding binomials and neglecting powers of r^2 and other negligible quantities, we get with sufficient accuracy when $r < \frac{1}{2}$,

$$M = \frac{(2-3r^2)(2N-5)}{8(1-r^2)^3} \pm .533(1+\frac{1}{2}r^2)\sqrt{(N-3)}. \quad (11)$$

The foregoing results have been tested by means of series adjusted with several different formulas, and the agreement between computation and observation was as close as could fairly be expected. To illustrate, let us take from the *Smithsonian Report* of 1873, p. 335, the signs of residuals which occur in an experience table of mortality adjusted by the 21-term formula of Table B. (ANALYST, May 1877, p. 82.) Here we have $r = .372$ and $N = 70$. The probable total of signs falling within groups of only one or two, as computed by (11), is

$$M = 41.8 \pm 4.7, \quad \text{while the}$$

obs'd number as counted in the table, omitting the first and last signs, is 46.

1	-	15	+	29	-	43	-	57	+
2	+	16	-	30	+	44	-	58	-
3	-	17	-	31	+	45	-	59	-
4	+	18	+	32	+	46	+	60	+
5	+	19	+	33	-	47	+	61	-
6	-	20	+	34	-	48	-	62	-
7	+	21	-	35	-	49	+	63	-
8	-	22	-	36	+	50	+	64	+
9	+	23	-	37	+	51	-	65	-
10	-	24	-	38	-	52	-	66	+
11	+	25	+	39	-	53	+	67	+
12	-	26	+	40	-	54	-	68	-
13	+	27	-	41	+	55	+	69	+
14	-	28	+	42	+	56	-	70	-

The agreement is sufficient to justify the conclusion that the adjusting process probably has not introduced any new errors into the series, and that the adjustment is good as far as it goes. The same series adjusted by the 9-term formula of Table B., for which $r = .537$, gives by computation

$$M = 53.1 \pm 5.0,$$

while observation shows $M = 53$.

A fictitious series of 100 terms was constructed by the aid of the system of errors of equal frequency, in *Interpolation* etc., p. 24, and was adjusted by the 25-term formula of Table C. Here the series was periodic, and $r = .311$, and formula (8) gave

$$M = 58.0 \pm 5.6.$$

The number actually observed was $M = 56$.

A few general remarks on the subject of tests of good adjustment will be appropriate here. If we could construct the true series, so as to have $r=0$, then the most probable value of the arithmetical mean of $\left(\frac{v}{\varepsilon_1}\right)^2$ would be unity, as in formula (2), and the most probable number M of signs falling in groups of only one or two signs each, in a periodic series, would be $\frac{1}{2}N$, as in formula (9). Practically, this state of things cannot be quite realized, for it is probable that, even by repeated adjustments, we can never reduce the value of r much below somewhere from $\frac{1}{3}$ to $\frac{1}{4}$. But any adjusted series ought to satisfy, approximately at least, the condition that the observed mean of $\left(\frac{v}{\varepsilon_1}\right)^2$ should be as in formula (3). If it is greater than the upper value

$$(1-r^2)\left(1 + .6745\sqrt{\frac{2}{N}}\right),$$

there will be some reason to fear that the series has been smoothed out too much, and that large values of v have been caused by departures from the normal form of the series. The same will be true if the observed number M is smaller than the lower value given by formula (8)

$$\frac{(2-3r^2)N}{4(1-r^2)^3} - .533(1+\frac{1}{2}r^2)\sqrt{N},$$

and similarly when (11) is used.

The mere fact that the conditions (3) and (8) are satisfied, however, is not of itself a sufficient evidence of good adjustment. For suppose the adjustment were the least possible, so that the adjusted series did not differ sensibly from the unadjusted one; then we shall have every where $v=0$ and $\epsilon' = \epsilon$ and $r=1$, so that the mean of $(v \div \epsilon_1)^2$ would be zero and the expression (3) would be also zero, and the condition would be fully satisfied though no adjustment had been made. The same may be said respecting a condition employed by another writer, and remarked upon somewhat imperfectly in *Interpolation* etc., pp. 14 to 16. The condition reduces to this, that when a law of mortality or other similar series has been adjusted by means of an empirical equation containing k constants or parameters, the mean of $(v \div \epsilon_1)^2$ ought to be approximately

$$1 - \frac{k}{N}.$$

This is true if the equation is capable of expressing the true law of the series, and if the values of the k constants have been determined by the method of least squares. When they are found by any other method, the mean will be a little larger. Now in a rate of mortality the precise analytical form of the series is not known in advance, and the number k is one of the things to be determined. Suppose $k=N$. The given series can be represented exactly by an algebraic curve which has N parameters. Here then we shall have every where $v=0$, so that the mean of $(v \div \epsilon_1)^2$ is zero and $1 - (k \div N)$ is also zero, and this condition of good adjustment is satisfied where no adjustment at all has been made. The inference is, that an equation which satisfies this condition cannot be considered as the true one, or the best one possible, unless it shall also appear that the condition cannot be satisfied by any other equation of simpler form or containing fewer constants.

We conclude then, that if we use an adjustment formula such as (1), a sufficient test will require that the mean of $(v \div \epsilon_1)^2$ or the value of M should be brought as near to their ultimate values 1 and $\frac{1}{2}N$ respectively, as they can be without violating the conditions (3) and (8). In other words, the most accurate adjustment will probably be made by that formula whose error-ratio r is the least, provided that the conditions (3) and (8) or (11) are approximately satisfied.

ENUMERATION OF PRIMES.

BY PROF. W. W. JOHNSON, ST. JOHN'S COLLEGE, ANNAPOLIS, MD.

MR. J. W. L. GLAISHER'S enumeration of primes, referred to at page 9, Vol. II of the *ANALYST*, has been completed, and a "preliminary account" of the results appeared in the *Proceedings of the Cambridge Philosophical Society*, Vol. III.

In this paper, Mr. Glaisher gives tables of the form described below, for the first three millions of natural numbers comprised in Burckhardt's *Tables of Divisors*, and for the 7th, 8th and 9th millions which are covered by Dase's *Tables*.

Calling the hundred numbers between $100n$ and $100(n+1)$ a "century", the tables show, for each of the thirty groups of 100,000 in these six millions, the number of centuries which contain no prime, the number which contain a single prime, two primes, three, and so on; also the total number of primes in each group. These tables are given below, the ten numbers in each column being the numbers of primes in each of the ten groups of the million to which it refers, and the total at the foot, the number of primes in the entire million. The differences are appended to exhibit more strikingly the remarkable inequality which exists in the distribution of primes.

	1st million	2nd million	3rd million
1st group	9,593 — 1,203	7,216 + 9	6,874 — 17
2nd "	8,392 — 379	7,225 — 144	6,857 — 8
3rd "	8,013 — 150	7,081 + 22	6,849 — 58
4th "	7,863 — 185	7,103 — 75	6,791 — 21
5th "	7,678 — 118	7,028 — 55	6,770 + 39
6th "	7,560 — 115	6,973 + 42	6,809 — 44
7th "	7,445 — 37	7,015 — 83	6,765 — 49
8th "	7,408 — 85	6,932 + 25	6,716 + 30
9th "	7,323 — 99	6,957 — 54	6,746 — 38
10th "	7,224 — 8	6,903 — 29	6,708
Total	78,499	70,433	67,885

	7th million	8th million	9th million
1st group	6,397	6,369	6,250
2nd "	6,402 + 5	6,306 — 63	6,301 + 51
3rd "	6,425 + 23	6,348 + 42	6,283 — 18
4th "	6,337 — 88	6,299 — 49	6,285 + 2
5th "	6,347 + 10	6,301 + 2	6,245 — 40
6th "	6,402 + 55	6,305 + 4	6,326 + 81
7th "	6,338 — 64	6,347 + 42	6,281 — 45
8th "	6,375 + 37	6,245 — 102	6,299 + 19
9th "	6,411 + 36	6,364 + 119	6,220 — 79
10th "	6,365 — 46	6,274 — 90	6,270 + 50
	+ 4	— 24	
Total	63,799	63,158	62,760

Excepting the first two centuries, which contain 26 and 21 primes respectively, there are no centuries containing more than 17 primes; and excluding the first group of 100,000 numbers, there are no centuries containing more than 15 except two containing 17 primes, one between 700,000 and 800,000 and the other in the third million, and one containing 16 primes in the second million. Two centuries containing 15 primes occur in the eighth million, but none with more than 14 occur in the seventh or ninth. No centuries free from primes occur in the first million, and but one each in the second and third, but the seventh million contains six and the eighth and ninth four each, of such centuries.

Mr. Glaisher defers a comparison of his results with the logarithmic integral formula until after the completion of a sufficiently extended table of values of the integral.

PROPOSITION. BY L. W. MEECH, A. M.—If a, b, c denote three lines drawn from the angles of a plane triangle to the center of the inscribed circle whose radius is r , then will the reciprocals

$$\frac{1}{a'}, \frac{1}{b'}, \frac{1}{c'}, \frac{1}{r}$$

be the four sides of a trapezium inscribed in a semicircle; the latter side coinciding with the diameter.

SHORT METHOD OF ELLIPTIC FUNCTIONS.

BY LEVI W. MEECH, A. M., HARTFORD, CONN.

(Continued from page 168, Vol. IV.)

NOTE 1. The inverse formula demonstrated in this Section, $\Delta = \sqrt{b} \times \theta_3 x \div \theta x$ virtually accomplishes the object of various elaborate Researches, for the addition, subtraction, multiplication or division of Elliptic Functions; or for finding the roots of modular equations. For example, let it be required to resolve the equation, $m.F(e, \theta) = F(e, \theta') + F(e, \theta'') + F(e, \theta''')$; or $mAx = A(x' + x'' + x''')$; or $mx = x' + x'' + x'''$. The value of x , found through x' , x'' , x''' , and substituted in the inverse formula will evidently define the unknown amplitude θ , of the function $mF(e, \theta)$ equal to the sum of the other three. Since $A''y = Ax'$; where $x' = A''y \div A$; different moduli can be included in the operation also.

NOTE 2. Double Periodicity: $\sin x = \sin(x + 2m\pi)$; $\sin x = \sin(x + n \log q. \sqrt{-1})$ or $\sin x = \sin(x + 2m\pi - n \log q. \sqrt{-1})$; where m, n are arbitrary integers; 2π is the *real* period, as in trigonometry, and $-\log q. \sqrt{-1}$, the *imaginary* period.

NOTE 3. The meaning of *Theta function* is thus defined:

$$\begin{aligned} \theta x &= \theta(x) = 1 - 2q \cos 2x + 2q^4 \cos 4x - 2q^9 \cos 6x + \dots \\ \theta_3 x &= \theta(\frac{x}{3}) = 1 + 2q \cos 2x + 2q^4 \cos 4x + 2q^9 \cos 6x + \dots \\ \theta(0) &= \sqrt{bA}; \quad \theta(\frac{\pi}{2}) = \sqrt{A}. \end{aligned}$$

The last two equations are readily verified by taking the square root of the series for A near the close of Section V, and changing from terms of t to terms of q . Among other formulas from Abel and Jacobi, we find,

$$\begin{aligned} \theta_1 x &= H(x) = 2q^{\frac{1}{4}} \sin x - 2q^{\frac{9}{4}} \sin 3x + 2q^{\frac{25}{4}} \sin 5x - \dots \\ \theta_2 x &= H(\frac{x}{2}) = 2q^{\frac{1}{4}} \cos x + 2q^{\frac{9}{4}} \cos 3x + 2q^{\frac{25}{4}} \cos 5x + \dots \\ \sin \theta &= \frac{1}{\sqrt{e}} \cdot \frac{\theta_1 x}{\theta x} = \frac{4}{eA} \left[\frac{q^{\frac{1}{4}} \sin x}{1-q} + \frac{q^{\frac{9}{4}} \sin 3x}{1-q^3} + \frac{q^{\frac{25}{4}} \sin 5x}{1-q^5} + \dots \right] \\ \cos \theta &= \sqrt{\frac{b}{e}} \cdot \frac{\theta_2 x}{\theta x} = \frac{4}{eA} \left[\frac{q^{\frac{1}{4}} \cos x}{1+q} + \frac{q^{\frac{9}{4}} \cos 3x}{1+q^3} + \frac{q^{\frac{25}{4}} \cos 5x}{1+q^5} + \dots \right] \\ \Delta = \sqrt{b} \cdot \frac{\theta_3 x}{\theta x} &= \frac{1}{A} + \frac{4}{A} \left[\frac{q \cos 2x}{1+q^2} + \frac{q^2 \cos 4x}{1+q^4} + \dots \frac{q^n \cos 2nx}{1+q^{2n}} + \dots \right] \\ \theta &= \int \Delta dx = x + 2 \left[\frac{q \sin 2x}{1+q^2} + \frac{1}{2} \frac{q^2 \sin 4x}{1+q^4} + \dots \frac{1}{n} \frac{q^n \sin 2nx}{1+q^{2n}} + \dots \right] \\ q &= \frac{e^2}{16} \left(1 + \frac{1}{2} e^2 + \frac{21}{64} e^4 + \dots \right). \quad \log q = -\frac{\pi A'}{A} = -\frac{\pi F(b, \frac{1}{2}\pi)}{F(e, \frac{1}{2}\pi)}. \\ \frac{1}{2}t &= q - 2q^5 + 5q^9 - 10q^{13} + 18q^{17} - \dots \end{aligned}$$

VII. SUCCESSIVE AND SIMULTANEOUS QUADRATURES. — A new scale of amplitudes, possessing important advantages, may be combined with the same scale of moduli already described in Sections III and IV. Thus let, in descending order,

$$\begin{aligned}\tan g &= \sqrt[4]{(1-e^2)} = \sqrt[4]{b}; \\ \tan g^0 &= \sqrt{(\sin 2g)} = \sqrt[4]{b^0}; \\ \tan g^{00} &= \sqrt{(\sin 2g^0)} = \sqrt[4]{b^{00}}; \\ \tan g^{000} &= \sqrt{(\sin 2g^{00})} = \sqrt[4]{b^{000}}; \\ \tan u &= \sqrt[4]{b} \cdot \tan \theta, & F &= \int \frac{d\theta}{\sqrt{(1-e^2 \sin^2 \theta)}}; \\ \tan u^0 &= \frac{\tan 2u}{\sqrt[4]{b^0}}, & F &= \frac{2}{1+b} \int \frac{du}{\sqrt{(1-e^{02} \cos^2 2u)}}; \\ \tan u^{00} &= \frac{\tan 2u^0}{\sqrt[4]{b^{00}}}, & F &= \frac{2}{1+b} \cdot \frac{2}{1+b^0} \int \frac{\frac{1}{2} du^0}{\sqrt{(1-e^{002} \cos^2 2u^0)}}; \\ \tan u^{000} &= \frac{\tan 2u^{00}}{\sqrt[4]{b^{000}}}, & F &= \frac{2}{1+b} \cdot \frac{2}{1+b^0} \cdot \frac{2}{1+b^{00}} \int \frac{\frac{1}{4} du^{00}}{\sqrt{(1-e^{0002} \cos^2 2u^{00})}}.\end{aligned}$$

By way of demonstration we observe that,

$$d2u = \frac{d \tan 2u}{1 + \tan^2 2u} = \frac{\sqrt[4]{b^0} \cdot d \tan u^0}{1 + b^0 \tan^2 u^0} \cdot \cos^2 2u = \frac{1}{1 + \tan^2 2u} = \frac{1}{1 + b^0 \tan^2 u^0}.$$

Substituting in the second expression of F , since $b^{02} = 1 - e^{02}$,

$$\frac{du}{\sqrt{(1 - e^{02} \cos^2 2u)}} = \frac{\frac{1}{2} \sqrt[4]{b^0} \cdot d \tan u^0}{\sqrt{[(1 + b^0 \tan^2 u^0)(b^{02} + b^0 \tan^2 u^0)]}}.$$

Now $\sqrt[4]{b^0}$ may be divided out. Then since $d \tan u^0 = (1 + \tan^2 u^0) du^0$, and $\tan^2 u^0 = \frac{1 - \cos 2u^0}{1 + \cos 2u^0}$, by trigonoetry,

$$\frac{du}{\sqrt{(1 - e^{02} \cos^2 2u)}} = \frac{du^0}{\sqrt{\{ [1 + b^0 + (1 - b^0) \cos 2u^0] \cdot [1 + b^0 - (1 - b^0) \cos 2u^0] \}}}.$$

The product of the sum and difference of the two quantities under the last radical, is known to be the difference of their squares. Passing out as a factor $(1 + b^0)$, also multiplying by $2 \times \frac{1}{2}$, and observing that $e^{00} = (1 - b^0) \div (1 + b^0)$, we obtain the third form of F above. By similar substitutions, the second expression of F was derived from the first; after which the law of derivation is evidently uniform.

On referring to the scale of amplitudes in Section IV, we observe that, if developed as in Section V, the leading periodic term of the integral there, may be denoted by $C \sin 2\theta^0$, or $C' \sin 2\theta^{00}$, ... In Section VII, the corresponding term is $-\frac{1}{2} C \sin 4u$, or $-\frac{1}{4} C' \sin 4u^0$, ... So that the new scale

of amplitudes makes the periodic portion of the integral continually smaller than the former method, in the ratio of $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \dots$

And if, in the preceding denominators, we desire to change from a cosine to a sine, the relation of the first two equations will indicate the process. Thus in the fourth expression of F , let $\tan u^{00} = \sqrt{b^{00}} \cdot \tan \Psi$; whence by comparison, $\tan \Psi = (\tan 2u^0) \div b^{00}$; and

$$F = \frac{2}{1+b} \cdot \frac{2}{1+b^0} \cdot \frac{1}{4} \int \frac{d\Psi}{\sqrt{(1-e^{002}\sin^2\Psi)}}.$$

In the preceding system, the amplitudes depend one upon another in successive order. Let us now represent the same amplitudes by simultaneous terms. For a first example, let T be such that

$$u^0 = \tan^{-1}(\tan T \cdot \tan u) + \tan^{-1}(\cot T \cdot \tan u).$$

Taking the trigonometric tangent of both sides

$$\tan u^0 = \frac{(\tan T + \cot T) \tan u}{1 - \tan^2 u} = \frac{\tan 2u}{\sin 2T} = \frac{\tan 2u}{\sqrt{b^0}}.$$

Thus the auxiliary T is found by the relation $\sin 2T = \sqrt{b^0}$. And if

$$u^0 = u_1 + u_2, \tan u_1 = \tan T \cdot \tan u, \tan u_2 = \cot T \cdot \tan u.$$

For a second application, we assume

$$u^{00} = \tan^{-1}(\tan T^0 \cdot \tan u^0) + \tan^{-1}(\cot T^0 \cdot \tan u^0).$$

Taking the tangent and proceeding as before, we find $\sin 2T^0 = \sqrt{b^{00}}$. Also substituting for $\tan u^0$ its former equal $(\tan 2u) \div \sqrt{b^0}$,

$$u^{00} = \tan^{-1} \left(\frac{\tan 2u}{\tan T^0 \cdot \sqrt{b^0}} \right) + \tan^{-1} \left(\frac{\tan 2u}{\cot T^0 \cdot \sqrt{b^0}} \right).$$

Here let $\sin 2T_1 = \tan T^0 \cdot \sqrt{b^0}$; $\sin 2T_2 = \cot T^0 \cdot \sqrt{b^0}$. By comparison with the first example, u^{00} can now be resolved into the four simultaneous terms $u^{00} = u_1 + u_2 + u_3 + u_4$; where

$$\tan u_1 = \tan T_1 \tan u; \quad \tan u_2 = \cot T_1 \tan u;$$

$$\tan u_3 = \tan T_2 \tan u; \quad \tan u_4 = \cot T_2 \tan u.$$

In like manner for a third example, u^{000} can be resolved into eight similar terms, by the relations,

$$\sin 2T^{00} = \sqrt{b^{000}}; \sin 2T_1^{00} = \tan T^{00} \cdot \sqrt{b^{00}}; \sin 2T_2^{00} = \cot T^{00} \cdot \sqrt{b^{00}}.$$

$$\sin 2T_1^{00} = \tan T_1^{00} \cdot \sqrt{b^0}; \sin 2T_2^{00} = \cot T_1^{00} \cdot \sqrt{b^0};$$

$$\sin 2T_3^{00} = \tan T_2^{00} \cdot \sqrt{b^0}; \sin 2T_4^{00} = \cot T_2^{00} \cdot \sqrt{b^0}.$$

$$u^{000} = u_1 + \dots + u_8; \tan u_1 = \tan T_1 \tan u; \tan u_2 = \cot T_1 \tan u; \dots$$

For a fourth example, beginning with $\sin 2T^{000} = \sqrt{b^{0000}}$, we may extend the series to sixteen similar terms, whose sum is u^{0000} ; and so on. In all the examples, $\tan u$ may be replaced by its equal $\sqrt{b} \cdot \sin \theta$. And the computed value of u , or of u^0 or of u^{00} , etc., is to be substituted in the proper equation of F , of the series shown at the beginning of this Section.

VIII. QUADRANTAL QUADRATURE.—It will be instructive to observe how the preceding system and other known scales of moduli and amplitude naturally originate in the integration for F by quadrantal quadrature. Let us first represent the binomial coefficients by the well known integrals with respect to an auxiliary arc ω . Thus,

$$dF = \frac{d\theta}{\sqrt{(1 - e^2 \sin^2 \theta)}} = d\theta (1 + \frac{1}{2}e^2 \sin^2 \theta + \frac{3}{8}e^4 \sin^4 \theta + \dots),$$

$$dF = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} d\omega (1 + \sin^2 \omega \cdot e^2 \sin^2 \theta + \sin^4 \omega \cdot e^4 \sin^4 \theta + \dots) d\theta.$$

Taking the sum of this geometric series, and integrating in respect to θ ,

$$dF = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \frac{d\omega d\theta}{1 - e^2 \sin^2 \omega \sin^2 \theta},$$

$$F = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \frac{d\omega}{\sqrt{(1 - e^2 \sin^2 \omega)}} \tan^{-1} [\sqrt{(1 - e^2 \sin^2 \omega)} \cdot \tan \theta].$$

As heretofore, let F be replaced by Ax ; let $A = \sqrt{(1 - e^2 \sin^2 \omega)}$; then $\int d\omega \div A = A \cdot \frac{1}{2} \pi = \int A dx'$. Dividing out A , and regarding ω or A as an implicit function of x' , we have

$$x = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} dx' \cdot \tan^{-1} (A \cdot \tan \theta).$$

While x denotes the true integral, let Ψ denote its approximate value as found by the common trapezoidal formulas of quadrature. Thus for the two sides of one trapezoid,

$$\Psi = \frac{1}{2} \tan^{-1} (A_0 \tan \theta) + \frac{1}{2} \tan^{-1} (A_1 \tan \theta), \text{ or}$$

$$\tan (2\Psi - \theta) = b \tan \theta; A_0 = 1, A_1 = b.$$

The original scale of Landen and Lagrange described in Sections III, IV, is here evidently reproduced in amplitude.

Taking the middle ordinate of one trapezoid,

$$\Psi = \tan^{-1} (A_{\frac{1}{2}} \tan \theta), \text{ or } \tan \Psi = \sqrt{b} \cdot \tan \theta.$$

Here we recognize the first new amplitude u of Section VII. Again taking the middle ordinates in two trapezoids,

$$\Psi = \frac{1}{2} \tan^{-1} (A_{\frac{1}{4}} \tan \theta) + \frac{1}{2} \tan^{-1} (A_{\frac{3}{4}} \tan \theta).$$

Observing that $A_{\frac{1}{4}}, A_{\frac{3}{4}}$ are complementary, or $A_{\frac{1}{4}} A_{\frac{3}{4}} = b$; we have,

$$\tan 2\Psi = \frac{(A_{\frac{1}{4}} + A_{\frac{3}{4}}) \tan \theta}{1 - b \tan^2 \theta} = \frac{(A_{\frac{1}{4}} + A_{\frac{3}{4}})}{2\sqrt{b}} \tan 2u = \frac{\tan 2u}{\sqrt{b^0}}.$$

This equation evidently coincides with that of the second new amplitude, and is the type of those that follow, in the beginning of Section VII.

Let us digress to verify the equality of the last two members. For if equal, $A_{\frac{1}{4}} + A_{\frac{3}{4}} = 2\sqrt{b} \div \sqrt{b^0} = \sqrt{b} \cdot \sqrt{[2(1 + b)]}$; $A_{\frac{1}{4}} A_{\frac{3}{4}} = b$. Whence

$\mathcal{A}_{\frac{1}{2}}$ is found by a quadratic, and $\mathcal{A}_{\frac{1}{2}}^2 = \sqrt{b} \cdot [1 + b - \sqrt{b} + (1 - \sqrt{b})\sqrt{1 + b}]$. The value of $\mathcal{A}_{\frac{1}{2}}^2$ found by the independent method of bisection, being precisely the same, verifies the equality.

On taking the sides of two trapezoids, we again find the values of $\mathcal{A}$ to occur in the geometric progression 1, $\sqrt{b}$, b ; thus,

$$\mathcal{F} = \theta + 2 \tan^{-1}(\sqrt{b} \cdot \tan \theta) + \tan^{-1}(b \tan \theta) = \tan^{-1}[(\tan 2u^{\circ}) \div b^{\infty}].$$

Generally let p denote the number of trapezoids, p being an odd number. Taking the first side or θ with twice the other alternate sides, we have $\mathcal{F}$ the amplitude of summation in the first Theorem of Jacobi. Thus,

$$\mathcal{F} = \theta + 2 \tan^{-1}(\mathcal{A}_2 \tan \theta) + \dots + 2 \tan^{-1}(\mathcal{A}_{\frac{p-1}{2}} \tan \theta); \text{ or}$$

$$\mathcal{F} = \theta + 2\theta_2 + 2\theta_4 + 2\theta_6 + \dots + 2\theta_{p-1}.$$

When $p = 3$, for example, let the amplitudes a_1, a_2 be found for the relations $F(e, a_1) = \frac{1}{3}F(e, \frac{1}{2}\pi)$; $F(e, a_2) = \frac{2}{3}F(e, \frac{1}{2}\pi)$; which can be resolved by Legendre's Tables, or by Section VI. Generally let m denote any whole number less than p ; then $F(e, a_m) = \frac{m}{p}F(e, \frac{1}{2}\pi)$.

$$F(e, \theta) = \beta F(h, \mathcal{F}); \tan \theta_m = \frac{\mathcal{A}_m}{p} \tan \theta = \frac{\cos a_m}{\sin a_{p-m}} \cdot \tan \theta.$$

$$h = e^p \sin^4 a_1 \sin^4 a_3 \sin^4 a_5 \dots \sin^4 a_{p-2}.$$

$$\beta = \frac{\sin^2 a_1 \sin^2 a_3 \dots \sin^2 a_{p-2}}{\sin^2 a_2 \sin^2 a_4 \dots \sin^2 a_{p-1}} = \frac{p \cdot F(h, \frac{1}{2}\pi)}{F(e, \frac{1}{2}\pi)}.$$

The decreased modulus h may also be found by trial from the equation

$$\frac{F(h, \frac{1}{2}\pi)}{F[\sqrt{1-h^2}, \frac{1}{2}\pi]} = \frac{F(e, \frac{1}{2}\pi)}{pF[\sqrt{1-e^2}, \frac{1}{2}\pi]}.$$

In the Philosophical Transactions for 1831, Mr. Ivory has simplified the demonstration of Jacobi's Theorem, and extended it to the case where p is an even number, that is to regular trapezoidal quadrature. Thus when p is even,

$$\mathcal{F} = \theta + 2 \tan^{-1}(\mathcal{A}_2 \tan \theta) + \dots + 2 \tan^{-1}(\mathcal{A}_{\frac{p-2}{2}} \tan \theta) + \tan^{-1}(\mathcal{A}_p \tan \theta).$$

The second Theorem of Jacobi can be applied to large moduli, by first changing e to b ; and from $F(b, \theta)$ finding $\beta', h', \mathcal{F}'$ according to a modification of the first Theorem; then changing back from h' to the larger h , where $h = \sqrt{1-h'^2}$, so that $F(e, \theta) = (1 \div \beta')F(h, \mathcal{F}')$. When h becomes virtually 1, the integration is effected by logarithms.

In Section VI, was shown the discovery of Jacobi that the exponents of q in the Theta function increase as the series of squares 1, 4, 9, 16, ... A comparison of results by Jacobi discloses another property equally remarkable. Let $q, {}^{\circ}q, q^{\infty}, \dots$ denote the values of q respectively defined by $b, b^{\circ}, b^{\infty}, \dots$ in the decreasing scale of moduli. Then rigorously $q = q, q^{\circ} = q^2, q^{\infty} = q^4, q^{\infty\infty} = q^8, \dots$ That is, the exponents of q increase regularly in geometrical progression.

Thus in the descending scale of $\varphi = 45^\circ, 9^\circ 52\frac{3}{4}', 25\frac{3}{8}', 2\frac{7}{8}'', \dots$ $\text{Log } q = 2.63563 \pm$; $\text{Log } q^\circ = 3.27127 = \text{Log } q^2$; $\text{Log } q^{\circ\circ} = 11.0850 = \text{Log } q^8$.

The like property belongs to other scales of moduli; and it happily meets the difficulty of finding small arcs to a sufficient number of places from the cosine in the common logarithmic tables.

For large Moduli. Where e is near to 1, we may change to the complementary modulus by assuming as in Section II, $\cos \theta \cdot \cos \theta' = 1$; thus if $d' = \sqrt{1-b^2 \sin^2 \omega}$;

$$dF = \frac{d\theta}{\sqrt{1-e^2 \sin^2 \theta}} = \frac{\sqrt{1-b^2 \sin^2 \theta'} \cdot d\theta'}{\sqrt{1-b^2 \sin^2 \theta'}} = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \frac{\sqrt{1-b^2 \sin^2 \omega} d\omega d\theta'}{1-b^2 \sin^2 \omega \sin^2 \theta'},$$

$$F = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \frac{\sqrt{1-b^2 \sin^2 \omega}}{d'} \tan^{-1}(d' \tan \theta') = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \frac{\sqrt{1-b^2 \sin^2 \omega}}{d'} \tan^{-1} \frac{d' \sin \theta}{\sqrt{1-b^2 \sin^2 \omega}}.$$

By applying common quadrature to the last member, dividing by the common coefficient of $\tan^{-1}$, and then taking the trigonometric tangent of both sides we might obtain formulas corresponding to the second Theorem of Jacobi. Or developing the arc in terms of its tangent, we may apply quadrature to the following sum of the series:

$$F = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \frac{d\omega}{d'} \cdot \frac{1}{2} \log \frac{1+d' \sin \theta}{1-d' \sin \theta}.$$

$$F = \sin \theta + \frac{1}{3}(1-\frac{1}{2}b^2)\sin^3 \theta + \frac{1}{5}(1-b^2+\frac{3}{8}b^4)\sin^5 \theta + \dots$$

$$= a \sin \theta + a_1 \log \frac{1+a_2 \sin \theta}{1-a_2 \sin \theta}, \text{ nearly};$$

where a, a_1, a_2 are to be found from correlative values of F and θ , or from the first three coefficients of powers of $\sin \theta$. When more terms are required, the last formula but two will have the advantage of a common factor, by treating the coefficient of $\log$ as the variable of quadrature, and proceeding with b , as in the former method with e .

IX. INTEGRATION OF FUNCTIONS OF THE SECOND AND THE THIRD ORDER. — The common methods of quadrature are occasionally employed for all the Elliptic Functions. For illustration, required the value of F when $e = \sin 45^\circ$, and $\theta = 90^\circ$. In a first trial, let the 90° be divided into three equal parts of 30° each. Computing for the middle of each part, by the formula for quadrature, we have

$$F(e, \frac{1}{2}\pi) = \frac{1}{3} \frac{\pi}{2} \left(\frac{1}{\sqrt{1-e^2 \sin^2 15^\circ}} + \frac{1}{\sqrt{1-e^2 \sin^2 45^\circ}} + \frac{1}{\sqrt{1-e^2 \sin^2 75^\circ}} \right) + R.$$

The result so found is $1.854053 + R$. The true result is 1.854075 ; or the residue R is 0.000022 . By dividing the amplitude into four or more parts, the integral may thus be found to any degree of accuracy.

Besides the two series for E already given in Section V, we here insert from Jacobi, another series in terms of A, q, x , as previously defined :

$$E(e, \theta) = \frac{2x \cdot E(e, \frac{\pi}{2})}{\pi} + \frac{2\pi}{A} \frac{q \sin 2x - 2q^4 \sin 4x + 3q^9 \sin 6x - \dots}{1 - 2q \cos 2x + 2q^4 \cos 4x - 2q^9 \cos 6x + \dots}$$

And in terms of the scale of Section IV,

$$E(e, \theta) = F(e, \theta) [1 - \frac{1}{2}e^2(1 + \frac{1}{2}e^2 + \frac{1}{4}e^4 e^{e^2} + \frac{1}{8}e^6 e^{e^2} e^{e^2} + \frac{1}{16}e^8 e^{e^2} e^{e^2} e^{e^2} + \dots)] \\ + e [\frac{1}{2}\sqrt{e^2} \sin \varphi^e + \frac{1}{4}\sqrt{e^4} \sin \varphi^{e^2} + \frac{1}{8}\sqrt{e^6} \sin \varphi^{e^2} e^{e^2} + \dots]$$

Among formulas of quadrantal integration, the following is readily verified by differentiation with respect to θ :

$$E(e, \theta) = \cot \theta (1 - \Delta) + \frac{2}{\pi} \int_0^{\frac{\pi}{2}} d\omega \sqrt{1 - e^2 \sin^2 \omega} \cdot \tan^{-1} [\sqrt{1 - e^2 \sin^2 \omega} \cdot \tan \theta].$$

Lastly the value of E may be found by taking a sufficient number of simultaneous terms from Section VII or VIII to represent $dF = A d\mathcal{V} \div n$, and making the obvious substitution for $dF = d\theta \div \Delta$,

$$dE = \sqrt{1 - e^2 \sin^2 \theta} \cdot d\theta \times \Delta \div \Delta = (1 - e^2 \sin^2 \theta) A \times d\mathcal{V} \div n.$$

Thus, when e is so small that two single, with one double term will suffice, we take

$$\mathcal{V} = \theta + 2 \tan^{-1} (\sqrt{b} \cdot \tan \theta) + \tan^{-1} (b \tan \theta). \quad \text{Then}$$

$$dE = (1 - e^2 \sin^2 \theta) \frac{1}{4} A \left(1 + \frac{2\sqrt{b}}{1 - (1-b) \sin^2 \theta} + \frac{b}{1 - (1-b^2) \sin^2 \theta} \right) d\theta;$$

the integral of which reduces to

$$E = \frac{1}{4} A [2(1+b)(1+b+4\sqrt{b})\theta + \frac{1}{2} e^2 \sin 2\theta - 2b \tan^{-1} (\sqrt{b} \cdot \tan \theta) \\ + (1-b) \tan^{-1} (b \tan \theta)].$$

The integral of the Third Species Π may be found in like manner. For illustration, when the above three terms will give a sufficient approximation; substituting for $dF = \frac{1}{4} A \cdot d\mathcal{V}$, we have, since $d\Pi =$

$$\frac{d\theta}{(1+n \sin^2 \theta) \sqrt{1 - e^2 \sin^2 \theta}}; \quad d\Pi = \frac{d\theta}{1+n \sin^2 \theta} \left(1 + \frac{2\sqrt{b}}{1 - (1-b) \sin^2 \theta} + \frac{b}{1 - e^2 \sin^2 \theta} \right) \frac{A}{4}.$$

Integrating by the method of rational fractions,

$$\Pi = \frac{A}{4} \left[1 + \frac{2n\sqrt{b}}{n+1-b} + \frac{nb}{n+e^2} \right] \frac{\tan^{-1} [\sqrt{(1+n)} \cdot \tan \theta]}{\sqrt{(1+n)}} + \frac{A}{4} \left[\frac{2-2b}{1-b+n} \right. \\ \left. \times \tan^{-1} (\sqrt{b} \cdot \tan \theta) + \frac{e^2 \tan^{-1} (b \tan \theta)}{n+e^2} \right].$$

Among other formulas we next insert that of Jacobi: Let

$$+n = -e^2 \sin^2 a; \quad F(e, \theta) = Ax; \quad F(e, a) = Ax'; \quad \text{then,}$$

$$\Pi(n, e, \theta) = Ax + \frac{1}{\cot a \cdot E(e, a)} \left[4x \left(\frac{q \sin 2x'}{1-q^2} + \frac{q^2 \sin 4x'}{1-q^4} + \dots \right) + \frac{1}{2} \log \frac{\theta(x-x')}{\theta(x+x')} \right]$$

Again from Legendre's *Traite des Fonctions Elliptiques*, vol. 1, p. 141;— $\Pi(n, e, \frac{1}{2}\pi) = F(e, \frac{1}{2}\pi) + [\tan a \div \sqrt{(1 - e^2 \sin^2 a)}] [F(e, \frac{1}{2}\pi) E(e, a) - E(e, \frac{1}{2}\pi) F(e, a)]$. Here as before, $n = -e^2 \sin^2 a$.

Lastly in very convergent series, when either n or e is comparatively small, and the amplitude is 90° ; we first find

$$m = \frac{\sqrt{1+n}-1}{\sqrt{1+n}+1} \cdot \left(\frac{2}{e^2}-1 \right); \quad m' = \left(\frac{\sqrt{1+n}-1}{\sqrt{1+n}+1} \right)^2 = k^2;$$

$$A = \frac{2}{\pi} F(e, \frac{1}{2}\pi); \quad B = \frac{2}{\pi} E(e, \frac{1}{2}\pi); \quad p_1 = 2mA - (2m+2k)B; \quad p_2 = \frac{4}{3}mp_1 - \frac{2}{3}m'A; \quad p_3 = \frac{8}{5}mp_2 - \frac{8}{5}m'p_1; \quad p_4 = \frac{12}{7}mp_3 - \frac{12}{7}m'p_2; \dots$$

$$\Pi(n, e, \frac{\pi}{2}) = \frac{\pi}{2\sqrt{1+n}} \left[A - p_1 + p_2 - p_3 + p_4 - \dots \right]$$

For demonstration, we have from Analytic Trigonometry,

$$\frac{1}{1+n \sin^2 \theta} = \frac{1}{\sqrt{1+n}} \left(1 + 2k \cos 2\theta + 2k^2 \cos 4\theta + 2k^3 \cos 6\theta + \dots \right)$$

$$\frac{1}{\sqrt{1-e^2 \sin^2 \theta}} = A - 2A_1 \cos 2\theta + 4A_2 \cos 4\theta - 6A_3 \cos 6\theta + \dots$$

Substituting, multiplying and integrating through a quadrant,

$$\Pi(n, e, \frac{\pi}{2}) = \frac{\pi}{2\sqrt{1+n}} \left(A - 2kA_1 + 4k^2A_2 - 6k^3A_3 + \dots \right)$$

Here k is put for $[\sqrt{1+n}-1] \div [\sqrt{1+n}+1]$; $p_1 = 2kA_1$; $p_2 = 4k^2A_2$; etc. And the above derivation of $p_1, p_2, \dots$ is readily found from that of $A_1, A_2, \dots$ shown in Section V.

The general integration, when the parameter n is greater than e , can be made to depend on a parameter less than e , by the following relation, where $nn' = e^2$; $(1+n)(1+n') = a$;

$$\Pi(n) + \Pi(n') = F + \frac{1}{\sqrt{a}} \tan^{-1} \frac{\sqrt{a} \cdot \tan \theta}{A}.$$

This can be verified by differentiation. When $n = -e^2$, and when $n = -1$, the integrals are shown at (16), (20), in Section I. When $n = n' = \pm e$,

$$\Pi(n) = \frac{1}{2}F + \frac{1}{2\sqrt{a}} \tan^{-1} \frac{\sqrt{a} \cdot \tan \theta}{\sqrt{1-e^2 \sin^2 \theta}}.$$

Again in the special case of $e = 1$, we have

$$\Pi = \int_0^\theta \frac{d\theta}{(1-h \sin^2 \theta) \cos \theta} = \frac{1}{2(1-h)} \left(\sqrt{h} \log \frac{1-\sqrt{h} \sin \theta}{1+\sqrt{h} \sin \theta} + 2 \log \tan(45^\circ + \frac{1}{2}\theta) \right).$$

Lastly by transformation, we have found the following integral, subject to the particular relation,

$$h = 1 \pm b = 1 \pm \sqrt{1-e^2}; \quad A = \sqrt{1-e^2 \sin^2 \theta};$$

$$\Pi = \int_0^\theta \frac{d\theta}{(1-h \sin^2 \theta) A} = \frac{h-2}{2h-2} F(e, \theta) + \frac{1}{4h-4} \log \frac{A+h \sin \theta \cos \theta}{A-h \sin \theta \cos \theta}.$$

The logarithm evidently disappears when $\theta = 90^\circ$.

REMARKS ON MR. MEECH'S ARTICLE ON ELLIPTIC FUNCTIONS.

BY CHAS. H. KUMMELL, U. S. LAKE SURVEY, DETROIT, MICHIGAN.

1. IN Section II, the spherical triangle is used to prove the theorem of addition of the first species, but in this proof $\cos C$ is taken negative. (In this Mr. Meech agrees with all authors I have seen, for instance see Schellbach, page 106.) This has always puzzled me and I finally discovered the following to me more natural demonstration of this important theorem.

Let there be a spherical triangle whose sides are a , b , c and opposite angles φ , ψ , $\pi - \mu$ respectively, or let μ be the exterior angle for which we should have if the triangle was plane

$$\mu = \varphi + \psi, \quad (1)$$

or $\sin \mu = \sin \varphi \cos \psi + \cos \varphi \sin \psi. \quad (2)$



It is required to find analogous relations for the spherical triangle. From μ let fall a perpendicular on c , then we have, if the segment adjacent to b is denoted by d ,

$$\tan d = \tan b \cos \varphi,$$

$$\tan(c-d) = \tan a \cos \psi;$$

hence

$$\sin d = \frac{\tan b \cos \varphi}{\sqrt{(1 + \tan^2 b \cos^2 \varphi)}}; \quad \cos d = \frac{1}{\sqrt{(1 + \tan^2 b \cos^2 \varphi)}};$$

$$\sin(c-d) = \frac{\tan a \cos \psi}{\sqrt{(1 + \tan^2 a \cos^2 \psi)}}; \quad \cos(c-d) = \frac{1}{\sqrt{(1 + \tan^2 a \cos^2 \psi)}};$$

consequently

$$\sin c = \frac{\tan b \cos \varphi + \tan a \cos \psi}{\sqrt{[(1 + \tan^2 b \cos^2 \varphi)(1 + \tan^2 a \cos^2 \psi)]}}. \quad (3)$$

Place

$$\frac{\sin c}{\sin \mu} = \frac{\sin b}{\sin \psi} = \frac{\sin a}{\sin \varphi} = \epsilon; \quad (4)$$

also $\cos c = \sqrt{(1 - \epsilon^2 \sin^2 \mu)} = \Delta \mu; \cos b = \Delta \psi; \cos a = \Delta \varphi; \quad (5)$

substituting into (3) and reducing we obtain

$$\sin \mu = \frac{\sin \varphi \cos \psi \Delta \psi + \sin \psi \cos \varphi \Delta \varphi}{1 - \epsilon^2 \sin^2 \varphi \sin^2 \psi}. \quad (2')$$

This is analogous to (2). The following are deduced from this;

$$\cos \mu = \frac{\cos \varphi \cos \psi - \sin \varphi \sin \psi \Delta \varphi \Delta \psi}{1 - \epsilon^2 \sin^2 \varphi \sin^2 \psi}; \quad (6)$$

$$\Delta\mu = \frac{\Delta\varphi \Delta\psi - \varepsilon^2 \sin\varphi \sin\psi \cos\varphi \cos\psi}{1 - \varepsilon^2 \sin^2\varphi \sin^2\psi}. \quad (7)$$

We have

$$\cos\mu d\mu = \frac{d\sin\mu}{d\varphi} d\varphi + \frac{d\sin\mu}{d\psi} d\psi. \quad (8)$$

But we obtain after reduction

$$\begin{aligned} \frac{d\sin\mu}{d\varphi} &= \frac{\cos\varphi \cos\psi - \sin\varphi \sin\psi \Delta\varphi \Delta\psi}{\Delta\varphi (1 - \varepsilon^2 \sin^2\varphi \sin^2\psi)^2} \left(\Delta\varphi \Delta\psi - \varepsilon^2 \sin\varphi \sin\psi \cos\varphi \cos\psi \right) \\ &= \frac{\cos\mu \Delta\mu}{\Delta\varphi}; \text{ also } \frac{d\sin\mu}{d\psi} = \frac{\cos\mu \Delta\mu}{\Delta\psi}. \end{aligned}$$

Hence in (8)

$$\frac{d\mu}{d\varphi} = \frac{d\varphi}{\Delta\varphi} + \frac{d\psi}{\Delta\psi}. \quad (9)$$

$$\therefore F_0^{\mu}(\varepsilon) = F_0^{\varphi}(\varepsilon) + F_0^{\psi}(\varepsilon); \quad (1')$$

$$\text{also } F_0^{\psi}(\varepsilon) = F_{\psi}^{\mu}(\varepsilon). \quad (10)$$

If $\varepsilon = 0$ (1') becomes the same as (1).

2. In Section III, the method of evaluating elliptics of the first species by means of ascending moduli and corresponding amplitudes is explained as Legendre and most authors have done it. I think however the following method invented by myself will be found more convenient in practice.

Suppose we have to evaluate

$$\int_0^{\varphi} \frac{d\varphi}{\sqrt{(a^2 - c^2 \sin^2\varphi)}} = \frac{1}{a} F_0^{\varphi}\left(\frac{c}{a}\right). \quad (10)$$

$$\text{Place } a' = \frac{1}{2}(a+c); c' = \sqrt{(ac)}, \text{ and } \sin(2\varphi' - \varphi) = (c \div a) \sin\varphi. \quad (11)$$

We have then

$$\frac{1}{a} F_0^{\varphi}\left(\frac{c}{a}\right) = \int_0^{\varphi} \frac{d\varphi}{\sqrt{(a^2 - c^2 \sin^2\varphi)}} = \int_0^{\varphi'} \frac{d\varphi}{\sqrt{(a'^2 - c'^2 \sin^2\varphi)}} = \frac{1}{a'} F_0^{\varphi'}\left(\frac{c'}{a'}\right). \quad (12)$$

Again let $a'' = \frac{1}{2}(a' + c')$; $c'' = \sqrt{(a'c')}$, and $\sin(2\varphi'' - \varphi') = (c' \div a') \sin\varphi'$, (11') then we have, proceeding in this manner

$$\begin{aligned} \frac{1}{a} F_0^{\varphi}\left(\frac{c}{a}\right) &= \int_0^{\varphi} \frac{d\varphi}{\sqrt{(a^2 - c^2 \sin^2\varphi)}} = \int_0^{\varphi'} \frac{d\varphi}{\sqrt{(a'^2 - c'^2 \sin^2\varphi)}} = \frac{1}{a'} F_0^{\varphi'}\left(\frac{c'}{a'}\right) \\ &= \int_0^{\varphi''} \frac{d\varphi}{\sqrt{(a''^2 - c''^2 \sin^2\varphi)}} = \frac{1}{a''} F_0^{\varphi''}\left(\frac{c''}{a''}\right) \\ &= \dots \dots \dots \\ &= \int_0^{\varphi^{(n)}} \frac{d\varphi}{\sqrt{(a^{(n)2} - c^{(n)2} \sin^2\varphi)}} = \frac{1}{a^{(n)}} F_0^{\varphi^{(n)}}\left(\frac{c^{(n)}}{a^{(n)}}\right) \end{aligned}$$

and if $a^{(n)} = c^{(n)}$ within the accuracy of the computation,

$$\begin{aligned} \frac{1}{a} F_0^{\phi} \left(\frac{c}{a} \right) &= \int_0^{\phi^{(n)}} \frac{d\varphi}{c^{(n)} \cos \varphi} = \frac{1}{c^{(n)}} \log \tan \left(\frac{\pi}{4} + \frac{\varphi^{(n)}}{2} \right) \\ &= \frac{1}{c^{(n)}} F_0^{\phi^{(n)}}(1). \end{aligned} \quad (13)$$

The quantity $c^{(n)}$ is the limit reached by taking successively arithmetic and geometric means, or it may be called

the arithmetic-geometric mean of a and c ,

and in ANALYST, Vol. IV, No. 5, p. 157, I have ventured on the symbol

$$\text{arithmetic-geometric mean of } a \text{ and } c = \left\| \frac{1}{2} \left(a + \frac{c}{\times} \right)^{\frac{1}{2}} \right\|.$$

In Vol. IV, No. 4, page 121, I have exhibited the very convenient computation of such a quantity. The scale of amplitudes being computed simultaneously the complete value is found very readily.

3. Similarly, for the descending scale of moduli the following method, first given by Gauss in 1818, may be used.

Suppose we have to evaluate

$$\frac{1}{a_0} F_0^{\phi_0} \left[\sqrt{1 - \frac{b_0^2}{a_0^2}} \right] = \int_0^{\phi_0} \frac{d\varphi}{\sqrt{(a_0^2 \cos^2 \varphi + b_0^2 \sin^2 \varphi)}}. \quad (14)$$

$$\text{Place } a_1 = \frac{1}{2}(a_0 + b_0); \quad b_1 = \sqrt{(a_0 b_0)}; \quad \tan(\varphi_1 - \varphi_0) = (b \div a) \tan \varphi_0, \quad (15)$$

$$\begin{aligned} \text{then } \frac{1}{a_0} F_0^{\phi_0} \left[\sqrt{1 - \frac{b_0^2}{a_0^2}} \right] &= \int_0^{\phi_0} \frac{d\varphi}{\sqrt{(a_0^2 \cos^2 \varphi + b_0^2 \sin^2 \varphi)}} \\ &= \frac{1}{2} \int_0^{\phi_1} \frac{d\varphi}{\sqrt{(a_1^2 \cos^2 \varphi + b_1^2 \sin^2 \varphi)}} = \frac{1}{2a_1} F_0^{\phi_1} \left[\sqrt{1 - \frac{b_1^2}{a_1^2}} \right]. \end{aligned} \quad (16)$$

$$\text{Placing } a_2 = \frac{1}{2}(a_1 + b_1); \quad b_2 = \sqrt{(a_1 b_1)}; \quad \tan(\varphi_2 - \varphi_1) = (b_1 \div a_1) \tan \varphi_1 \quad (17)$$

we have likewise

$$\frac{1}{2a_1} F_0^{\phi_1} \left[\sqrt{1 - \frac{b_1^2}{a_1^2}} \right] = \frac{1}{2^2 a_2} F_0^{\phi_2} \left[\sqrt{1 - \frac{b_2^2}{a_2^2}} \right]. \quad (16')$$

$$\text{If } a_n = b_n = \left\| \frac{1}{2} \left(a_0 + \frac{b_0}{\times} \right)^{\frac{1}{2}} \right\| \quad (18)$$

$$\text{then } \frac{1}{a_n} F_0^{\phi_n} \left[\sqrt{1 - \frac{b_n^2}{a_n^2}} \right] = \frac{1}{a_n} F_0^{\phi_n}(0) = \frac{\varphi_n}{a_n} = \varphi_n \div \left\| \frac{1}{2} \left(a_0 + \frac{b_0}{\times} \right)^{\frac{1}{2}} \right\|. \quad (19)$$

$$\text{Therefore } \frac{1}{a_0} F_0^{\phi_0} \left[\sqrt{1 - \frac{b_0^2}{a_0^2}} \right] = \varphi_n \div 2^n \left\| \frac{1}{2} \left(a_0 + \frac{b_0}{\times} \right)^{\frac{1}{2}} \right\|. \quad (20)$$

This method of evaluating is especially convenient if φ_0 is some multiple of $\frac{1}{2}\pi$, as two examples in the ANALYST prove.

I have prepared formulas by means of the same substitutions for evaluating the second and third species, which I may communicate at some future time.

CONTINUED ROOTS.

BY T. S. E. DIXON, CHICAGO, ILLINOIS.

THE expression $\sqrt{q-p\sqrt{q-p\sqrt{q-\&c.}}}$, from its analogy to the continued fraction $\frac{q}{p+q}$ would be called a *Continued Root*. The two expressions are equivalents, each being an expression for the roots of the quadratic equation $x^2+px=q$.

Continued roots, however, seem to have a wider application than continued fractions, since they may be formed so as to express not only the roots of a quadratic equation, but also the roots of an equation of any degree of the form $x^n \pm px = \pm q$.

Thus $\sqrt[n]{q+p\sqrt[n]{q+p\sqrt[n]{q+\&c.}}} =$ the roots of the cubic equation $x^3 - px = q$, $\sqrt[n]{q+p\sqrt[n]{q+p\sqrt[n]{q+\&c.}}} =$ the roots of the equation $x^n - px = q$.

A combination of the two expressions may also be used to represent the roots of these equations, affording radicals of one less degree. Thus in the equation $x^3 - px = q$, $x = \sqrt[p+q]{\sqrt[p+q]{p+q}}$

The roots of any trinomial equation, of whatever degree, may be expressed in terms of the known quantities under the form of a continued root. Thus if $x^m+px^n=q$, $x = \sqrt[m]{q-p.\sqrt[m+n]{q-p.\sqrt[m+n]{q-\&c.}}}$.

Again, if $x^{m+n}+px^n=q$, $x = \sqrt[n]{q \over p + \sqrt[n-m]{p \over p + \sqrt[n-m]{q \over p + \&c.}}}$

Methods analogous to those which give approximate values to continued fractions may also be applied to continued roots. These roots also give a method for the solution of biquadratic equations.

If $x^2-px=q$ (1), $x = \sqrt{q+p\sqrt{q+p\sqrt{q+\&c.}}}$, $x^2=q+p\sqrt{q+p\sqrt{q+\&c.}}$, $x^4=q^2+p^2q+(2pq+p^3)\sqrt{q+p\sqrt{q+\&c.}}$, whence $x^4-(2pq+p^3)x=q^2+p^2q$ (2). Given $x^4-Ax=B$. Then $2pq+p^3=A$, $q^2+p^2q=B$, whence the cubic $p^6+4Bp^2=A^2$. Obtaining now the values of p and q , we have from (1), $x = \frac{1}{2}[p \pm \sqrt{p^2+4q}]$.

If the biquadratic is of the general form $x^4+Ax^2+Bx=C$, add to eqn. (2) $rx^2=q+prx$, transpose and $x^4+rx^2-(p^3+2pq+r)x=q^2+p^2q+qr$, whence the cubic $p^6+2Ap^4+(A^2+4C)p^2=B^2$; where x is found as above.

It is interesting to note that continued roots have their converse, which by analogy would be called *Continued Powers*. Thus the four expressions following are equivalents, each being an expression for the roots of the equation $x^2 + px = q$.

$$\frac{q}{p+q}, \quad \frac{-p+q}{-p+q}, \quad \sqrt{q-p}\sqrt{q-\&c.}, \quad \frac{q}{p} - \frac{1}{p} \left(\frac{q}{p} - \frac{1}{p} \right) \left(\frac{q}{p} - \frac{1}{p} \right) \&c.$$

$\frac{q}{p+\&c.}, \quad \frac{-p+q}{-p+\&c.},$

LEAST SQUARES.

BY R. J. ADCOCK. (CONTINUED FROM P. 184, VOL. IV.)

LIMITS OF ERRORS.—By first and second formulas on pages 183 and 184,

$$p = \frac{n^*}{4\pi m S(d_1^2)} \quad (1)$$

is the probability that any point of the total number $4\pi S(d_1^2)m$, shall be at or on any required point, line or surface, and

$$p = \left(\frac{n}{4\pi m S(d_1^2)} \right)^n \quad (2)$$

is the probability for n points of the same number. Each of which values of p is greatest when $S(d_1^2) =$ a minimum.

Let it be assumed that, in different sets of n points similarly observed, the minimum $S(d_1^2)$ remains unchanged, or that the total number of points, in determining the probability in (1) or (2), remains the same, which is nearly true in practice and more nearly so as n is increased.

Now the number of points which are at the distance l from any point is $4\pi m l^2$, therefore, under the assumption, the probability that a random point shall be at the distance l from the foot of any one of the normals $d_1, d_2 \dots d_n$,

is
$$p = \frac{4\pi m n l^2}{4\pi m S(d_1^2)} = \frac{n l^2}{S(d_1^2)} \dots (3) \quad \text{Hence } l = \sqrt{\left(\frac{p S(d_1^2)}{n} \right)} \quad (4)$$

is the limit within which equal normals or errors are situated when their probability is p . $p = 1$, gives $l = \sqrt{[S(d_1^2) \div n]} \dots (5) =$ what is called the mean error, that is, if each error is of the same magnitude it will be $\sqrt{[S(d_1^2) \div n]}$. And $p = \frac{1}{2}$, gives $l \sqrt{[S(d_1^2) \div 2n]} \dots (6)$ which is called the probable error of a single observation, that is, if $\frac{1}{2}$ the errors are equal, each will be $\sqrt{[S(d_1^2) \div 2n]}$. Hence this value should include one half of the n errors, and a single one is as likely to be less than, as to exceed it.

*Instead of 1 as the numerator in the formula near the bottom of p. 183, Vol. IV, read n . Also, for π in the same formula and in the two formulas on p. 184, read 4π .

$\dagger S(d_1^2) = d_1^2 + d_2^2 + d_3^2 + \dots + d_n^2$.

Let it be assumed that any point of matter attracts any other point of matter as the product of their masses and inversely as the square of the distance between them, and that these attractions are all summed together to give the total attraction of a body on another body, which is the sum of the attractions of all the particles of the first body on all the particles of the second body. Hence the total attraction of a body on another body is the sum of the attractions of all the particles of the first body on all the particles of the second body. Let it be assumed that the attraction of a point of matter on another point of matter is as the product of their masses and inversely as the square of the distance between them. Hence the total attraction of a body on another body is the sum of the attractions of all the particles of the first body on all the particles of the second body. Let it be assumed that the attraction of a point of matter on another point of matter is as the product of their masses and inversely as the square of the distance between them. Hence the total attraction of a body on another body is the sum of the attractions of all the particles of the first body on all the particles of the second body.

Universal Proposition in Attraction.—If the points of every particle matter attract the points of every other particle of matter, then the mutual attractions between any two particles, whose dimensions are infinitely small in comparison with the distance between them, will be directly as the product of their masses and inversely as the square of the distance between them.

That it is as the product of the masses, when the distance is constant, results from the consideration that if the density of one be increased or diminished in any ratio, the number of attracting points and forces is increased or diminished in the same ratio without affecting their direction; the force of attraction varying then as the mass of one when the other is constant must vary as the product of the masses when neither is constant.

That it is in the inverse ratio of the square of the distance, follows first from the fact that any two points of matter of the same density, attract each other with a constant force for all distances, because the point of application of a force may be any where on its line of direction, distance being a quantity of a different kind has no effect on force; second each point of each particle being a centre of attraction for all other particles, the number of these equal forces applied between two particles is directly as the product of their masses and inversely as the square of the distance between them, as necessary consequence of their situation.

R. J. Adcock.

[My request of Mr. Adcock, the above note is re-stated, as amended by him and returned to us with the "proof" to which we referred at page 19 Vol. IV.]

In proof of Mr. Adcock's proposition, and as a correction of our assumption at p. 190, Vol. IV, that the attraction is as the sum of the masses,

insert the following extract from a paper on the subject furnished by Alexander Evans, Esq., of Elkton, Maryland.—Ed.]

In *Poisson's Mechanics*, Vol. I, Ch. VII, Section 242, will be found the following: "Seeing the smallness of the sun and planets with respect to the distances which separate them, we conceive that these attractions may be regarded with sufficient approximation as those of forces parallel and equal in the whole extent of the same planet, their resultant is therefore equal to their sum and the distance remaining the same "*la force motrice*" of each planet is proportional to the product of its mass and that of the sun." * * *

"Let f be the attractive force of one of the two bodies upon the other, that is to say, the numerical ratio of its intensity to that of the force chosen for unity: let also M and m be the masses of the sun and that of the planet, "*la force motrice*" of the planet will be fMm at the unit of distance, and will become $\frac{fMm}{r^2}$ at any distance r , * * *." "*La force motrice*" of the mass

M due to the attraction of m is also represented by $\frac{fMm}{r^2}$ in such a manner that the reaction of each planet upon the sun is equal and contrary to the action of the star upon the planet, but "*la force motrice*" $\frac{fMm}{r^2}$ acting upon the two masses M and m will impress upon them at every instant velocities infinitely small which are respectively proportional to these masses, or otherwise expressed, their "*forces accellatrices*" are $\frac{fm}{r^2}$ and $\frac{fM}{r^2}$."

That is, the forces of nature being measured by the velocity acquired in the unit of time, it is necessary to divide each "*force motrice*" $\frac{fMm}{r^2}$, by the respective masses, M and m , in order to obtain the velocity of each body, which varies inversely as its mass; these *accelerative velocities* therefore become $\frac{fm}{r^2}$ and $\frac{fM}{r^2}$; these are taken as the measure of the cele-

tial (as of all) forces, and amount together to $\frac{f(M+m)}{r^2}$, so that the masses of the two bodies may be supposed united in one body for the expression of the *accelerative* or velocity force (so to speak).—And Poisson says further:

"It is necessary to conceive that we impress at every instant upon the sun an infinitely small velocity equal and contrary to that which it receives from the attraction of the planet; but in order not to alter the relative movements of the two bodies it will be necessary at the same time to impress this velocity upon the planet, and this amounts to applying to it a "*force accellatrice*" equal and contrary to that of the sun: then in the movement of

which there is question the "force *acceleratrice*" of the planet m will be constantly directed towards the sun M and equal to the sum of the two forces $\frac{fM}{r^2}$ and $\frac{fm}{r^2}$; if then we wish to express it by $\frac{\mu}{r^2}$ as in number 225 it is necessary to take $\mu = f(M+m)$."

SOLUTION OF A PROBLEM.

BY MARCUS BAKER, U. S. COAST SURVEY.

If the centers of the escribed circles to any plane triangle be joined two and two a new triangle is formed, which call the *first derived triangle*; if in like manner the centers of the escribed circles to the first derived triangle be joined two and two another triangle is formed, which call the *second derived triangle*. Let this process be continued till we reach the n^{th} derived triangle. It is required to determine an expression for the area of this n^{th} derived triangle in terms of functions of the original triangle.

Let $\triangle_0$ represent the original triangle, K_0 its area, s its semiperimeter and R_0 the radius of its circumscribing circle; also let $\triangle_n$, K_n , s_n and R_n represent corresponding functions of the n^{th} derived triangle.

Let A, B, C represent the angles and a, b, c the sides of the original triangle, and O', O'', O''' the centers of the first set of escribed circles; then is $O''A \sin \frac{1}{2}A = s - c$ and $O'''A \sin \frac{1}{2}A = s - b$ whence $O''O''' = a \div \sin \frac{1}{2}A$;

$$\therefore \left. \begin{aligned} O''O''' &= \frac{a}{\sin \frac{1}{2}A} = 4R \cos \frac{1}{2}A; & O'''O' &= \frac{b}{\sin \frac{1}{2}B} = 4R \cos \frac{1}{2}B; \\ O'O'' &= \frac{c}{\sin \frac{1}{2}C} = 4R \cos \frac{1}{2}C \end{aligned} \right\}. \quad (1)$$

$$K_1 = \frac{O'O'' \cdot O''O''' \cdot O'''O'}{4R_1} = \frac{64R^3 \cos \frac{1}{2}A \cos \frac{1}{2}B \cos \frac{1}{2}C}{8R} = 8R^2 \cos \frac{1}{2}A \cos \frac{1}{2}B \cos \frac{1}{2}C.$$

But $4R \cos \frac{1}{2}A \cos \frac{1}{2}B \cos \frac{1}{2}C = s$ and therefore

$$K_1 = 2Rs. \quad (2)$$

It is an immediate inference from (2) that $K_2 = 2R_1s_1$, or generally that

$$K_n = 2R_{n-1}s_{n-1}. \quad (3)$$

From an examination of the figure we see that the angles of the first derived triangle are the arithmetic means of the angles of the original triangle taken two and two. The second derived triangle stands in the same relation to the first derived triangle that the first derived triangle stands to the original triangle, i. e., its angles are the arithmetic means of the angles of the next preceding triangle taken two and two.

The angles of the successive triangles will therefore have the following values:—

Δ	A	B	C
Δ_1	$(\frac{1}{2}) (\pi - A)$	$(\frac{1}{2}) (\pi - B)$	$(\frac{1}{2}) (\pi - C)$
Δ_2	$(\frac{1}{2})^2 (\pi + A)$	$(\frac{1}{2})^2 (\pi + B)$	$(\frac{1}{2})^2 (\pi + C)$
Δ_3	$(\frac{1}{2})^3 (3\pi - A)$	$(\frac{1}{2})^3 (3\pi - B)$	$(\frac{1}{2})^3 (3\pi - C)$
Δ_4	$(\frac{1}{2})^4 (5\pi + A)$	$(\frac{1}{2})^4 (5\pi + B)$	$(\frac{1}{2})^4 (5\pi + C)$
Δ_5	$(\frac{1}{2})^5 (11\pi - A)$	$(\frac{1}{2})^5 (11\pi - B)$	$(\frac{1}{2})^5 (11\pi - C)$
Δ_6	$(\frac{1}{2})^6 (21\pi + A)$	$(\frac{1}{2})^6 (21\pi + B)$	$(\frac{1}{2})^6 (21\pi + C)$
...	...	...	...
Δ_n	$(\frac{1}{2})^n (g\pi \pm A)$	$(\frac{1}{2})^n (g\pi \pm B)$	$(\frac{1}{2})^n (g\pi \pm C)$

The *upper* signs belong to *even* values of n and the *lower* to *odd* values of n , while g , the coefficient of π , is a number which it is impracticable if not impossible to express, but which is very easily derived from the corresponding coefficient in the preceding term and therefore easily tabulated.

Letting g stand for the coefficient of π in the n^{th} triangle, g' for the corresponding coefficient in the next preceding triangle, g'' in the next preceding &c., then

$$g = 2g' \pm 1, \quad g' = 2g'' \mp 1 \text{ \&c.,}$$

where the *upper* sign is to be used when g belongs to the *odd* triangle and the *lower* for the *even* triangle.

The following table gives the value of g for all values of n from 0 to 50.

n	g	n	g	n	g
0	0	17	43 691	34	6 399 391 061
1	1	18	87 381	35	12 798 782 123
2	1	19	175 763	36	25 597 564 245
3	3	20	351 525	37	51 195 128 491
4	5	21	703 051	38	102 390 256 981
5	11	22	1 406 101	39	204 780 513 963
6	21	23	2 812 203	40	409 561 027 925
7	43	24	5 624 405	41	819 122 052 851
8	85	25	11 248 811	42	1 638 244 111 701
9	171	26	22 497 621	43	3 276 488 223 403
10	341	27	44 995 243	44	6 552 976 446 805
11	683	28	99 990 485	45	13 105 952 893 611
12	1 365	29	199 980 971	46	26 211 905 787 221
13	2 731	30	399 961 941	47	52 423 811 574 443
14	5 461	31	799 923 883	48	104 847 623 148 885
15	10 923	32	1 599 847 765	49	209 695 246 297 771
16	21 845	33	3 199 695 531	50	419 390 492 595 541

Referring now to (1) we get by adding

$$s_1 = 2R (\cos \frac{1}{2}A + \cos \frac{1}{2}B + \cos \frac{1}{2}C). \quad (4)$$

$$\text{Hence } s_2 = 2R_1 [\cos (\frac{1}{2})^2(\pi - A) + \cos (\frac{1}{2})^2(\pi - B) + \cos (\frac{1}{2})^2(\pi - C)]$$

$$\dots \dots \dots s_{n-1} = 2R_{n-2} [\cos (\frac{1}{2})^{n-1}(g''\pi \pm A) + \cos (\frac{1}{2})^{n-1}(g''\pi \pm B) + \cos (\frac{1}{2})^{n-1}(g''\pi \pm C)] \quad (5')$$

$$s_n = 2R_{n-1} [\cos (\frac{1}{2})^n(g'\pi \mp A) + \cos (\frac{1}{2})^n(g'\pi \mp B) + \cos (\frac{1}{2})^n(g'\pi \mp C)]. \quad (5)$$

It is well known that $R_1 = 2R$

$$\text{whence } R_2 = 2R_1 = 2^2R$$

$$R_3 = 2R_2 = 2^3R$$

$$\dots \dots \dots R_{n-1} = 2R_{n-2} = 2^{n-1}R \quad (6')$$

$$R_n = 2R_{n-1} = 2^nR. \quad (6)$$

Substituting in (3) the values of s_{n-1} and $2R_{n-1}$ found in (5') and (6) we have

$$K_n = 2^{2n-1}R^2 [\cos (\frac{1}{2})^{n-1}(g''\pi \pm A) + \cos (\frac{1}{2})^{n-1}(g''\pi \pm B) + \cos (\frac{1}{2})^{n-1}(g''\pi \pm C)],$$

the argument for g'' being $n-2$.

Example. Suppose it is required to find the area of the 10th derived triangle, the original triangle being equilateral and its sides unity.

We have $R^2 = a^2 \div 4 \sin^2 A = \frac{1}{3}$; $g'' = 85$ and the upper sign is to be used because n is even; hence

$$K_{10} = 2^{19} \cos (\frac{1}{2})^9 (85\pi + \frac{1}{3}\pi) = 2^{19} \cos \frac{\pi}{3} = 2^{18} \cdot \frac{1}{2}.$$

Cor. If we refer to the angles of the n th triangle we see that as g is increased the angles differ less and less from one another and hence if g is indefinitely increased the inequality of the angles is indefinitely diminished or the limiting triangle is equilateral.

SOLUTIONS OF PROBLEMS IN NUMBER SIX, VOL. IV.

SOLUTIONS of problems in No. 6, Vol. IV, have been rec'd as follows:

From Marcus Baker, 182 and 183; Prof. W. P. Casey, 182 and 186; Prof. A. B. Evans, 183; Geo. Eastwood, 183 and 185; Prof. J. M. Greenwood, 182, 183, 184 and 185; H. Heaton, 182, 183, 184, 185 and 186; W. E. Heal, 182 and 185; Prof. E. W. Hyde, 182, 183 and 186; Prof. W. W. Johnson, 186; Prof. J. H. Kershner, 182, 183 and 185; Christine Ladd, 186; Prof. D. J. Mc. Adam, 182 and 185; Dr. A. B. Nelson, 183 and 185; Prof. A. T. Smith, 182; S. W. Salmon, 182, 183, 185, 186; Prof. Scheffer, 182, 183, 184, 185, 186; E. B. Seitz, 182—185, (for 186 see Chauv. Geom.)

182. "If $\tan \theta + \sin \theta = m$ and $\tan \theta - \sin \theta = n$, show that $m^2 - n^2 = 4\sqrt{mn}$."

SOLUTION BY S. W. SALMON, MT. OLIVE, NEW JERSEY.

Taking the sum, difference and product of the equations we have

$$2 \tan \theta = m + n \dots (a), \quad 2 \sin \theta = m - n \dots (b),$$

$$\text{and } mn = \tan^2 \theta - \sin^2 \theta = (\sin^2 \theta - \cos^2 \theta \sin^2 \theta) \div \cos^2 \theta = \sin^4 \theta \div \cos^2 \theta \\ = \tan^2 \theta \sin^2 \theta. \quad (c)$$

Multiplying (a) by (b), we have $4 \tan \theta \sin \theta = m^2 - n^2$; whence from (c), $m^2 - n^2 = 4 \sqrt{mn}$.

183. "Show that the altitude of the greatest equilateral triangle that can be circumscribed about a given triangle, is $\sqrt{[a^2 + b^2 - 2ab \cos(\frac{1}{3}\pi + C)]}$."

SOLUTION BY MARCUS BAKER, U. S. COAST SURVEY.

Let ABC be the given triangle and upon a and b describe arcs containing 60° each, O and O' being the centers of these arcs. Through C draw KM parallel to OO' . Draw MA and KB and prolong them to their intersection in N ; then is KMN the maximum equilateral triangle that can be circumscribed about ABC . This is evident since KM , parallel to OO' , is the longest line that can be drawn through C and terminated by the arcs O and O' respectively.



Now the altitude of the equilateral triangle KMN is equal to $\frac{1}{2}KM\sqrt{3} = OO'\sqrt{3}$. Considering the triangle $OO'C$ we see that $OC = a\sqrt{\frac{1}{3}}$, $O'C = b\sqrt{\frac{1}{3}}$ and $OCO' = \frac{\pi}{3} + C$; therefore we have

$$\overline{OO'}^2 = \frac{1}{3}(a^2 + b^2 - 2ab \cos(\frac{\pi}{3} + C)) \text{ or}$$

$$\overline{OO'}\sqrt{3} = \sqrt{[a^2 + b^2 - 2ab \cos(\frac{\pi}{3} + C)]}.$$

184. "In cutting the maximum rectangular parallelepipedon from a frustum of a cone, five pieces are cut off. Find the volume of each of these pieces."

SOLUTION BY E. B. SEITZ, GREENVILLE, OHIO.

The maximum parallelepipedon must have square bases, and the vertices of the upper base must be in the convex surface of the frustum.

Let R, r = the radii of the bases of the frustrum, h = its altitude, h_1 = the altitude of the whole cone, v = the volume of the frustrum, x = the altitude of the parallelopipedon, $2y$ = the diagonal of its base, v_1 = its volume.

$$\text{Then } x : R - y :: h : R - r, \text{ or } h(R - y) - (R - r)x = 0 \quad (1)$$

$$\text{and } v_1 = 2xy^2 = \text{a maximum.} \quad (2)$$

$$\text{Differentiating (1) and (2), we have } hdy + (R - r)dx = 0 \text{ and } 2xdy + ydx = 0, \text{ whence } hy - 2(R - r)x = 0. \quad (3)$$

From (1) and (3) we find

$$x = \frac{hR}{3(R - r)}, y = \frac{2}{3}R, \text{ and } v_1 = \frac{8hR^3}{27(R - r)}.$$

To get this parallelopipedon we may first cut from the top of the frustrum a piece whose volume is

$$v_2 = \frac{\pi h(8R^3 - 27r^3)}{81(R - r)}.$$

Two pieces from the opposite sides of the frustrum may then be cut off. To find the volume of one of these pieces, let it be intersected by a plane parallel to the base, at the distance z from the vertex of the cone; the area of the section will be

$$u = \frac{R^2 z^2}{h_1^2} \cos^{-1} \left(\frac{h_1 \sqrt{2}}{3z} \right) - \frac{R^2}{9h_1} \sqrt{(18z^2 - 4h_1^2)}.$$

Hence the volume is

$$v_3 = \int_{\frac{2}{3}h_1}^{h_1} u dz = \frac{hR^3}{81(R - r)} \left[27 \cos^{-1} \left(\frac{\sqrt{2}}{3} \right) - 2\pi + 8 - 6\sqrt{14} + 2\sqrt{2} \log \left(\frac{3 + \sqrt{7}}{2 + \sqrt{2}} \right) \right].$$

The volume of each of the remaining two pieces is

$$v_4 = \frac{1}{2}(v - v_1 - v_2 - 2v_3).$$

185. "Let the equation $ay^2 + bxy + cx^2 + dy + ex + f = 0$ represent a parabola referred to rectangular axes. Prove that the latus rectum = $(bd - 2ae) \div [(a + c)\sqrt{4a^2 + b^2}]$."

SOLUTION BY PROF. J. M. GREENWOOD, KANSAS CITY, MO.

The general equation of the second degree is a parabola when the first three terms form a perfect square, and may be written thus:

$$a^2x^2 + 2abxy + b^2y^2 + 2gx + 2fy + c = 0. \quad (1)$$

Comparing the given equation with (1), we have $c = a^2$, $a = b^2$, $g = \frac{1}{2}e$, $f = \frac{1}{2}d$ and $b = 2\sqrt{ac}$.

Salmon's Conics, page 186, fifth edition, the expression for the latus rectum is

$$p = \frac{2(fa - gb)}{(a^2 + b^2)^{\frac{3}{2}}}.$$

Writing $c = a^2$, $a = b^2$, $\frac{1}{2}e = g$, $\frac{1}{2}d = f$, and $b^2 = 4ac$, in the formula and reducing we get

$$p = (bd - 2ae) \div [(a + c)\sqrt{4a^2 + b^2}].$$

186. " $A_1, A_2; B_1, B_2$ and C_1, C_2 are three pairs of points in a plane such that the three lines A_1A_2, B_1B_2, C_1C_2 meet in a common point O . Let B_1C_1 and B_2C_2 meet in a , and determine in like manner the points b and c . Prove that a, b and c are in a straight line.

Note. We have thus 10 points situated 3 by 3 on 10 lines, each point being the intersection of 3 lines, so that for each point there are three pairs of points collinear with it, and three points not joined to it in the figure but collinear with each other. Devise a notation which shall express the mutual relation of these points and lines."

SOLUTION BY PROF. JOHNSON.

First suppose the three straight lines OA_1, OB_1, OC_1 not to be in the same plane; then, since the point a is on the line B_1C_1 , it is in the plane $A_1B_1C_1$, and since it is on the line B_2C_2 , it is in the plane $A_2B_2C_2$. In like manner b and c are at once in the plane $A_1B_1C_1$, and in the plane $A_2B_2C_2$; hence a, b and c are in the intersection of these two planes, that is in a straight line. But, when the three lines OA_1, OB_1 and OC_1 are in the same plane, the figure may be regarded as the projection of a similarly formed figure in space, and the points a, b and c will be the projections of three points in a straight line, and therefore will themselves be in a straight line.

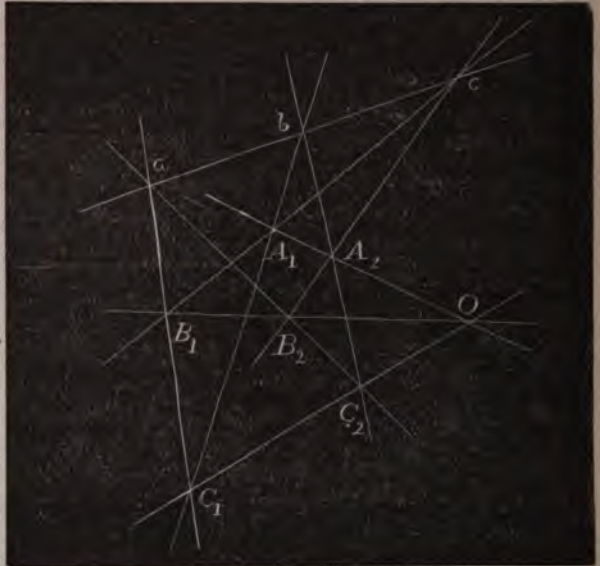
The mutual relations of the 10 points and the 10 lines of the figure are expressed by the following notation. Take any five symbols as a, b, c, d, e ; then 10 combinations, two by two, may be used to denote the 10 points, and their 10 combinations, three by three, to express the 10 lines; in such a manner that the three points on $(a b c)$ are $(a b)$ $(a c)$ and $(b c)$, and the lines which pass through $(a b)$ are $(a b c)$ $(a b d)$ and $(a b e)$. Thus a line passes through a point, if two of their symbols agree; two points are joined by a line in the figure, if they agree in one of their symbols; and the three

points which are not joined in the figure to the point denoted by a given pair of the symbols [as $(a\ b)$], are situated on the line denoted by the combination of the other three symbols. [as $(c\ d\ e)$.]

SOLUTION BY CHRISTINE LADD.

$A_1B_1C_1$ and $A_2B_2C_2$ are triangles having a centre of homology, O ; hence, as is well known, they have an axis of homology $a\ b\ c$.

The notation of anharmonic ratios is the simplest that has hitherto been devised for expressing the relations of the points and lines of a geometrical net, whether plane or in space. According to that notation, the anharmonic co-ordinates of any point P , referred to a given triangle ABC , and a given point, O , are said to be x, y, z , when the following values obtain for anhar-



monics of pencils: $(A.BOCP) = y \div z$, $(B.COAP) = z \div x$, $(C.AOBP) = x \div y$. The point itself is then denoted by the symbol

$$P = (x, y, z).$$

If the triangle $A_1B_1C_1$ be taken as triangle of reference, we have the following symbols for points:—

$$\begin{aligned} A_1 &= (1, 0, 0), & B_1 &= (0, 1, 0), & C_1 &= (0, 0, 1); \\ A_2 &= (2, 1, 1), & B_2 &= (1, 2, 1), & C_2 &= (1, 1, 2); \\ a &= (0, 1, -1), & b &= (-1, 0, 1), & c &= (1, -1, 0); \\ O &= (1, 1, 1). \end{aligned}$$

Any three points are colinear when the determinant of their co-ordinates vanishes.

If the anharmonic co-ordinates of a line are l, m, n , the line itself is symbolically denoted by

$$L = [l, m, n].$$

For the lines connected with the figure, we have the following symbols:—

$$B_1C_1 = [1, 0, 0], \quad C_1A_1 = [0, 1, 0], \quad A_1B_1 = [0, 0, 1];$$

$$O A_1 = [0, 1, -1], \quad O B_1 = [-1, 0, 1], \quad O C_1 = [1, -1, 0];$$

$$B_2C_2 = [-3, 1, 1], \quad C_2A_2 = [1, -3, 1], \quad A_2B_2 = [1, 1, -3];$$

$$a b c = [1, 1, 1];$$

and any three are concurrent when the determinant of their coordinates vanishes. A point, P , is on a line, L , if $lx + my + nz$ vanishes identically.

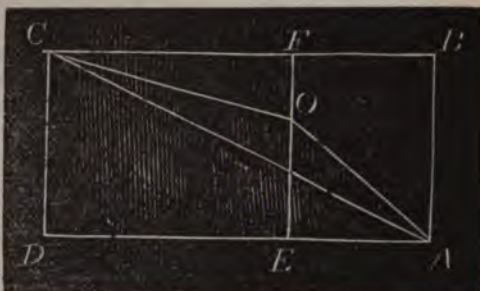
This notation is especially adapted to reciprocations. In order to pass from a point to a line, it is only necessary to exchange the symbols () and [].

[Professors Scheffer, Johnson and A. T. Smith, each answered the Query published at p. 176, Vol. IV, but want of room compels us to defer the publication of the answer to No. 2.]

PROBLEMS.

187. BY ISAAC H. TURRELL, CINCINNATI, OHIO.—Six circles may be described each of which shall touch four of the others. Prove that the lines joining the centers of the non-touching pairs, are concurrent.

188. BY HENRY C. ALLEN, NEW YORK CITY.—Two given rectangular planes, $ABFE$ and $EFCD$ lie contiguous, as in the figure. The plane BE is hard ground whereon one can travel at the rate $2m$; the other is soft ground whereon the rate of travel is m . Required the point O of crossing EF so that the time of journey from A to C shall be the quickest possible.



189. BY PROF. J. H. KERSHNER, MERCERSBURG, PA.—Find by Algebra the series whose sum is 127, when the sum of the squares of the seven terms is 5461.

190. BY PROF. ORSON PRATT, SEN.—Transform the equation,

$$x^8 + 9x^5 + 36x^7 + 85x^6 + 132x^5 + 141x^4 + 105x^3 + 54x^2 + 18x + 9 = 0,$$
 into an equation of the third degree.

191. BY W. E. HEAL. — Compute, in the most convenient way, the product of the differences of the equation $x^5 - ax^4 + bx^3 - cx^2 + dx - e = 0$.

192 BY GEO. M. DAY, LOCKPORT, N. Y. — Find the average area of all the circles inscribed in a given semicircle.

193. BY E. B. SEITZ, GREENVILL, OHIO. — A triangle is formed by joining three points taken at random in the surface of a given triangle. Find the chance that the circle circumscribing this triangle lies wholly within the given triangle.

194. BY PROF. D. J. MC. ADAM, WASHINGTON, PA. — Sum the series

$$1 - \frac{1}{2^2} + \frac{1}{4^2 \cdot 2^2} - \frac{1}{6^2 \cdot 4^2 \cdot 2^2} + \frac{1}{8^2 \cdot 6^2 \cdot 4^2 \cdot 2^2} - \&c.$$

195. BY PROF. J. SCHEFFER, COLLEGE OF ST. JAMES, MD. — A uniform rod rests with one extremity against a rough vertical wall, and with the other extremity on a rough horizontal plane, such that it is held in equilibrium by friction alone. The beam is *not* in a vertical plane. The coefficient of friction of the horizontal plane and vertical wall being respectively μ and μ' ; find the normal pressure of the rod upon the horizontal plane and vertical wall, and the exact position of the rod with reference to the two latter planes.

PUBLICATIONS RECEIVED.

A List of Writings Relating to the Method of Least Squares, with Historical and Critical Notes.

By MANSFIELD MERRIMAN, PH. D., Instructor in the Sheffield Scientific School of Yale College. 82 pp. 8vo. New Haven, Conn. 1877.

Science Observer. 8 pp. 8vo. Monthly. 50cts. per annum. Boston, Mass.

ERRATA.

On page 154, Vol. IV, line 12, for $\frac{3}{2}$, before (), read $\frac{1}{2}$.

" " 190, " " " 3, for y^2 read x^2 .

" " 191, " " " 10, from bottom, for $2ac$, read $2ae$.

" " 13, (current volume) line 4, from bottom, for q, o, qq^{oo} , read q, qo, q^{oo} .

THE ANALYST.

VOL. V.

MARCH, 1878.

No. 2.

ON THE MOTION OF THE CENTRE OF GRAVITY OF THE EARTH AND MOON.

BY G. W. HILL, PH. D.

THAT the motion of the centre of gravity of the earth and moon is sensibly the same as if the masses of these two bodies were concentrated at this centre has been confidently asserted over and over again. However a little scepticism on the matter may not be altogether ill-advised. Were this assertion true it would follow that, setting aside the action of the planets, we should get the sensibly exact mean angular motion of this centre about the sun, by first deriving the mean distance a' from the elliptic value of the radius vector

$$r' = a'[1 + \frac{1}{2}e'^2 + \text{periodic terms}],$$

and then n' from the equation $n' = \sqrt{\frac{M}{a'^3}}$, M denoting the sum of the masses of the sun, earth and moon.

Let us see whether this value is sensibly exact under the conditions we suppose.

Assume that the masses of the sun, earth and moon are denoted by m_1 , m_2 and m_3 , and their rectangular coordinates severally by ξ_1, η_1, ζ_1 ; ξ_2, η_2, ζ_2 ; ξ_3, η_3, ζ_3 . And let the rectangular coordinates of the moon relative to the earth be denoted by x, y and z ; those of the sun relative to the centre of gravity of the earth and moon by x', y' and z' ; and those of the centre of gravity of the three bodies by X, Y and Z . Then from an attentive consideration of the subjoined figure, where S, E and M denote the positions of the sun, earth and moon, C the centre of gravity of the last two bodies, and C' the centre of gravity of all three, it will be seen that, if we put



$$\mu = \frac{m_3}{m_2 + m_3}, \quad \mu' = \frac{m_2 + m_3}{m_1 + m_2 + m_3},$$

we shall have

$$\begin{aligned}\xi_1 &= \mu'x' + X, \\ \xi_2 &= (\mu' - 1)x' - \mu x + X, \\ \xi_3 &= (\mu' - 1)x' + (1 - \mu)x + X,\end{aligned}$$

with two groups, of three equations each, for the η and ζ , obtained from these by writing, in the second members, for x and X , y and Y , and again z and Z .

If we differentiate the equations just written, then square and add the results after having multiplied them severally by m_1 , m_2 and m_3 we shall get

$$m_1 d\xi_1^2 + m_2 d\xi_2^2 + m_3 d\xi_3^2 = m_1 \mu' dx'^2 + m_2 \mu dx^2 + M dX^2.$$

From this equation it is evident that, if Ω denote the potential function, the differential equations, determining the variables x , y , z , x' , y' , z' , are

$$\begin{aligned}m_2 \mu \frac{d^2 x}{dt^2} &= \frac{d\Omega}{dx}, & m_1 \mu' \frac{d^2 x'}{dt^2} &= \frac{d\Omega}{dx'}, \\ m_2 \mu \frac{d^2 y}{dt^2} &= \frac{d\Omega}{dy}, & m_1 \mu' \frac{d^2 y'}{dt^2} &= \frac{d\Omega}{dy'}, \\ m_2 \mu \frac{d^2 z}{dt^2} &= \frac{d\Omega}{dz}, & m_1 \mu' \frac{d^2 z'}{dt^2} &= \frac{d\Omega}{dz'}.\end{aligned}$$

Hence it may be gathered that the disturbing function for the motion of the sun relative to the centre of gravity of the earth and moon differs from the corresponding function for the motion of the moon relative to the earth only by a constant factor which depends on the masses.

The expression for Ω is

$$\Omega = \frac{m_1 m_2}{A_{1,2}} + \frac{m_1 m_3}{A_{1,3}} + \frac{m_2 m_3}{A_{2,3}},$$

where the A 's are given by the equations

$$\begin{aligned}A_{1,2}^2 &= (x' + \mu x)^2 + (y' + \mu y)^2 + (z' + \mu z)^2, \\ A_{1,3}^2 &= [x' - (1 - \mu)x]^2 + [y' - (1 - \mu)y]^2 + [z' - (1 - \mu)z]^2, \\ A_{2,3}^2 &= x^2 + y^2 + z^2.\end{aligned}$$

Let us put

$$r^2 = x^2 + y^2 + z^2, \quad r'^2 = x'^2 + y'^2 + z'^2, \quad rr'S = xx' + yy' + zz'.$$

Then

$$\begin{aligned}A_{1,2}^2 &= r'^2 + 2\mu rr'S + \mu^2 r^2, \\ A_{1,3}^2 &= r'^2 - 2(1 - \mu)rr'S + (1 - \mu)^2 r^2.\end{aligned}$$

Since the ratio $\frac{r}{r'}$ is only about $\frac{1}{400}$, and μ about $\frac{1}{80}$, it is convenient to expand, in $\mathcal{Q}$, the reciprocals of $\mathcal{A}_{1,2}$ and $\mathcal{A}_{1,3}$ in infinite series proceeding according to ascending powers of $\frac{r}{r'}$. This, in both cases, evidently depends on the developement of

$$(1-2ax+a^2)^{-\frac{1}{2}}$$

in powers of a . By the Theorem of Lagrange, in solving the equation $y-aF(y)=x$ with respect to y , we get

$$y = x + aF(x) + \frac{a^2}{1.2} \frac{d.F(x)^2}{dx} + \dots + \frac{a^n}{1.2.3 \dots n} \frac{d^{n-1}.F(x)^n}{dx^{n-1}} + \dots,$$

whence

$$\frac{dy}{dx} = 1 + a \frac{d.F(x)}{dx} + \frac{a^2}{1.2} \frac{d^2.F(x)^2}{dx^2} + \dots + \frac{a^n}{1.2.3 \dots n} \frac{d^n.F(x)^n}{dx^n} + \dots$$

Let us suppose that we have here $F(y) = \frac{1}{2}(y^2-1)$; the equation, on which y depends, becomes then $y - \frac{1}{2}a(y^2-1) = x$, and the resolution of this quadratic in y gives $1 - ay = \sqrt{1 - 2ax + a^2}$, and by differentiation $\frac{dy}{dx} = (1-2ax+a^2)^{-\frac{1}{2}}$. Consequently

$$\begin{aligned} (1-2ax+a^2)^{-\frac{1}{2}} &= 1 + \frac{a d.(x^2-1)}{2 dx} + \frac{a^2 d^2.(x^2-1)^2}{2.4 dx^2} + \dots + \frac{a^n d^n.(x^2-1)^n}{2.4.6 \dots 2n dx^n} + \dots \\ &= 1 + a \frac{2}{2} x \\ &\quad + a^2 \left[\frac{4.3}{2.4} x^2 - \frac{2}{1} \cdot \frac{2.1}{2.4} \right] \\ &\quad + a^3 \left[\frac{6.5.4}{2.4.6} x^3 - \frac{3}{1} \cdot \frac{4.3.2}{2.4.6} x \right] \\ &\quad + a^4 \left[\frac{8.7.6.5}{2.4.6.8} x^4 - \frac{4}{1} \cdot \frac{6.5.4.3}{2.4.6.8} x^2 + \frac{4.3}{1.2} \cdot \frac{4.3.2.1}{2.4.6.8} \right] \\ &\quad + \dots \end{aligned}$$

The law of the numerical coefficients in this series is so plain that we can set down as many terms as we have occasion for.

In making the application to the reciprocals of $\mathcal{A}_{1,2}$ and $\mathcal{A}_{1,3}$ we must put, in the first case, $a = -\frac{\mu^r}{r}$, in the second, $a = (1-\mu)\frac{r}{r'}$, and in both, $x = S$.

We obtain as the potential function proper for the relative motion of the moon about the earth

$$\frac{1}{m_2 \mu} \mathcal{Q} = \frac{m_2 + m_3}{r} + m_1 \left[\frac{1}{\mu} \frac{1}{\mathcal{A}_{1,2}} + \frac{1}{1-\mu} \frac{1}{\mathcal{A}_{1,3}} \right]$$

$$\begin{aligned}
 &= \frac{m_2 + m_3}{r} \\
 &+ m_1 \left\{ \left[(1-\mu)^{-1} + \mu^{-1} \right] \frac{1}{r'} \right. \\
 &\quad + \left[(1-\mu) + \mu \right] \frac{r}{r'^3} \left[\frac{4.3}{2.4} S^2 - \frac{2}{1} \cdot \frac{2.1}{2.4} \right] \\
 &\quad + \left[(1-\mu)^2 + \mu^2 \right] \frac{r^2}{r'^4} \left[\frac{6.5.4}{2.4.6} S^3 - \frac{3}{1} \cdot \frac{4.3.2}{2.4.6} S \right] \\
 &\quad + \left[(1-\mu)^3 + \mu^3 \right] \frac{r^3}{r'^5} \left[\frac{8.7.6.5}{2.4.6.8} S^4 - \frac{4}{1} \cdot \frac{6.5.4.3}{2.4.6.8} S^2 + \frac{4.3}{1.2} \cdot \frac{4.3.2.1}{2.4.6.8} \right] \\
 &\quad + \dots \dots \dots \left. \right\}.
 \end{aligned}$$

To get the similar function for the relative motion of the sun about the centre of gravity of the earth and moon, it is necessary to multiply the preceding expression by

$$\frac{m_2 \mu}{m_1 \mu'} = \frac{m_1 + m_2 + m_3}{m_1} \mu (1 - \mu).$$

The term of the potential function for the moon, factored by $r^3 \div r'^4$, gives rise to inequalities in the lunar coordinates factored by $a \div a'$. As this term has $1 - 2\mu$ as a factor, we see the correctness of the rule which directs to multiply this class of inequalities by $1 - 2\mu$ in order to include the effect of the disturbance of the relative motion of the sun about the earth by the lunar mass.

In treating the motion of the sun about the centre of gravity of the earth and moon it will suffice to take two terms of the preceding expression and put

$$\frac{1}{m_1 \mu'} \mathcal{Q} = \frac{M}{r'} + M \mu (1 - \mu) \frac{r^2}{r'^3} \left(\frac{3}{2} S^2 - \frac{1}{2} \right).$$

Let the longitudes of the sun and moon be denoted respectively by λ' and λ , and neglect the latitudes; then

$$\frac{1}{m_1 \mu'} \mathcal{Q} = \frac{M}{r'} + \frac{1}{2} M \mu (1 - \mu) \frac{r^2}{r'^3} \left[3 \cos 2(\lambda - \lambda') + 1 \right].$$

The differential equations, determining r' and λ' , are

$$\begin{aligned}
 \frac{d^2 r'}{dt^2} - r' \frac{d\lambda'^2}{dt^2} + \frac{M}{r'^2} + \frac{3}{2} n'^2 a' \mu (1 - \mu) \frac{a^2}{a'^2} \left[3 \cos 2\tau + 1 \right] &= 0, \\
 \frac{d}{dt} \left(r'^2 \frac{d\lambda'}{dt} \right) - \frac{3}{2} n'^2 a'^2 \mu (1 - \mu) \frac{a^2}{a'^2} \sin 2\tau &= 0,
 \end{aligned}$$

where, it will be noticed, we have put $r = a$, and, after differentiation, in the final small terms, $r' = a'$, $M \div a'^3 = n'^2$ and $\lambda - \lambda' = \tau$ the mean angular distance of the moon from the sun. The integration of the second eq'n gives

$$\frac{d\lambda'}{dt} = \frac{a_0^2 n'}{r'^2} - \frac{3}{4} \frac{n'^2}{n-n'} \mu (1-\mu) \frac{a^2}{a'^2} \cos 2\tau,$$

a_0 being the arbitrary constant. We can now eliminate $\frac{d\lambda'}{dt}$ from the first equation and we get

$$\frac{d^2 r'}{dt^2} + \frac{M}{r'^2} - \frac{a_0^4 n'^2}{r'^3} + \frac{3}{4} n'^2 a' \mu (1-\mu) \frac{a^2}{a'^2} \left[\frac{3n-n'}{n-n'} \cos 2\tau + 1 \right] = 0.$$

Let us suppose that this equation is satisfied by

$$r' = a_0 + a' a_1 \cos 2\tau,$$

a_1 being a coefficient to be determined. Substituting this value of r' in the differential equation we get the two equations of condition

$$\frac{M}{a_0^2} - n'^2 a_0 + \frac{3}{4} n'^2 a' \mu (1-\mu) \frac{a^2}{a'^2} = 0,$$

$$(4n^2 - 8nn' + 3n'^2) a_1 - \frac{3}{4} n'^2 \frac{3n-n'}{n-n'} \mu (1-\mu) \frac{a^2}{a'^2} = 0.$$

Whence may be derived

$$\frac{a_0}{a'} = 1 + \frac{1}{4} \mu (1-\mu) \frac{a^2}{a'^2},$$

$$a_1 = \frac{3}{4} \frac{m^2(3-m)}{(1-m)(4-8m+3m^2)} \mu (1-\mu) \frac{a^2}{a'^2},$$

where, as is usually done in the lunar theory, we have put $n' \div n = m$. The value of r' , thus obtained, being substituted in the expression for $\frac{d\lambda'}{dt}$, we get

$$\frac{d\lambda'}{dt} = n' - \frac{3}{4} n' \frac{m}{1-m} \frac{4-2m+m^2}{4-8m+3m^2} \mu (1-\mu) \frac{a^2}{a'^2} \cos 2\tau.$$

Integrating

$$\lambda' = \varepsilon' + n't - \frac{3}{4} \frac{m^2}{(1-m)^2} \frac{4-2m+m^2}{4-8m+3m^2} \mu (1-\mu) \frac{a^2}{a'^2} \sin 2\tau.$$

The numerical values of the constant quantities, which enter into these formulæ, are

$$m = 0.0748, \quad \mu = \frac{1}{82.4869}, \quad \frac{a}{a'} = 0.002587, \\ n' = 1295977''.4.$$

They give us

$$r' = a' [1.00000 \ 00200 + 0.00000 \ 00003 \cos 2\tau], \\ \lambda' = \varepsilon' + n't - 0''.0001 \sin 2\tau.$$

The periodic terms of these equations are too small for consideration, but the constant term of $r' \div a'$ may be noticed. If we should obtain the value

of α' from measured values of r' , on the assumption that the value of the constant term is unity, it would be too large by the 0.00000 002 part. And this value substituted in the equation $n' = \sqrt{\frac{M}{a'^3}}$, would give n' too small by the 0.00000 003 part, or n' would be too small by 0".03895; or the error in the mean longitude of the sun would amount to nearly 4" in a century, a quantity which could not, in the present state of astronomy, be neglected. However, it is only fair to state that astronomers proceed in a way the reverse of this; that is they observe n' and thence deduce α' , and in this case the term 0.00000 002 is without significance, since the logarithms of the radii vectores in the ephemerides are usually given to 7 decimals only.

A CASE OF SYMBOLIC VS. OPERATIVE EXPANSION.

BY A. S. HATHAWAY, CORNELL UNIV., ITHACA, NEW YORK.

Denote by D^m the *symbolic* expansion of $\left(a_1 \frac{d}{da_1} + a_2 \frac{d}{da_2} + \dots\right)^m$, the general term of which is $A_m \alpha_1^r \alpha_2^s \dots \left(\frac{d}{da_1}\right)^r \left(\frac{d}{da_2}\right)^s \dots$, where $A_m = \frac{m!}{r!s!\dots}$, $r+s+\dots = m$; and by $(D)^m$, the operation D or $\left(a_1 \frac{d}{da_1} + a_2 \frac{d}{da_2} + \dots\right)$ repeated m times, whose general term is

$$A_m \left(\left(a_1 \frac{d}{da_1}\right)\right)^r \left(\left(a_2 \frac{d}{da_2}\right)\right)^s \dots;$$

the *extra* parenthesis here, and in what follows, inclosing a symbol which combines *operatively*. Then $(D)^2$ differs from D^2 in the production of an extra term $a_i \frac{d}{da_i}$ by each portion $(D)a_i \frac{d}{da_i}$ [from $\left(a_i \frac{d}{da_i}\right)a_i \frac{d}{da_i}$] of the complete operation $(D)D$ or $(D)^2$; so that the complete $(D)^2 = D^2 + D$. And in general, $(D)D^{m-1}$ differs from D^m by an extra term

$$(r+s+\dots)A_{m-1}\alpha_1^r\alpha_2^s\dots\left(\frac{d}{da_1}\right)^r\left(\frac{d}{da_2}\right)^s\dots$$

derived from each portion

$$(D)A_{m-1}\alpha_1^r\alpha_2^s\dots\left(\frac{d}{da_1}\right)^r\left(\frac{d}{da_2}\right)^s\dots \text{ of the complete operation}$$

$(D)D^{m-1}$; so that, since $r+s+\dots = m-1$, $(D)D^{m-1} = D^m + (m-1)D^{m-1}$.
 $\therefore D^m = [(D-m+1)]D^{m-1} = [(D-m+1)][(D-m+2)]\dots[(D-1)]D \text{ or } (D)^m. \quad (1)$

Inversely, to find $(D)^m$ in terms of $D^m, D^{m-1}, \dots, D$, we have m equations linear in $(D)^m, (D)^{m-1}, \dots, (D)$ (obtained by assuming for m in (1) the values $1, 2, 3, \dots, m$) from which, if $(m-k)_i$ = the sum of the products of the natural numbers from 1 to $m-k$ inclusive taken i at a time,

$$(D)^m = \begin{vmatrix} D^m & -(m-1)_1 & (m-1)_2 & \dots & \mp(m-1)_{m-1} \\ D^{m-1} & 1 & (m-2)_1 & \dots & \pm(m-2)_{m-2} \\ D^{m-2} & 0 & 1 & \dots & \mp(m-3)_{m-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ D & 0 & 0 & \dots & 1 \end{vmatrix}$$

By another familiar process applied to the symbols $D^m [= (D)^{m'}]$ and $(D)^m$ we get also

$$(D)^m = \sum \frac{D^i 0^m}{i!} D^i.$$

Examples of (1).—1. To change the independent variables in D^m from

$a_1, a_2, \dots$ to $\theta_1, \theta_2, \dots$, where $a_1 = \varepsilon^{\theta_1}, a_2 = \varepsilon^{\theta_2}, \dots$; so that

$$a_1 \frac{d}{da_1} = \frac{d}{d\theta_1}, \quad a_2 \frac{d}{da_2} = \frac{d}{d\theta_2}, \dots \quad \left(\left(a_1 \frac{d}{da_1} + a_2 \frac{d}{da_2} + \dots \right) \right)^{m'}$$

becomes by the transformation

$$\left(\frac{d}{d\theta_1} + \frac{d}{d\theta_2} + \dots \right)^{m'}. \quad \therefore \text{by (1) } D^m = \left(\frac{d}{d\theta_1} + \frac{d}{d\theta_2} + \dots \right)^{m'}.$$

For a special case see Todhunter's *Dif. Cal.* Art. 208.

2. By Euler's theorem concerning a homogeneous function $\varphi(a_1, a_2, \dots)$, of n dimensions, $F((D))\varphi = F(n)\varphi$; $\therefore (D)^{m'}\varphi = n^{m'}\varphi$; $\therefore$ by (1)

$$D^m \varphi = n^{m'} \varphi.$$

SOLUTION OF A PROBLEM.

BY PROF. E. W. HYDE, UNIVERSITY OF CINCINNATI.

Problem.—To show that $\cos^p \varphi \sin^q \varphi$ can be expanded into a series of *cosines* of multiples of φ when q is *even*, and into a series of *sines* of multiples of φ when q is *odd*.

First, suppose q even and $= 2n$, say. Then

$$\begin{aligned} \cos^p \varphi \sin^{2n} \varphi &= \cos^p \varphi (\sin^2 \varphi)^n = \cos^p \varphi (1 - \cos^2 \varphi)^n \\ &= \cos^p \varphi - n \cos^{p+2} \varphi + \frac{n(n-1)}{2!} \cos^{p+4} \varphi - \&c. \end{aligned}$$

Each term of this series can be expanded into a series of cosines of multiples of φ .

Second, suppose q odd and $= 2n + 1$ say, and also let p be even and $= 2m$. Then

$$\begin{aligned}\cos^{2m}\varphi \sin^{2n+1}\varphi &= (\cos^2\varphi)^m \sin^{2n+1}\varphi = \sin^{2n+1}\varphi (1 - \sin^2\varphi)^m \\ &= \sin^{2n+1}\varphi - m \sin^{2n+3}\varphi + \frac{m(m-1)}{2!} \sin^{2n+5}\varphi - \&c.\end{aligned}$$

Since each term of the right hand member is an odd power of $\sin \varphi$, each term may be expanded into a series of sines of multiples of φ .

Finally let $q = 2n + 1$, and $p = 2m + 1$ so that both are odd. Then, if we suppose $m < n$,

$$\begin{aligned}\cos^{2m+1}\varphi \sin^{2n+1}\varphi &= (\sin \varphi \cos \varphi)^{2m+1} (\sin^2\varphi)^{n-m} \\ &= \left(\frac{1}{2}\right)^{m+n+1} (\sin 2\varphi)^{2m+1} (1 - \cos 2\varphi)^{n-m};\end{aligned}$$

while if we suppose $m > n$ we have

$$\begin{aligned}\cos^{2m+1}\varphi \sin^{2n+1}\varphi &= (\sin \varphi \cos \varphi)^{2n+1} (\cos^2\varphi)^{m-n} \\ &= \left(\frac{1}{2}\right)^{m+n+1} (\sin 2\varphi)^{2n+1} (1 + \cos 2\varphi)^{m-n}.\end{aligned}$$

From the similarity of these two forms it evidently makes no difference whether m be greater or less than n , we will use then the first expression, i. e., regard m as less than n .

$$\begin{aligned}\therefore \cos^{2m+1}\varphi \sin^{2n+1}\varphi &= \left(\frac{1}{2}\right)^{m+n+1} (\sin 2\varphi)^{2m+1} \\ &\quad \times \left[1 - (n-m) \cos 2\varphi + \frac{(n-m)(n-m-1)}{2!} \cos^2 2\varphi - \&c. \right].\end{aligned}$$

Now the product of $(\sin 2\varphi)^{2m+1}$ into any even power of $\cos 2\varphi$ comes under the second case, and we have therefore only to consider the products containing *odd* powers.

The exponent of the highest odd power will be either $n-m$ or $n-m-1$; call it $2m' + 1$: then treating that product as before we have, supposing $m' < m$,

$$(\cos 2\varphi)^{2m'+1} (\sin 2\varphi)^{2m+1} = \left(\frac{1}{2}\right)^{m+m'+1} (\sin 4\varphi)^{2m'+1} [1 - \cos 4\varphi]^{m-m'}.$$

Repeating this process successively we may finally obtain either

$$\sin[2^r\varphi] [1 \mp \cos(2^r\varphi)] = \sin(2^r\varphi) \mp \frac{1}{2} \sin(2^{r+1}\varphi),$$

or

$$\sin(2^r\varphi) \cos(2^r\varphi) = \frac{1}{2} \sin(2^{r+1}\varphi).$$

To show that this will be the result we will find the degrees of the successive left hand members. That of the first is

$$2m+1+2n+1 = 2(m+n+1).$$

That of the second, supposing $n-m$ to be odd, is

$$2m+1+n-m = m+n+1.$$

That of the third, supposing $m-m'$ odd, is

$$n-m+m - \frac{n-m-1}{2} = \frac{n+m+1}{2}.$$

Thus the degree each time is half what it was before. If $n-m, m-m', m'-m'',$ etc., be even, we shall have for the successive degrees,

$$\begin{aligned} \text{1st, } 2m+1+2n+1 &= 2(m+n+1), \\ \text{2nd, } 2m+1+2m'+1 &= 2(m'+m+1) = m+n, \\ \text{3rd, } 2m'+1+2m''+1 &= 2(m''+m'+1) = \frac{1}{2}(m+n-2), \\ \text{4th, } 2m''+1+2m''' +1 &= 2(m''' + m'' + 1) = \frac{1}{4}(m+n-2), \text{ etc.} \end{aligned}$$

After the second, the degree is each time half the preceding. If the exponents $n-m, m-m',$ etc., were some even and some odd the result would be between these two.

A numerical example is added.—Let $m = 9$ and $n = 14$. Then

$$\begin{aligned} \cos^{19}\varphi \sin^{29}\varphi &= \left(\frac{1}{2}\right)^{24}(\sin 2\varphi)^{19}[1-\cos 2\varphi]^5, \\ (\cos 2\varphi)^5(\sin 2\varphi)^{19} &= \left(\frac{1}{2}\right)^{12}(\sin 4\varphi)^5[1-\cos 4\varphi]^7, \\ (\cos 4\varphi)^7(\sin 4\varphi)^5 &= \left(\frac{1}{2}\right)^6(\sin 8\varphi)^5[1+\cos 8\varphi], \\ \cos 8\varphi (\sin 8\varphi)^5 &= \left(\frac{1}{2}\right)^3(\sin 16\varphi)[1-\cos 16\varphi]^2, \\ \cos 16\varphi \sin 16\varphi &= \frac{1}{2} \sin 32\varphi. \end{aligned}$$

If in the expansion of the right-hand member of the first equation we take $(\cos 2\varphi)^3$ instead of $(\cos 2\varphi)^5$ we have

$$\begin{aligned} (\cos 2\varphi)^3(\sin 2\varphi)^{19} &= \left(\frac{1}{2}\right)^{11}(\sin 4\varphi)^3[1-\cos 4\varphi]^8, \\ (\cos 4\varphi)^7(\sin 4\varphi)^3 &= \left(\frac{1}{2}\right)^5(\sin 8\varphi)^3[1+\cos 8\varphi]^2, \\ \cos 8\varphi (\sin 8\varphi)^3 &= \left(\frac{1}{2}\right)^2(\sin 16\varphi)[1-\cos 16\varphi], \\ &= \left(\frac{1}{2}\right)^2[\sin 16\varphi - \frac{1}{2}\sin 32\varphi]. \end{aligned}$$

DISCUSSION OF AN EQUATION.

BY JOHN BORDEN, CHICAGO, ILL.

As equations of the fifth or less degree are reducible to the form

$$y^n + py + q = 0, \quad (1)$$

n being an integer and positive, its discussion is of interest.

QUESTION.—Determine the number of real roots, their limits, and sign, for all real values of p and q .

ANSWER:—I. If n is odd and p is positive, there is only one real root, whose sign is contrary to that of q and whose limits are $q \div p \times -0$, and $q \div p \times -1$.

II. If n is odd and p is negative, there are three real roots when

$$\frac{p^n}{q^{n-1}} = \frac{n^n}{(n-1)^{n-1}},$$

two of which have the same sign as q , and become equal to each other, and

of the arithmetical value of $n \div (n-1)$, when $p^n \div q^{n-1} = n^n \div (n-1)^{n-1}$, and become imaginary for all values of p and q which make $p^n \div q^{n-1} < n^n \div (n-1)^{n-1}$. Their limits are, for one, $q \div p \times -n \div (n-1)$, and $q \div p \times -1$; and for the other, $q \div p \times -n \div (n-1)$, and $q \div p \times -\infty$. The other real root remains real for all values of p and q and its sign is contrary to that of q , and its limits are $q \div p \times +0$, and $q \div p \times +\infty$.

III. If n is even and q is positive, there are two real roots when

$$\frac{p^n}{q^{n-1}} = \frac{n^n}{(n-1)^{n-1}},$$

both of the same sign and which is contrary to that of p . They become equal and also of the arithmetical value of $n \div (n-1)$, when $p^n \div q^{n-1} = n^n \div (n-1)^{n-1}$; and they become imaginary for all values of p and q which make $p^n \div q^{n-1} < n^n \div (n-1)^{n-1}$. Their limits are, for one, $q \div p \times -n \div (n-1)$ and $q \div p \times -1$, and for the other, $q \div p \times -n \div (n-1)$ and $q \div p \times -\infty$.

IV. If n is even and q is negative, there are always two real roots having contrary signs. Their limits are, for one, $q \div p \times -0$ and $q \div p \times -1$, and for the other, $q \div p \times +0$ and $q \div p \times +\infty$.

V. Eq. (1) has no other real roots than as above.

PROOF:—In Eq. (1) make $y = \frac{q}{p} \cdot x$. (2)

Then $x^n + \frac{p^n}{q^{n-1}} \cdot x + \frac{p^n}{q^{n-1}} = 0$; (3)

which gives $\frac{p^n}{q^{n-1}} = -\frac{x^n}{x+1}$. (4)

If n is odd, the sign of the first term of Eq. (4) depends on p .

Suppose p is positive, then no positive value of x can satisfy Eq. (4). Therefore x must be negative. In such case (4) assumes the arithm'l form

$$\frac{p^n}{q^{n-1}} = \frac{x^n}{1-x}. \quad (5)$$

Now any value of x greater than one is evidently impossible. For $x = 1$, $p^n \div q^{n-1} = \infty$. For $x = 0$, $p^n \div q^{n-1} = 0$. Therefore in Eq. (3), in this case, the only real value of x is negative and its limits are -0 , and -1 .

Therefore in this case the only real value of y in Eq. (1) has for its limits $q \div p \times -0$, and $q \div p \times -1$. As p is supposed to be positive, it is evident that the sign of the real values of y is contrary to that of q . This proves I.

Next suppose p is negative— n still remaining odd. Then for a positive value of x , Eq. (4) assumes the arithmetical form,

$$\frac{p^n}{q^{n-1}} = \frac{x^n}{x+1}, \quad (6)$$

and for a negative value of x , Eq. (4) assumes the arithmetical form

$$\frac{p^n}{q^{n-1}} = \frac{x^n}{x-1}. \quad (7)$$

Now, in Eq. (6), for $x=0$, $p^n \div q^{n-1} = 0$, and for $x=\infty$, $p^n \div q^{n-1} = \infty$. Therefore the corresponding values of y in Eq. (1) are $q \div p \times +0$ and $q \div p \times +\infty$. As p is here assumed to be negative this value of y will have a sign contrary to that of q . This proves one branch of Prop. II.

In Eq. (7) x can never be less than unity, because, for $x=1$, $p^n \div q^{n-1} = \infty$; also, for $x=\infty$, $p^n \div q^{n-1} = \infty$. Now it will be found, on trial, that as the value of x increases above unity the value of $p^n \div q^{n-1}$ decreases, also, that as the value of x decreases from ∞ the value of $p^n \div q^{n-1}$ decreases. Here then are two values of x approaching each other, and it will be found that $p^n \div q^{n-1}$ is a minimum when $= n^n \div (n-1)^{n-1}$, and that, in such case, the two values of x have become each equal to $n \div (n-1)$. At this point these two real roots vanish and become imaginary. Apply the rule for maxima and minima and this will appear. Therefore in Eq. (7) there are two values for x , one lying between 1 and $n \div (n-1)$ and the other between $n \div (n-1)$ and ∞ ; and these two negative values for x , in the present case, give as the corresponding values of y ; one, $q \div p \times -n \div (n-1)$ and $q \div p \times -1$, the other $q \div p \times -n \div (n-1)$ and $q \div p \times -\infty$. As p is here supposed to be negative the signs of these two values of y are the same as that of q . Thus is proved the other branch of Prop. II.

To illustrate; in $y^3 + py + q = 0$, it is well known that there is only one real root when p is positive, and that, when p is negative, there are three real roots when $p^3 \div q^2$ is equal or greater than $3^3 \div 2^2$ or $\frac{27}{4}$.

So also in $y^5 + py + q = 0$, it will be found as above that when p is positive there is only one real root, and that when p is negative there are three real roots when $p^5 \div q^4 =$ or $> 5^5 \div 4^4$. When $p^5 \div q^4 = 5^5 \div 4^4$, then two of the values of y are equal, such values being $y = q \div p \times -5 \div 4$.

If n is even, the sign of the first term of Eq. (4) depends on q . Suppose q is positive. Then no positive value of x can satisfy Eq. (4). But for x negative Eq. (4) reduces to the arithmetical form

$$\frac{p^n}{q^{n-1}} = \frac{x^n}{x-1}. \quad (8)$$

This is similar in form to Eq. (7), and in like manner proves Prop. III.

Suppose q is negative— n still remaining even. Then for x positive Eq. (4) assumes the arithmetical form

$$\frac{p^n}{q^n} = \frac{x^n}{x+1}. \quad (9)$$

Also for x negative Eq. (4) assumes the arithmetical form

$$\frac{p^n}{q^n} = \frac{x^n}{1-x}. \quad (10)$$

The value of x in Eq. (9) can lie any where between 0 and infinity; but the value of x in Eq. (10) can only lie between 0 and 1. This proves IV. And Prop. V. also follows from the foregoing.

Note. The cases in which n is fractional, and, in which p and q (both or either) are imaginary, have not, though they might have, been considered.

Also, an equation of the form, $z^n + p'z^{n-1} + q' = 0$ is merely the reciprocal form of Eq. (1). For if $z = \frac{1}{y}$, then $y^n + \frac{p'}{q'}y + \frac{1}{q'} = 0$, which is the same as Eq. (1).

THE CENTRE OF GRAVITY OF THE APPARENT DISK OF A PLANET.

BY PROF. A. HALL, U. S. NAVAL OBSERVATORY, WASHINGTON, D. C.

WHEN a planet is observed from the Earth its disk will not be completely illuminated except at the time of opposition. At other times it will appear gibbous or crescent. If we assume the planet to be a sphere, which, according to the best observations is the case with Mercury, Venus and Mars, the apparent disk will be composed of a semi-circle and a semi-ellipse which are separated by the line of cusps. If a right line be drawn through the centre of the line of cusps and at right angles to it, the centre of gravity of the apparent disk is in this line. In the plane triangle between the Sun, the Earth and the planet let φ be the angle at the planet; let a be the radius of the semi-circular part of the disk, and designate by m the distance of the centre of gravity from the centre of the line of cusps. Then when the planet is gibbous we shall have

$$m = \frac{8a}{3\pi} \sin \frac{1}{2}\varphi^2;$$

and when the planet is crescent,

$$m = \frac{8a}{3\pi} \cos \frac{1}{2}\varphi^2,$$

where π is the number 3.1416. The angle φ and the radius a can be computed from the data given in the ephemerides, and the distance m is known therefore for any time.

In order to show the use of the quantity m , suppose the angle of position and distance of a satellite of a planet have been observed with a filar micrometer in the following manner. The wire is laid across the apparent disk of the planet in such a way that this disk is divided into two parts of equal area. The wire therefore always passes through the centre of gravity of the apparent disk, and it is necessary before using the observations for computing orbits to reduce these measures to the true centre of the planet. If we denote by p and s the observed angle of position and the distance of the satellite, and by θ the angle of position of the line of cusps, the errors of p and s arising from the unsymmetrical form of the disk will be,

$$\Delta s = m \cdot \sin(p - \theta),$$

$$\Delta p = \frac{m}{s} \cdot \cos(p - \theta),$$

where Δp is expressed in parts of radius.

To complete the solution we have to find the value of θ . This is given from the spherical triangle between the pole of the Equator and the geocentric places of the Sun and the planet. In this triangle the angle at the planet is $90^\circ - \theta$; and if we denote by d the angular distance between the Sun and the planet, and by $\alpha, \delta, \alpha', \delta'$, the geocentric right ascensions and declinations of the Sun and planet, we shall have,

$$\cos d = \sin \delta \sin \delta' + \cos \delta \cos \delta' \cos(\alpha' - \alpha),$$

$$\sin d \cos \theta = \cos \delta \sin(\alpha' - \alpha),$$

$$\sin d \sin \theta = \sin \delta \cos \delta' - \cos \delta \sin \delta' \cos(\alpha' - \alpha).$$

These equations give the angle θ without ambiguity.

In the case of a spheroidal planet, as Jupiter or Saturn, the apparent outline of the planet will be composed of ellipses, but the determination of these ellipses though possible is much more troublesome, and for these planets the correction for defective illumination is hardly sensible.

A PROBLEM AND ITS SOLUTION.

BY E. B. SEITZ, GREENVILLE, OHIO.

ON the surface of a sphere, radius unity, three small circles, radii a, b, c , are described, each touching the other two; on the surface enclosed by them three small circles, radii x_1, y_1, z_1 , are described, each touching the other two and two of the first set of circles; on the surface enclosed by the first set of inscribed circles three small circles, radii x_2, y_2, z_2 , are described,

each touching the other two and two of the first set of inscribed circles; and so on. Find the radii x_n, y_n, z_n , of the n th set of inscribed circles.

SOLUTION.—We will first prove that, if four small circles each touch a fifth in the same way, and $t_1 t_2, t_2 t_3, t_3 t_4, t_4 t_1, t_1 t_3, t_2 t_4$ are the arcs of great circles which touch and join the four circles, two and two, we have

$$\sin \frac{1}{2} t_1 t_2 \sin \frac{1}{2} t_3 t_4 + \sin \frac{1}{2} t_2 t_3 \sin \frac{1}{2} t_4 t_1 = \sin \frac{1}{2} t_1 t_3 \sin \frac{1}{2} t_2 t_4.$$

Let A, B, C, D (fig. 1) be the poles of the four circles, O that of the fifth, a, b, c, d the points at which the four circles touch the fifth.

From t_1, t_2 draw arcs of great circles through A, B , and produce them till they meet at N ; then Nt_1 and Nt_2 are quadrants.

Put arc $Oa = R, Aa = r_1, Bb = r_2, AB = d$. Then in the spherical triangle Oab we have

$$\sin \frac{1}{2} ab = \sin R \sin \frac{1}{2} Ob. \quad (1)$$

In the triangle AOB we have

$$\sin^2 \frac{1}{2} Ob = \frac{\cos(r_1 - r_2) - \cos d}{2 \sin(R + r_1) \sin(R + r_2)}. \quad (2)$$

In the triangle ANB we have

$$\sin^2 \frac{1}{2} t_1 t_2 = \sin^2 \frac{1}{2} N = \frac{\cos(r_1 - r_2) - \cos d}{2 \cos r_1 \cos r_2}. \quad (3)$$

From (1), (2), and (3) we readily find

$$\sin \frac{1}{2} ab = \sin \frac{1}{2} t_1 t_2 \sin R \sqrt{\left[\frac{\cos r_1 \cos r_2}{\sin(R + r_1) \sin(R + r_2)} \right]}.$$

In like manner we can find the value of $\sin \frac{1}{2} bc, \sin \frac{1}{2} cd$, &c.. But since the chords joining the points a, b, c, d , form the sides and diagonals of an inscribed quadrilateral, and each chord is twice the sine of half the corresponding arc, we have

$$\sin \frac{1}{2} ab \sin \frac{1}{2} cd + \sin \frac{1}{2} ad \sin \frac{1}{2} bc = \sin \frac{1}{2} ac \sin \frac{1}{2} bd. \quad (4)$$

Substituting the values of $\sin \frac{1}{2} ab, \sin \frac{1}{2} bc$, &c., in (4), and omitting the common factor

$$\sin^2 R \sqrt{\left[\frac{\cos r_1 \cos r_2 \cos r_3 \cos r_4}{\sin(R + r_1) \sin(R + r_2) \sin(R + r_3) \sin(R + r_4)} \right]},$$

we have

$$\sin \frac{1}{2} t_1 t_2 \sin \frac{1}{2} t_3 t_4 + \sin \frac{1}{2} t_2 t_3 \sin \frac{1}{2} t_4 t_1 = \sin \frac{1}{2} t_1 t_3 \sin \frac{1}{2} t_2 t_4. \quad (5)$$

Let A, B, C (fig. 2) be the poles of the small circles, radii a, b, c ; O, Q, R the poles of the small circles, radii x_1, y_1, z_1 , MN the arc of a great circle which joins the circles A and O . From M and N draw arcs of great circles through A and O , and produce them till they meet at P .



Let t_1, t_2, t_3 be the lengths of the arcs of great circles which touch the circles A and O , B and Q , C and R respectively.

Put arc $AB = a + b$, $AC = a + c$, $BC = b + c$, $BO = b + x_1$, $CO = c + x_1$, $AO = s$, $MN = t_1$, $\angle CBO = \theta$, $\angle ABO = \phi$, $\angle ABC = \beta$.

Since each of the circles O, Q, R , is touched by four circles which touch each other consecutively we have, by (5),



$$\sin \frac{1}{2}t_1 \sin \frac{1}{2}t_2 = \sqrt{[(\tan a \tan b)(\tan x_1 \tan y_1)] + \sqrt{[(\tan a \tan y_1)(\tan b \tan x_1)]}}$$

$$= 2\sqrt{(\tan a \tan b \tan x_1 \tan y_1)}, \quad (6)$$

$$\sin \frac{1}{2}t_1 \sin \frac{1}{2}t_3 = 2\sqrt{(\tan a \tan c \tan x_1 \tan z_1)}, \quad (7)$$

$$\sin \frac{1}{2}t_2 \sin \frac{1}{2}t_3 = 2\sqrt{(\tan b \tan c \tan y_1 \tan z_1)}. \quad (8)$$

From (6), (7) and (8) we find

$$\sin \frac{1}{2}t_1 = \sqrt{2 \tan a \tan x_1}. \quad (9)$$

$$\cos t_1 = 1 - 2 \sin^2 \frac{1}{2}t_1 = 1 - 4 \tan a \tan x_1. \quad (10)$$

In the spherical triangle AOP we have

$$\cos s = \sin a \sin x_1 + \cos a \cos x_1 \cos t_1 = \cos a \cos x_1 - 3 \sin a \sin x_1. \quad (11)$$

In the triangle ABO we have

$$\cos \phi = \frac{\cos s - \cos(a+b) \cos(b+x_1)}{\sin(a+b) \sin(b+x_1)} = 1 - \frac{4 \operatorname{cosec}^2 b}{(\cot a \cot b)(\cot x_1 \cot b)};$$

$$\sin \frac{1}{2}\phi = \frac{\sqrt{2 \operatorname{cosec} b}}{\sqrt{[(\cot a + \cot b)(\cot x_1 + \cot b)]}}. \quad (12)$$

Similarly we find

$$\sin \frac{1}{2}\theta = \frac{\operatorname{cosec} b}{\sqrt{[(\cot b + \cot c)(\cot x_1 + \cot b)]}}, \quad (13)$$

$$\sin \frac{1}{2}\beta = \frac{\operatorname{cosec} b}{\sqrt{[(\cot a + \cot b)(\cot b + \cot c)]}}, \quad (14)$$

$$\cos \frac{1}{2}\beta = \sqrt{\left[\frac{\cot a \cot b + \cot a \cot c + \cot b \cot c - 1}{(\cot a + \cot b)(\cot b + \cot c)} \right]}. \quad (15)$$

But $\sin \frac{1}{2}(\theta + \phi) = \sin \frac{1}{2}\beta$, whence by developing, transposing, squaring, and reducing, we find

$$\sin^2 \frac{1}{2}\theta + \sin^2 \frac{1}{2}\phi - \sin^2 \frac{1}{2}\beta + 2 \cos \frac{1}{2}\beta \sin \frac{1}{2}\theta \sin \frac{1}{2}\phi = 0. \quad (16)$$

Substituting the values of $\sin \frac{1}{2}\theta$, &c., in (16), clearing and reducing, we get an equation of the first degree, whence we readily find

$$\cot x_1 = \cot a + 2 \cot b + 2 \cot c + 2\sqrt{(2 \cot a \cot b + 2 \cot a \cot c + 2 \cot b \cot c - 2)}.$$

Similarly we have

$$\cot y_1 = 2\cot a + \cot b + 2\cot c + 2\sqrt{(2\cot a \cot b + 2\cot a \cot c + 2\cot b \cot c - 2)},$$

$$\cot z_1 = 2\cot a + 2\cot b + \cot c + 2\sqrt{(2\cot a \cot b + 2\cot a \cot c + 2\cot b \cot c - 2)}.$$

$$\text{Put } p_n = 2\cot x_n + 2\cot y_n + 2\cot z_n + 2\sqrt{(2\cot x_n \cot y_n + 2\cot x_n \cot z_n + 2\cot y_n \cot z_n - 2)}$$

and $q_n = 2\cot x_n + 2\cot y_n + 2\cot z_n$. Then we have

$$\cot x_1 = p - \cot a, \quad \cot y_1 = p - \cot b, \quad \cot z_1 = p - \cot c. \quad (17)$$

By substitution we find $p_1 = 11p - 2q$, $q_1 = 6p - q$, and similarly,

$$p_2 = 11p_1 - 2q_1, \quad q_2 = 6p_1 - q_1. \quad \text{From these r}$$

$$\text{lations we find} \quad p_2 - 10p_1 + p = 0, \quad (e_1)$$

$$\text{and similarly,} \quad p_3 - 10p_2 + p_1 = 0, \quad (e_2)$$

$$p_4 - 10p_3 + p_2 = 0, \quad (e_3)$$

$$\dots \dots \dots$$

$$p_{n+1} - 10p_n + p_{n-1} = 0. \quad (e_n)$$

From (17) we have $\cot x_1 + \cot a = \cot y_1 + \cot b = \cot z_1 + \cot c = p$,

and similarly $\cot x_2 + \cot x_1 = \cot y_2 + \cot y_1 = \cot z_2 + \cot z_1 = p_1$,

$$\dots \dots \dots$$

$$\cot x_n + \cot x_{n-1} = \cot y_n + \cot y_{n-1} = \cot z_n + \cot z_{n-1} = p_{n-1}.$$

Subtracting the first of these equations from the second, the resulting equation from the third, and so on to the last, we find

$$\cot x_n \pm \cot a = \cot y_n \pm \cot b = \cot z_n \pm \cot c = p_{n-1} - p_{n-2} + p_{n-3} - \dots \pm p, \quad (18)$$

the double sign being taken plus, when n is odd, and minus when n is even.

Let $u_n = \cot x_n \pm \cot a = \cot y_n \pm \cot b = \cot z_n \pm \cot c$. Then we have

$$u_n = p_{n-1} - p_{n-2} + p_{n-3} - \dots \pm p, \quad (19)$$

$$u_{n+1} = p_n - p_{n-1} + p_{n-2} - \dots \pm p_1 \pm p, \quad (20)$$

$$u_{n+2} = p_{n+1} - p_n + p_{n-1} - \dots \pm p_2 \pm p_1 \pm p. \quad (21)$$

Adding (19) and (21), subtracting (20) $\times 10$ from the resulting equation, and substituting from $(e_1), (e_2), (e_3) \dots (e_n)$, we find

$$u_{n+2} - 10u_{n+1} + u_n = \pm 2q, \quad (22)$$

$$\text{and similarly we get } u_{n+3} - 10u_{n+2} + u_{n+1} = \mp 2q. \quad (23)$$

$$\text{Adding (22) and (23), } u_{n+3} - 9u_{n+2} + 9u_{n+1} - u_n = 0, \quad (24)$$

an equation in Finite Differences.

$$\text{Integrating (24), } u_n = C_1(5+2\sqrt{6})^n + C_2(5-2\sqrt{6})^n + C_3(-1)^n. \quad (25)$$

$$\text{When } n = 0, \quad u_0 = C_1 + C_2 + C_3 = 0. \quad (26)$$

$$\text{" } n = 1, \quad u_1 = C_1(5+2\sqrt{6}) + C_2(5-2\sqrt{6}) - C_3 = p. \quad (27)$$

$$\text{" } n = 2, \quad u_2 = C_1(5+2\sqrt{6})^2 + C_2(5-2\sqrt{6})^2 + C_3 = 10p - 2q. \quad (28)$$

From (26), (27), and (28) we find $C_1 = \frac{1}{24}p\sqrt{6} - \frac{1}{24}q(\sqrt{6}-2)$,

$$C_2 = -\frac{1}{24}p\sqrt{6} + \frac{1}{24}q(\sqrt{6}+2), \text{ and } C_3 = -\frac{1}{6}q.$$

Substituting in (25) we find $\cot x_n \pm \cot a = \cot y_n \pm \cot b = \cot z_n \pm \cot c$

$$= \frac{1}{6}[(5+2\sqrt{6})^n + (5-2\sqrt{6})^n \pm 2](\cot a + \cot b + \cot c)$$

$$+ \frac{1}{6}[(5+2\sqrt{6})^n - (5-2\sqrt{6})^n] \sqrt{(3\cot a \cot b + 3\cot a \cot c + 3\cot b \cot c - 3)}.$$

case also, $R = F$. But for the same track and train, F may be considered constant; therefore R is constant, that is, a brake is equally effective whether applied at the top or side of a wheel.

But our correspondent has proved, by experiment, that a force of ten pounds at A , in the direction AE , is held in equilibrium by a force of one pound at G in the direction GH ; of two pounds at I , in the direction IL , and of about four pounds at R , in the direction RL ; whence he concludes that the effect of the brake to arrest the motion of the train must depend on its position relative to the point of contact between the wheel and rail.

In the first of the experiments above named we have a lever of the second order; resistance at G , power at A , and fulcrum at F ; the experiment therefore does not apply to the case under consideration, for no amount of pressure at G can prevent a force at A , in the direction AE , from producing rotation about the point F .

In the second experiment we have a lever of the first order, fulcrum at A ; consequently if the weight of the wheel and the power at I are sufficient to develop two pounds of friction at F , the forces will be in equilibrium as shown by the experiment, but not otherwise.

In the third experiment, we must again regard F as the fulcrum, and consequently the lever will be of the first order; therefore, when the forces at A and R are in equilibrium they must have the proportion of FL to FA .

An analysis of the experiments therefore shows, that the *first* is not applicable; that the *second* confirms Eq. (1) for the point I , and that the *third* proves, what is otherwise apparent, that a brake applied at F would be useless if applied to prevent rotation about the point F . It is apparent, however, that if rotation about A is prevented, no rotation about F can occur, and hence the brake is only used to prevent rotation about A ; and for this purpose, it has been shown, its position, with respect to the point F , is immaterial.

A QUESTION AND ITS SOLUTION.

BY GEORGE EASTWOOD, SAXONVILLE, MASS.

QUESTION.—When only the four sides of a quadrangle are given, how may it be known that the figure can be circumscribed by a circle?

SOLUTION.—Let $ABCD$ be a quadrangle whose sides AB , BC , CD , DA are known by measurement. Produce the opposite sides AB , DC , to meet in the point E , and the opposite sides DA , CB , to meet in the point F .

Draw EH and FI to bisect the angles E and F , and let O be the point of their intersection. Draw, also, the diagonals AC , BD . Then, evidently, we have

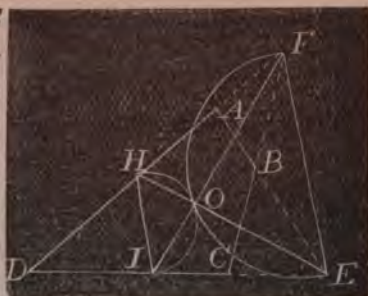
$$\begin{aligned}\angle ABC &= BFO + FOE + OEB \\ &= \frac{1}{2} \angle F + FOE + \frac{1}{2} \angle E,\end{aligned}$$

$$\begin{aligned}\angle ADC &= OIC - \frac{1}{2} \angle F \\ &= FOE - \frac{1}{2} \angle E - \frac{1}{2} \angle F\end{aligned}$$

$\therefore \angle ABC + \angle ADC = 2FOE = \text{two right}$

angles when FOE is a right angle, and the points A, B, C, D will be on the circumference of a circle. Hence this criterion:—

Prolong the opposite sides AB, DC and DA, CB to meet in E and F . Draw the bisectors EH, FI , and join IH, EF , upon which describe semicircles. Then, if these semicircles touch each other in the point of intersection (O) of the bisectors, FO will be perpendicular to EO , angles $A+B$ will = angles $B+D$, and the quadrangle can be circumscribed by a circle.



ON THE ROOTS OF EQUATIONS.

BY PROFESSOR WORPITZKY, BERLIN, PRUSSIA.*

SUPPOSE a portion of an area z to be ascertained which contains a single root of the equation $f(z) = 0$; the root itself $z = c$ being unknown.

Moreover, the function, $f(z)$, is supposed to be monogen in all parts of the area; finally determined; and with the exception of $z = c$ to be nowhere equal to zero.

Then $f(z) = (z-c)\varphi(z)$ may be substituted, in which equation the function, $\varphi(z)$, possesses exactly the same attributes, except that $\varphi(z) = C$; that is a finite value different from c : and after the well known formula of *Lauch* concerning marginal integrals we obtain

$$\frac{2\pi i}{C} = \int \frac{dz}{f(z)}.$$

Hence C may be ascertained without knowing the value of c .

On the other hand;

$$\begin{aligned}f'(z) &= \varphi(z) + (z-c)\varphi'(z); \\ f'(c) &= \varphi(c) = C.\end{aligned}$$

*As this paper from Prof. Worpitzky, in German, was accompanied by a letter to Alex. Evans, Esq., of Elkton, Md., referring to his translation at p. 66, Vol. IV, ANALYST, the original was sent by us to Mr. Evans, to whom we are indebted for this translation.—Ed.

Hence every single root c , of the equation $f(z) = 0$, is also a root of the equation $f'(z) - C = 0$, if the true value, viz.,

$$C = \frac{2\pi i}{\int_c \frac{dz}{f(z)}},$$

be substituted for C .

The fencing in, (*einhegung*), or determination of each single root c , and the numerical calculation of the marginal integrals around the areas in which they lie, need not be explained here, as their practical admissibility (*ausföhrbarkeit*) is theoretically unquestioned.

Nor need the train of consequences, which might be drawn from the foregoing, be enumerated here, the following only need be taken into consideration. (*platzfinden*.)

The processes (*mittel*) are known by which to reduce one algebraic equation to another, to wit, to $f(z) = 0$, which contains all the roots of the former as single roots; we also know how to determine (*einzuhegen*) each one of the latter.

Whenever this is done our formula in question presents the root $z = c$ as a root of the equation $f'(z) - C = 0$, which is lower than the former by one degree.

This equation may then be treated in like manner as the former, and then we arrive, by a finite number of entirely definite and practical operations, at an equation of the first degree, viz.; $z = c$, in which the value of c is composed of the marginal integrals in a form similar to a continued fraction.

Moreover we obtain $z - c$ as a common divisor of $f(z)$ and of $f'(z) - C$.

RECTIFICATION OF THE HYPERBOLA.

BY ARTEMAS MARTIN, M. A., ERIE, PA.

THE equation to the hyperbola, in rectangular coordinates referred to its center, is

$$b^2x^2 - a^2y^2 = a^2b^2. \quad (1)$$

Denoting by H_x the arc whose abscissa is x ,

$$\begin{aligned} dH_x &= \left(1 + \frac{dy^2}{dx^2}\right)^{\frac{1}{2}} dx = \left(\frac{[1 + (b^2 \div a^2)]x^2 - a^2}{x^2 - a^2}\right)^{\frac{1}{2}} dx, \\ &= \left(\frac{e^2x^2 - a^2}{x^2 - a^2}\right)^{\frac{1}{2}} dx = a \left(\frac{e^2v^2 - 1}{v^2 - 1}\right)^{\frac{1}{2}} dv, \end{aligned}$$

where e , the eccentricity, $= \frac{(a^2 + b^2)^{\frac{1}{2}}}{a}$, and $av = x$.

Put $e = \frac{1}{e}$ and $v = \left(\frac{1-e_1^2 v_1^2}{1-v_1^2} \right)^{\frac{1}{2}}$, then by substitution,

$$dH_x = \frac{a(1-e_1^2)dv_1}{(1-v_1^2)^{\frac{1}{2}}(1-e_1^2 v_1^2)^{\frac{1}{2}}}. \quad (2)$$

Let $A = v_1 \left(\frac{1-e_1^2 v_1^2}{1-v_1^2} \right)^{\frac{1}{2}}$, then

$$dH_x = a dA - a \left(\frac{1-e_1^2 v_1^2}{1-v_1^2} \right)^{\frac{1}{2}} dv_1 + \frac{a(1-e_1^2)dv_1}{(1-v_1^2)^{\frac{1}{2}}(1-e_1^2 v_1^2)^{\frac{1}{2}}};$$

$$\therefore H_x = aA - aE(e_1, v_1) + \int \frac{a(1-e_1^2)dv_1}{(1-v_1^2)^{\frac{1}{2}}(1-e_1^2 v_1^2)^{\frac{1}{2}}}, \quad (3)$$

where $E(e_1, v_1)$ denotes an elliptic arc, semimajor axis unity, eccentricity e_1 and abscissa v_1 .

Let $e_1 = \frac{1-\sqrt{1-e_2^2}}{1+\sqrt{1-e_2^2}}$ and $v_1 = \frac{2v_2}{1-e_1^2} \left(\frac{1-v_2^2}{1-e_2^2 v_2^2} \right)^{\frac{1}{2}}$, and we get

$$\begin{aligned} \frac{a(1-e_1^2)dv_1}{(1-v_1^2)^{\frac{1}{2}}(1-e_1^2 v_1^2)^{\frac{1}{2}}} &= 2ae_1 dv_1 + 2a \left(\frac{1-e_1^2 v_1^2}{1-v_1^2} \right)^{\frac{1}{2}} dv_1 \\ &\quad - 2a(1+e_1) \left(\frac{1-e_2^2 v_2^2}{1-v_2^2} \right)^{\frac{1}{2}} dv_2. \end{aligned} \quad (4)$$

$$\therefore H_x = aA + 2ae_1 v_1 + aE(e_1, v_1) - 2a(1+e_1)E(e_2, v_2). \quad (5)$$

See *Wright's Solutions of the Cambridge Problems*, vol. ii, pp. 148-9.

A PROBLEM IN LEAST SQUARES.

BY R. J. ADCOCK, MONMOUTH, ILL.

FIND the most probable position of the straight line determined by the measured coordinates, each measure being equally good or of equal weight, $(x_1, y_1), (x_2, y_2), \dots (x_n, y_n)$ of n points, that is find a and b in the equation

$$y = ax + b. \quad (1)$$

Since $y_1 - ax_1 - b, y_2 - ax_2 - b, \dots y_n - ax_n - b$ are the distances, parallel to the axis y , from the n points to the required line,

$$\left[(y_1 - ax_1 - b)^2 + (y_2 - ax_2 - b)^2 + \dots + (y_n - ax_n - b)^2 \right] \frac{1}{1+a^2} = S(d_1^2) = u, \quad (2)$$

is the sum of the squares of the normals from the n points to the required line, which sum by the principle of least squares, ANALYST, p. 183, Vol. IV, must be a minimum. Therefore

$$\frac{du}{db} = \frac{2nb - 2S(y_1) + 2aS(x_1)}{1+a^2} = 0, \text{ or } S(y_1) - aS(x_1) - nb = 0, \quad (3)$$

where $S(y_1) = y_1 + y_2 + \dots + y_n$. And

$$\begin{aligned} \frac{du}{da} = & \frac{2aS(x_1^2) - 2S(x_1y_1) + 2bS(x_1)}{1+a^2} \\ & - \frac{[S(y_1^2) + a^2S(x_1^2) + nb^2 - 2aS(x_1y_1) - 2bS(y_1) + 2abS(x_1)]2a}{(1+a^2)^2} = 0, \\ \text{or } & a^2S(x_1y_1) - a^2bS(x_1) + a[S(x_1^2) - S(y_1^2)] - nab^2 + bS(x_1^2) + 2abS(y_1) \\ & - S(x_1y_1) = 0, \quad (4) \end{aligned}$$

where $S(x_1y_1) = x_1y_1 + x_2y_2 + \dots + x_ny_n$.

Eliminating b from (3) and (4) and solving for a ,

$$\begin{aligned} a = & \frac{S(y_1^2) - S(x_1^2) + [S(x_1)]^2 - [S(y_1)]^2}{2[S(x_1y_1) - S(x_1)S(y_1)]} \\ & \pm \sqrt{\left[1 + \left(\frac{S(y_1^2) - S(x_1^2) + [S(x_1)]^2 - [S(y_1)]^2}{2[S(x_1y_1) - S(x_1)S(y_1)]}\right)^2\right]}. \quad (5) \end{aligned}$$

Equation (3) shows that the line passes through the centre of gravity of the n points, and therefore by (2) it must be a principal axis of them. And by (5) there are two positions of the line, at right angles to each other, which make $S(d_1^2)$ a minimum. The first value of a which gives the least minimum is the one which the problem requires.

$x_1 = 1, y_1 = 1, x_2 = 3, y_2 = 2, x_3 = 5, y_3 = 4$, gives

$$\begin{aligned} a = & \frac{1+4+16-1-9-25+(1+3+5)^2-(1+2+4)^2}{2[(1+6+20)-(1+3+5)(1+2+4)]} \pm \sqrt{\left(1 + \frac{1}{16}\right)} \\ = & -\frac{1}{4} \pm \frac{1}{4}\sqrt{17} = .78078 \text{ or } 1.2808. \end{aligned}$$

NOTE ON THE QUANTITY g , PAGE 25, BY PROF. JOHNSON.—The value of this quantity is

$$g_n = \frac{1}{2}(2^n \pm 1);$$

for, beginning with $g_0 = 0$, we have, from the given relation

$$g_{n-1} = 2g_n \pm 1,$$

$g_1 = 1, g_2 = 2-1, g_3 = 2^2-2+1$, and in general $g_n = 2^{n-1} - 2^{n-2} + 2^{n-3} \dots$, whence

$$g_n = \frac{2^n \pm 1}{2+1}.$$

In the table an error occurs in the value of g_{19} which introduces a cumulative error in the subsequent values, and another error was made in forming g_{28} ; the final value should be

$$g_{50} = 375,299,968,947,541.$$

NOTE, BY E. B. SEITZ.—The value of g , in Mr. Baker's solution, p. 25, may be found as follows:

Let $g = u_n$; then $u_n = u_{n-1} \pm 1 \dots (1)$, and $u_{n-1} = u_{n-2} \mp 1, \dots (2)$ the upper sign being used when n is odd, and the lower when n is even. Adding (1) and (2), we have $u_n - u_{n-1} - 2u_{n-2} = 0$, an equation in Finite Differences, whose solution gives $u_n = C_1 \cdot 2^n + C_2(-1)^n \dots (3)$

When $n = 1$, $u_1 = 2C_1 - C_2 = 1 \dots (4)$, and when $n = 2$, $u_2 = 4C_1 + C_2 = 1 \dots (5)$. From (4) and (5) we find $C_1 = \frac{1}{3}$, and $C_2 = -\frac{1}{3}$; therefore $g = u_n = \frac{1}{3}(2^n \pm 1)$, the double sign being used as above. Hence the angles of the n th triangle are

$$\frac{1}{3}\pi \pm (\frac{1}{2})^n (\frac{1}{3}\pi - A), \quad \frac{1}{3}\pi \pm (\frac{1}{2})^n (\frac{1}{3}\pi - B), \quad \frac{1}{3}\pi \pm (\frac{1}{2})^n (\frac{1}{3}\pi - C).$$

[Mr. Baker has sent a "Revised Solution" of the question, in which he represents g by the same formula obtained above, and also notices the error in the table, pointed out by Prof. Johnson, but our space will not permit its publication.]

ANSWER TO QUERY (SEE P. 176, VOL. IV), BY THE EDITOR.

FROM the fixed point E draw a line EAB perpendicular to the fixed line CI and intersecting it in B , at any distance a from the point E , and let A represent the middle point of EB .

Let FGH be any position of the right angle, the side GH intersecting the line CI in L , and let the fix'd length of $GE = a$; then, to find the locus of P , the middle point of GF , draw PQ perpendicular to AB , intersecting the side



GH in m ; draw Gn perpendicular to CI , and put $AQ = x$ and $PQ = y$.

Because $AE = \frac{1}{2}a$, $EQ = \frac{1}{2}a + x$, and $BQ = \frac{1}{2}a - a + x = x - \frac{1}{2}a$; $\therefore Gn = 2(x - \frac{1}{2}a)$ and $Fn = \sqrt{a^2 - [2(x - \frac{1}{2}a)]^2} = \pm 2\sqrt{(x^2 - ax)}$.

Hence, from the similar triangles Fng and EQM , and Fng and PGm ,

$$Fn : Gn :: EQ : Qm, \text{ or } 2\sqrt{(x^2 - ax)} : 2(x - \frac{1}{2}a) :: x + \frac{1}{2}a : Qm = \frac{x^2 - \frac{1}{4}a^2}{\sqrt{(x^2 - ax)}},$$

$$Fn : FG :: PG : Pm, \text{ or } 2\sqrt{(x^2 - ax)} : a :: \frac{1}{2}a : Pm = \frac{\frac{1}{4}a^2}{\sqrt{(x^2 - ax)}}.$$

$$\text{But } Qm + Pm = y = \frac{x^2 - \frac{1}{4}a^2}{\sqrt{(x^2 - ax)}} + \frac{a^2}{\sqrt{(x^2 - ax)}} = \frac{x^2}{\sqrt{(x^2 - ax)}}; \therefore y^2 = \frac{x^3}{x - a},$$

which is the equation to the cissoid.

[This query was also answered, differently, by Mr. Seitz, of Greenville, Ohio, Prof. T. A. Smith, of Beloit, Wisconsin, and by Professors Scheffer and Johnson. Prof. Johnson, besides his own demonstration, has sent the following, taken from Briot and Bouquet.]

A being the fixed point and BC the fixed line, and ADB the right angle of which the side $BD = AC = 2a$. Let P and O be the middle points of BD and AC . Draw a circle with centre C and radius a and draw a tangent at Q parallel to BC ; also draw the line BR parallel to AQ . ABD and ABC are equal right triangles, hence if AC and BD meet in L , ABL is an isosceles triangle and AO being equal to BP , OP is parallel to AB and P lies on the line OR . Let this line cut the circle in S and join SC , BPR and OCS are then equal isosceles triangles; hence $OS = PR$ and P describes a cissoid according to the usual definition.



To this is added the following construction for the tangent to a cissoid.

Draw BI perpendicular to DC and AI to AD , then their intersection I is the "instantaneous center" of the motion, and IP is normal to the path of P that is to the cissoid.

SOLUTIONS OF PROBLEMS IN NUMBER ONE.

SOLUTIONS of problems in number one have been received as follows:

From Henry C. Allen, 188; George H. Bangs, 188, 189; Marcus Baker, 188; George Eastwood, 195; A. S. Hathaway, 189, 190, 191; W. E. Heal, 190, 191; Dr. David S. Hart, 189, 190; Prof. J. H. Kershner, 187, 188, 189; Christine Ladd, 187; Prof. H. T. J. Ludwick, 192; Prof. D. J. McAdam, 188, 194; Artemas Martin, 192; Prof. Orson Pratt, 190; P. Richardson, 189; Prof. J. Scheffer, 188, 189, 190, 194, 195; E. B. Seitz, 187, 188, 190, 192, 193; I. H. Turrell, 187.

[In proposing 188 it should have been stated that the proposer desired a solution which should be independent of Calculus.]

187. "Six circles may be described each of which shall touch four of the others. Prove that the lines, joining the centers of the non-touching pairs, are concurrent."

SOLUTION BY ISAAC H. TURRELL, CINCINNATI, OHIO.

The standard case will be when the contacts are all external; for instance when three circles, touching one another externally, are inscribed within the space enclosed by three others, also tangent to one another externally, each circle of either group touching two of the other group, and conversely.

Denoting the circles of the outer group and their centers by a, b, c , and of the inner by x, y, z , there will be three pairs of non-touching circles, viz; ax, by, cz , and it is to be shown that the lines ax, by, cz are concurrent.

Let P, P', P'', O, O', O'' , be the external centers of similitude of the pairs ab, ac, bc, ax, by, cz , respectively; and since, when two circles touch in the same way, another pair of circles, the external center of similitude of either pair lies on the radical axis of the other pair, the points O'', O', P'' , lie on the radical axis of ax , and O'', O, P' , lie on the radical axis of by ; similarly the points O', O, P , lie on the radical axis of cz .

Again, the radical axis of two circles is perpendicular to the line joining their centers, and these radical axes form the triangle $OO'O''$; hence (since the perpendiculars of a triangle meet in a point), the lines joining the centers of the non-touching circles are concurrent.

In case of *internal* contact among any of the circles, the same method of proof will apply, if the corresponding *internal* centers of similitude are employed instead of the external centers.

NOTE.—The foregoing properties of circles in contact combined with the "Theorem of Pascal applied to the Circle," affords a simple and remarkably elegant solution, by pure geometry, of the following noted problem, namely:

Within the space enclosed by three circles which touch one another externally, to inscribe three circles tangent to one another, each of which shall touch two of the given circles.

[Miss Ladd has submitted an elaborate and very elegant solution of this question which we hope to publish in a future number.]

188. "Two given rectangular planes $ABFE$ and $EFCD$ lie contiguous, as in the figure. The plane BE is hard ground whereon one can travel at the rate $2m$; the other is soft ground whereon the rate of travel is m .

Required the point O of crossing EF so that the time of journey from A to C shall be the quickest possible."

SOLUTION BY THE EDITOR.

Take a point O' on the line EF , join AO' and $O'C$ and draw $O'g$ and Oh respectively perpendicular to AO and CO' , then is $\angle OO'g = \angle OAE$.

Suppose OO' to be diminished to infinity, then will the ultimate ratio of Ag to AO' , of Ch to CO , and of $\angle O'Oh$ to $\angle OCF$ be a ratio of equality; and therefore the time in Ag will equal the time in AO' , and the time in Ch will equal the time in CO , therefore the time in Og must equal the time in $O'h$; but the rate in Og is $2m$ while the rate in $O'h$ is m , $\therefore$

$$Og = 2O'h \quad (1)$$

And because of the equality of the angles OAE and $OO'g$, and OCF and $O'Oh$, if we put $OO' = r$, we shall have $Og = r \sin OAE$ and $O'h = r \sin OCF$; $\therefore$, by (1),

$$\sin OAE = 2 \sin OCF. \quad (2)$$

Put $AO = R$ and $CO = R'$, then we have $OE = R \sin OAE$ and $OF = R' \sin OCF = R' \frac{1}{2} \sin OAE$ by (2), $= \frac{1}{2} R' \sin OAE$;

$$\therefore R \times OF = \frac{1}{2} R' \times OE. \quad (3)$$

If we put $AE = a$, $ED = b$, $EF = c$ and $EO = x$, we readily get $R = \sqrt{a^2 + x^2}$, and $R' = \sqrt{b^2 + (c-x)^2}$, which values substi'd in (3) give

$$\sqrt{a^2 + x^2}(c-x) = \frac{1}{2} \sqrt{b^2 + (c-x)^2} x,$$

an equation of the fourth degree from which the value of x may be found.

SOLUTION BY GEO. H. BANGS, NEW YORK CITY.

Let the route AOC be that of the least time. Produce AO to H and CO to G . (G being a point in DA produced.) Then if the rate on OH = the rate on OA the line AH will give the least time from A to H ; and if the rate on OG = the rate on OC the line CG will give the least time from C to G . Now it is plain that the lines AH and CG may be taken as the representatives of times for units of space. Therefore, in this case, $AH : CG :: \frac{1}{2}m : m$; or $AH = \frac{1}{2}CG$, whence it readily appears that $OH = \frac{1}{2}OC$ and $OA = \frac{1}{2}OG$. Hence, putting $AE = a$, $DE = b$, $EF = c$ and $EO = x$, we get

$$\sqrt{x^2 + a^2}(x-c) = \frac{1}{2} \sqrt{x^2 + (x-c)^2} x.$$

[Mr Seitz also gave a solution of this question without formal application of Calculus.]

189. "Find by Algebra the series whose sum is 127, when the sum of the squares of the seven terms is 5461."

SOLUTION BY PROF. J. H. KERSHNER, MERCERSBERG, PA.

$$\text{Let} \quad x + xy + xy^2 + xy^3 + xy^4 + xy^5 + xy^6 = 127, \quad (1)$$

$$x^2 + x^2y^2 + x^2y^4 + x^2y^6 + x^2y^8 + x^2y^{10} + x^2y^{12} = 5461. \quad (2)$$

Dividing (2) by (1) we get

$$x - xy + xy^2 - xy^3 + xy^4 - xy^5 + xy^6 = 43; \quad (3)$$

adding (1) and (3) and taking half their sum, we have

$$x + xy^2 + xy^4 + xy^6 = 85. \quad (4)$$

$$\text{Subt. (3) from (4), } xy + xy^3 + xy^5 = 42. \quad (5)$$

Multiply (5) by y and subtract the product from (4) and we get

$$x = 85 - 42y, \text{ or } y = 2 + [(1-x) \div 42]. \quad \therefore x = 1, \text{ and } y = 2.$$

190. "Transform the equation, $x^9 + 9x^8 + 36x^7 + 85x^6 + 132x^5 + 141x^4 + 105x^3 + 54x^2 + 18x + 9 = 0$, into an equation of the third degree."

SOLUTION BY DAVID S. HART, A. M., M. D., STONINGTON, CONN.

Subtract $(x+1)^9 = 0$ from the given equation, and there remains

$$x^6 + 6x^5 + 15x^4 + 21x^3 + 18x^2 + 9x + 8 = 0. \quad (2)$$

Subtract $(x+1)^6 = 0$ from (2) and there remains

$$x^3 + 3x^2 + 3x + 7 = 0; \quad (3)$$

and from (3) subtract $(x+1)^3 = 0$ and there remains $6 = 0$.

Therefore $(x+1)^9 + (x+1)^6 + (x+1)^3 + 6 = 0$, or, putting $(x+1)^3 = y$,

$$y^3 + y^2 + y + 6 = 0,$$

which is an equation of the third degree.

191. "Compute, in the most convenient way, the product of the differences of the equation $x^5 - ax^4 + bx^3 - cx^2 + dx - e = 0$."

SOLUTION BY A. S. HATHAWAY, ITHACA, NEW YORK.

The product of the differences of the roots is equal to the square root of the product of the squares of the differences, but this last product is the discriminant of the equation.

Let $x = y \div z$, then multiply by z^5 and we obtain

$$y^5 - ay^4z + by^3z^2 - cy^2z^3 + dyz^4 - ez^5 = 0. \quad (1)$$

Differentiate with respect to y , and also with respect to z ;

$$5y^4 - 4ay^3z + 3by^2z^2 - 2cyz^3 + dz^4 = 0, \quad (2)$$

$$-ay^4 + 2by^3z - 3cy^2z^2 + 4dyz^3 - 5ez^4 = 0. \quad (3)$$

Multiply (2) by $-a$, (3) by 5 , subtract, and divide $dy \, z$;

$$(10b - 4a^2)y^3 + (3ab - 15c)y^2z + (20d - 2ac)yz^2 + (ad - 25e)z^3 = 0. \quad (4)$$

Again, multiply (2) by $-ay + 2bz$, (3) by $5y - 4az$, subt., and divide by z^2 ;

$$(3ab-15c)y^3+(20d+10ac-6b^2)y^2z+(4bc-15ad-25e)yz^2+(20ae-2bd)z^3=0. \quad (5)$$

Now multiply (2) by $-ay^2+2byz-3cz^2$, (3) by $5y^2-4ayz+3bz^2$, subtract, and divide by z^3 , and we obtain

$$(20d-2ac)y^3+(4bc-15ad-25e)y^2z+(20ae-10bd-6c^2)yz^2+(3cd-15be)z^3=0. \quad (6)$$

Lastly, multiply (2) by $-ay^3+2by^2z-3cyz^2+4dz^3$, (3) by $5y^3-4ay^2z+3byz^2-2cz^3$, subtract, and divide by z^4 , and we get

$$(ad-25e)y^3+(20ae-2bd)y^2z+(3cd-15be)yz^2+(10ce-4d^2)z^3=0. \quad (7)$$

Now the discriminant of (1) is the resultant of (2) and (3) which is the determinant of the coefficients of (4), (5), (6), (7); but this discriminant is equal to the product of the squares of the differences of the roots; therefore for the product of the differences of the roots we obtain the following expression:

$$\begin{vmatrix} 10b-4a^2 & 3ab-15c & 20d-2ac & ad-25e \\ 3ab-15c & 20d+10ac-6b^2 & 4bc-15ad-25e & 20ae-2bd \\ 20d-2ac & 4bc-15ad-25e & 20ae+10bd-6c^2 & 3cd-15be \\ ad-25e & 20ae-2bd & 3cd-15be & 10ce-4d^2 \end{vmatrix}^{\frac{1}{2}}$$

192 "Find the average area of all the circles inscribed in a given semicircle."

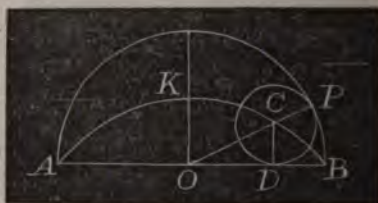
SOLUTION BY PROF. H. T. J. LUDWICK, MT. PLEASANT, N. C.

Let O represent the center of the given semicircle and C the center of an inscribed circle. Put $AO = a$, $OD = x$ and $DC = y$.

The locus of C is $x^2+2ay=a^2$. Area of circle $= \pi y^2 = \pi(a^2-x^2)^2 \div 4a^2$.

Length of arc $AKB = \frac{1}{a} \int_0^a (a^2+x^2)^{\frac{1}{2}} dx$. Denoting by A the average area,

$$\begin{aligned} A &= \frac{\pi}{4a^2} \int_0^a (a^2-x^2)^2 (a^2+x^2)^{\frac{1}{2}} dx \div \int_0^a (a^2+x^2)^{\frac{1}{2}} dx \\ &= \frac{\pi a^2}{96} \left(\frac{39 \log(1+\sqrt{2}) - 5\sqrt{2}}{\sqrt{2} + \log(1+\sqrt{2})} \right). \end{aligned}$$



193. "A triangle is formed by joining three points taken at random in the surface of a given triangle. Find the chance that the circle circumscribing this triangle lies wholly within the given triangle."

SOLUTION BY E. B. SEITZ, GREENVILLE, OHIO.

Let ABC be the given triangle, PQR the triangle formed, O the center of the inscribed circle of ABC , and M the center of the circle circ'bing PQR .

Draw the triangle DEF , making its sides parallel to those of ABC , and at a distance from them equal to MP , and draw ONS and MH perpendicular to AB and PQ .

Now while PQ is given in length and direction, and the angle PRQ is given, if MP is less than the radius of the inscribed circle of ABC , the area of the triangle DEF represents the number of ways the three points can be taken, so that the circumsc'g circle will lie wholly within the given $\triangle$.



Let $PQ = 2x$, $OS = r$, perimeter of $ABC = s$, area of segment $PRQ = t$, area of sector $PMQ = v$, area of $\triangle PMQ = u$, area of $\triangle ABC = A$, $\angle PMH = \theta$, $\varphi = \sin^{-1}(x \div r)$, and $\psi =$ the angle which PQ makes with some fixed line. Then we shall have $PM = x \operatorname{cosec} \theta$, $ON = r - x \operatorname{cosec} \theta$, area $DEF = (r - x \operatorname{cosec} \theta)^2 (A \div r^2)$, $v = \theta x^2 \operatorname{cosec}^2 \theta$, $u = x^2 \cot \theta$, $t = v - u$. An element of the triangle at Q is $4x dx d\psi$, or $4r^2 \sin \varphi \cos \varphi d\varphi d\psi$, and at R it is dt . The limits of ψ are 0 and 2π ; of θ , φ and $\pi - \varphi$; and of x , 0 and r , or of φ , 0 and $\frac{1}{2}\pi$. Hence, doubling, since R may lie on either side of PQ , we have for the required chance

$$p = \frac{2}{A^3} \int_0^r 4x dx \int_{\theta=\varphi}^{\theta=\pi-\varphi} dt \int_0^{2\pi} d\psi (r - x \operatorname{cosec} \theta)^2 \frac{A}{r^2} = \frac{2\pi^2 r^4}{5A^2} = \frac{8\pi^2 r^2}{5s^2}.$$

194. "Sum the series $1 - \frac{1}{2^2} + \frac{1}{4^2 \cdot 2^2} - \frac{1}{6^2 \cdot 4^2 \cdot 2^2} + \frac{1}{8^2 \cdot 6^2 \cdot 4^2 \cdot 2^2} - \&c.$ "

SOLUTION BY PROF. J. SCHEFFER.

If by means of the formula

$$\int \sin^n x dx = \frac{-\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x dx}{n},$$

we develop the successive integrals $\int_0^\pi \sin^2 x dx$, $\int_0^\pi \sin^4 x dx$, $\int_0^\pi \sin^6 x dx$, etc., we find that the given series is equal to the definite integral

$$\frac{1}{\pi} \int_0^\pi \cos(\sin x) dx.$$

The series converges rapidly. Eight terms suffice to obtain the sum correctly for at least 10 decimals. The sum is $= .7651976865$.

Such integrals as the last are always, and without difficulty, obtained by series.

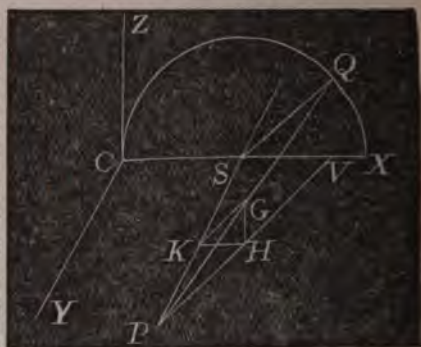
195. "A uniform rod rests with one extremity against a rough vertical wall, and with the other extremity on a rough horizontal plane, such that it is held in equilibrium by friction alone. The beam is *not* in a vertical plane. The coefficient of friction of the horizontal plane and vertical wall being respectively μ and μ' ; find the normal pressure of the rod upon the horizontal plane and vertical wall, and the exact position of the rod with reference to the two latter planes."

SOLUTION BY GEO. EASTWOOD SAXONVILLE, MASS.

Let the plane XCZ represent the vertical wall and the plane XCY the horizontal plane.

Let PQ be a position of the rod when it is bordering on motion. Draw PS perpendicular to CX , and join SQ .

If we suppose for a moment, to fix our thoughts, that P is a fixed point, the locus of the point Q will be a semi-circle CQX , having the point S for its centre. Let G be the centre of gravity of the rod, PHV the projection of PQ in the plane of XY , H and V being the projections of the points G and Q .



Draw HK parallel to CX , which will evidently be at right angles to PS , and join KG . Let W be the weight of the rod, μ' the coefficient of friction between the rod and vertical wall, and R' their mutual pressure; then the force $\mu'R'$ will act in the tangent to the locus at the point Q , that is, at right angles to SQ and in the plane CQX . Put $PG=GQ=\beta$, $\angle QPS=\alpha$, $\angle QSV=\theta$, R =reaction of horizontal plane, and μR = the force of friction at P .

Now the horizontal forces acting on the rod are, μR at P , and $\mu'R' \sin \theta$ at Q ; $\therefore$

$$\mu R = \mu' R' \sin \theta. \quad (1)$$

The vertical forces acting on the rod are R at P and W at G , and $\mu'R' \cos \theta$ at Q ; $\therefore$

$$R + \mu' R' \cos \theta = W. \quad (2)$$

Next, conceive a line to be drawn perpendicular to PS , and let moments be taken about these two lines. Then

$$W.HK = \mu' R'.SQ, \quad (3)$$

$$W.PK = R'.QV + \mu' R' \cos \theta. \quad (4)$$

Geometrical relations give, $HK=GK \cos \theta = \beta \sin \alpha \cos \theta$, $SQ=2\beta \sin \alpha$, $PS \cos \theta = 2\beta \cos \alpha \cos \theta$, $PK = \beta \cos \alpha$, $QV = SQ \sin \theta = 2\beta \sin \alpha \sin \theta$.

Making these substitutions in (3) and (4), the latter equations become

$$W\beta \sin \alpha \cos \theta = 2\beta \mu' R' \sin \alpha, \quad (5)$$

$$W\beta \cos \alpha = 2\beta R' \sin \theta + 2\beta \mu' R' \cos \alpha \cos \theta. \quad (6)$$

Dividing (6) by (5), and reducing, we finally obtain $\tan \alpha = \mu' \tan \theta$.

From eq's (1) and (2), $\sin \theta = \mu R \div \mu' R' \dots (7)$, $\cos \theta = (W - R) \div \mu' R'$. (8)
Hence $\tan \alpha = \mu' \mu R \div (W - R)$, and therefore

$$\sin \alpha = \frac{\mu' \mu R}{[(W - R)^2 + (\mu' \mu R)^2]^{\frac{1}{2}}} \dots (9), \cos \alpha = \frac{W - R}{[(W - R)^2 + (\mu' \mu R)^2]^{\frac{1}{2}}} \quad (10)$$

The substitution of (8) in (5) gives

$$W \cdot \frac{W - R}{\mu' R'} = 2\mu' R', \dots W - R = \frac{2(\mu' R')^2}{W} \dots (11), \text{ and } R = \frac{W^2 - 2(\mu' R')^2}{W}$$

In (6), substitute the values of $\sin \theta$, $\cos \theta$, $\sin \alpha$, $\cos \alpha$, as found in (7), (8), (9) and (10) and reduce; then

$$W(W - R) = 2\mu^2 R^2 + 2(W - R)^2. \quad (13)$$

By means of (11) and (12), equation (13) becomes

$$2(\mu' R')^2 = 2\mu^2 \left(\frac{W^2 - 2(\mu' R')^2}{W} \right)^2 + \frac{8(\mu' R')^4}{W^2}, \text{ which reduces to}$$

$$R'^4 - \frac{(4\mu^2 + 1)W^2}{4\mu'^2(\mu + 1)} R'^2 = - \frac{\mu^2 W^4}{4\mu'^4(\mu^2 + 1)}. \quad (14)$$

The resolution of equation (14) gives

$$R' = \pm \frac{W}{2\mu'} \left[\frac{(4\mu + 1) \mp (1 - 8\mu^2)^{\frac{1}{2}}}{2(\mu^2 + 1)} \right]^{\frac{1}{2}}.$$

This value of the reaction of the vertical plane, substituted in (12) determines the reaction of the horizontal plane, and by means of these values and the given coeff's μ' and μ , the angles α and θ can be readily found.

PROBLEMS.

196. BY P. M. MC. KAY, LOOXAHOMA, MISS.—The sides of a triangle are respectively $x^2 + x + 1$, $2x + 1$ and $x^2 - 1$, x being any number greater than one; prove that the angle opposite the side $x^2 + x + 1$ is equal to 120° .

197. BY LIEUT. FRED. SCHWATKA, SPOTTED TAIL AG'Y, D. T.— AB is the diameter of a circle whose center is C , D is the middle point of CB , and is the center of a circle whose diameter is CB , and E is the middle p't of AC . EF is perp. to AB and intersects the circle C in F , and EG is tangent to the circle D at G . Prove that EFG is an equilateral triangle.

198. BY ARTEMAS MARTIN, M. A., ERIE, PA.—Find the average dist. between two points taken at random in the surface of a rectan., sides a & b .

199. BY AMATEUR.—Find an infinite series the sum of which is a square,—the sum of n terms of which is a square,—and the sum of the remaining terms, after the n terms are taken away, is a square.

200. BY R. J. ADCOCK. — Prove that the locus of points, the sum of the squares of their distances from fixed points on the surface of a sphere.

201. BY PROF. ASAPH HALL.—Given

$$x = a \cos A \pm r_1,$$

$$y = a \sin A \pm r_2,$$

where r_1 and r_2 are the probable errors of x and y . Required errors of a and A .

QUERY, BY CHRISTINE LADD.—Which of the two following is correct?

“The point $x'y'z'$ is a multiple point of the order k , if all the coefficients of the order $k-1$ vanish for that point.” (*Salmon's Plane Curves*, Art. 73.)

“Il s'ensuit que, pour un point multiple de l'ordre n , il est nécessaire, mais non suffisant, que les dérivées partielles de la fonction f de degré n s'annulent toutes pour les co-ordonnées du point.” (*Calcul Différentiel*, Art. 186.)

PUBLICATIONS RECEIVED.

The National Quarterly Review. (Jan., 1878.) DAVID A. GORTON & Co., Philadelphia. Besides the vigorous and scholarly literary articles which this periodical contains, this No. contains an interesting review of *The Progress of Modern Science*.
The Mathematical Visitor, No. 2. ARTEMAS MARTIN, M. A., Editor and Publisher. Pa. 4to. 38 pp. This is an annual publication, devoted exclusively to mathematical problems, and contains a variety of problems, and many elegant solutions.

ERRATA.

On page 161 (Vol. IV), line 15, for θ' , θ'' , θ''' , read θ^0 , θ^{00} , θ^{000} .
 “ “ “ “ 2, from bottom, for 48, at end of line, read 49.
 “ “ 162, “ 5, read $A = \frac{2}{1+b} \cdot \frac{2}{1+b^0} \cdot \frac{2}{1+b^{00}} = \left[\frac{2}{1+b} \right]^3$.
 “ “ “ “ 14, for $\sqrt{1+e^2 \sin^2 \theta}$, read $\sqrt{1-e^2 \sin^2 \theta}$.
 “ “ “ 7 (Vol. V), “ 12, for thirty, read sixty.
 “ “ “ “ 15, for tables, read totals.
 “ “ 9, “ 3, from bottom, for $\Delta d\psi$, read $A \Delta d\psi$.
 “ “ 11, “ 3, “ “ for $\sqrt{b \sin \theta}$, read $\sqrt{b \tan \theta}$.
 “ “ 15, “ 3, for the factor $2\pi \div A$, read $4 \div A$.
 “ “ 47, “ 8, for ψ , read ϕ .
 “ “ “ line 22, for $(\cot a \cot b)(\cot x_1 \cot b)$, read $(\cot a + \cot b)(\cot x_1 + \cot b)$.
 “ “ 48, in (20), for $\pm p$ read $\mp p$, and in (21), for $\pm p_1$, read, $\mp p_1$.
 “ “ 53, line 1, for e' , read e_1 , and in line 4, from bot., for $1-e$ read $1-e_1$.

THE ANALYST.

VOL. V.

MAY, 1878.

No. 3.

ON REPEATED ADJUSTMENTS, AND ON SIGNS OF RESIDUALS.

BY E. L. DE FOREST, M. A., WATERTOWN, CONN.

LET any consecutive terms in a series be denoted by

$$\dots u_{-2}, u_{-1}, u_0, u_1, u_2, \dots \&c.$$

Suppose them to be adjusted by the formula

$$u'_0 = l_0 u_0 + l_1(u_1 + u_{-1}) + l_2(u_2 + u_{-2}) + l_3(u_3 + u_{-3}) + l_4(u_4 + u_{-4}), \quad (1)$$

and let the adjusted terms be again adjusted by the formula

$$u''_0 = L_0 u'_0 + L_1(u'_1 + u'_{-1}) + L_2(u'_2 + u'_{-2}). \quad (2)$$

By the first operation, five consecutive adjusted terms will be

$$u'_{-2} = l_0 u_{-2} + l_1(u_{-1} + u_{-3}) + l_2(u_0 + u_{-4}) + l_3(u_1 + u_{-5}) + l_4(u_2 + u_{-6})$$

$$u'_{-1} = l_0 u_{-1} + l_1(u_0 + u_{-2}) + l_2(u_1 + u_{-3}) + l_3(u_2 + u_{-4}) + l_4(u_3 + u_{-5})$$

$$u'_0 = l_0 u_0 + l_1(u_1 + u_{-1}) + l_2(u_2 + u_{-2}) + l_3(u_3 + u_{-3}) + l_4(u_4 + u_{-4})$$

$$u'_1 = l_0 u_1 + l_1(u_2 + u_0) + l_2(u_3 + u_{-1}) + l_3(u_4 + u_{-2}) + l_4(u_5 + u_{-3})$$

$$u'_2 = l_0 u_2 + l_1(u_3 + u_1) + l_2(u_4 + u_0) + l_3(u_5 + u_{-1}) + l_4(u_6 + u_{-2}),$$

and by the second operation, a readjusted term will be

$$u''_0 = L_0 u'_0 + L_1(u'_1 + u'_{-1}) + L_2(u'_2 + u'_{-2}).$$

Substituting in this the values of $u'_0, u'_1, \&c.$, as above, we get

$$u''_0 = \left. \begin{aligned} &[L_0 l_0 + L_1(l_1 + l_1) + L_2(l_2 + l_2)] u_0 \\ &+ [L_0 l_1 + L_1(l_0 + l_2) + L_2(l_1 + l_3)] (u_1 + u_{-1}) \\ &+ [L_0 l_2 + L_1(l_1 + l_3) + L_2(l_0 + l_4)] (u_2 + u_{-2}) \\ &+ [L_0 l_3 + L_1(l_2 + l_4) + L_2 l_1] (u_3 + u_{-3}) \\ &+ (L_0 l_4 + L_1 l_3 + L_2 l_2) (u_4 + u_{-4}) \\ &+ (L_1 l_4 + L_2 l_3) (u_5 + u_{-5}) + L_2 l_4 (u_6 + u_{-6}). \end{aligned} \right\} \quad (3)$$

It thus appears that the re-adjusted term u''_0 is a linear function of the original terms $u_0, u_1, \&c.$, and that the expression (3) may itself be regarded as an adjustment formula which replaces and is equivalent to the two formulas (1) and (2) used successively. Furthermore, if the order of procedure

It must be remembered that all values of l beyond l_m are zero, and also that wherever the expression gives to l a negative index figure, it must be changed to positive, since $l_{-i} = l_i$.

In this way, by taking $m = 4$ and $n = 2$, we find that (6) reduces to (3).

If a series is adjusted twice by the same formula, then (5) is identical with (4), and (6) is somewhat simplified, because we now have

$$L_i l_k = L_k l_i,$$

so that (6) may be written as follows.

$$u''_0 = \left. \begin{aligned} & [l_0 l_0 + 2(l_1 l_1 + l_2 l_2 + l_3 l_3 + l_4 l_4 + \dots + l_m l_m)] u_0 \\ & + 2(l_0 l_1 + l_1 l_2 + l_2 l_3 + l_3 l_4 + \dots + l_{m-1} l_m) (u_1 + u_{-1}) \\ & + [l_1 l_1 + 2(l_0 l_2 + l_1 l_3 + l_2 l_4 + l_3 l_5 + \dots + l_{m-2} l_m) (u_2 + u_{-2}) \\ & + 2(l_1 l_2 + l_0 l_3 + l_1 l_4 + l_2 l_5 + \dots + l_{m-3} l_m) (u_3 + u_{-3}) \\ & + [l_2 l_2 + 2(l_1 l_3 + l_0 l_4 + l_1 l_5 + l_2 l_6 + \dots + l_{m-4} l_m) (u_4 + u_{-4}) \\ & + 2(l_2 l_3 + l_1 l_4 + l_0 l_5 + l_1 l_6 + \dots + l_{m-5} l_m) (u_5 + u_{-5}) \\ & \dots \\ & + 2l_{m-1} l_m (u_{2m-1} + u_{-(2m-1)}) + l_m l_m (u_{2m} + u_{-2m}). \end{aligned} \right\} \quad (7)$$

In the successive products ll in the horizontal lines of this formula, the sequence of the indices of the first l , and also of those of the second l , is easily seen. Let us apply this to a simple case. The 5-term formula of Table A. (ANALYST, May 1877, p. 81), in exact numbers is

$$u'_0 = \frac{1}{35} [17u_0 + 12(u_1 + u_{-1}) - 3(u_2 + u_{-2})] \quad (8)$$

so that the coefficients l , in decimals of five places are

$$l_0 = .48572 \quad l_1 = .34286 \quad l_2 = -.08572.$$

Forming the products

$$l_0 l_0, \quad l_0 l_1, \quad l_0 l_2, \quad l_1 l_1, \quad l_1 l_2, \quad l_2 l_2,$$

and putting $m = 2$, we get by formula (7)

$$u''_0 = .48572u_0 + .27430(u_1 + u_{-1}) + .03427(u_2 + u_{-2}) \\ - .05878(u_3 + u_{-3}) + .00735(u_4 + u_{-4}). \quad (9)$$

This is an adjustment formula such that if a given series is adjusted by it once, the result is the same as though it had been adjusted by (8) twice. Again, if we suppose a series to be adjusted by both (8) and (9) successively, the equivalent single formula is found by designating their coefficients by L and l respectively, forming all the products of every L by every l , and putting $m = 4$ and $n = 2$. Formula (6) then gives

$$u'''_0 = .41814u_0 + .29305(u_1 + u_{-1}) + .04827(u_2 + u_{-2}) - .03780(u_3 + u_{-3}) \\ - .01952(u_4 + u_{-4}) + .00756(u_5 + u_{-5}) - .00063(u_6 + u_{-6}), \\ \dots \quad (10)$$

and this result is denoted by u'''_0 because one adjustment of a series by this formula is equivalent to three successive adjustments by the original formula (8). In the same way we may combine (10) with (8), (9) or (10), and thus get the equivalents of four, five and six applications of (8). The process

has indeed been extended much farther. It is not so difficult or tedious as might be supposed, for it has been found that the coefficients of the extreme terms of the formula soon become so small as to disappear, that is, they do not affect the first five places of decimals. Thus in (10) the coefficient of u_6 is only seen in the fourth place. At the successive steps of the process the correctness of the work is readily tested by the condition that for every formula obtained we must have

$$1_0 + 2(1_1 + 1_2 + 1_3 + \&c.) = 1,$$

within such narrow limits of error as are to be expected from the use of logarithms in forming the products. And at each step, before proceeding farther, it is best to make the formula satisfy this condition exactly, by altering one, two or three of the coefficients, if necessary, to the extent of a single unit in the fifth decimal place. The various products have been carried to five, and sometimes to six, places of decimals. The resulting coefficients are shown in the accompanying table I., to three places only. The numbers of successive applications of the original formula (8) are found in the upper line of the table, and the corresponding coefficients are in the columns

TABLE I.

	1	2	3	4	5	6	8	12	16	32
l_0	.486	.486	.418	.396	.372	.354	.330	.296	.274	.230
l_1	.343	.274	.293	.280	.277	.271	.261	.246	.234	.205
l_2	-.086	.034	.048	.077	.090	.102	.116	.131	.139	.145
l_3		-.059	-.038	-.034	-.023	-.014	.002	.026	.042	.072
l_4		.008	-.019	-.024	-.030	-.032	-.031	-.024	-.015	.014
l_5			.008	.000	-.004	-.009	-.017	-.024	-.026	-.017
l_6			-.001	.004	.004	.003	.000	-.008	-.014	-.022
l_7				-.001	.001	.002	.003	.002	-.002	-.014
l_8					-.001	-.000	.001	.003	.003	-.004
l_9									.002	.002
l_{10}										.003
l_{11}										.001
r	.697	.629	.596	.574	.558	.544	.524	.497	.479	.438
ρ	.297	.102		.0513		.0331		.0160		.0058

below. The coefficients in the last column, for instance, are those of an adjustment formula such that if a given series is adjusted by it once, the result is the same as if it had been adjusted by the original formula (8) thirty-two times in succession, or by (9) sixteen times. This 32-fold formula, if given exactly, would include no less than 129 terms, with 65 different coefficients, but only 12 coefficients are large enough to appear in the third decimal place, and only four more would affect the fourth place. At the bottom of

the table are given the ratios of error and of irregularity

$$r = \frac{\varepsilon'}{\varepsilon}, \quad \rho = \frac{(d'_4)}{(d_4)}.$$

The ratio r diminishes slowly as the number of repetitions is increased, while ρ diminishes much faster. For any large number of repetitions, the form of the series of values of l approximates somewhat to that of the formulas of Table B. (ANALYST, p. 82.) For instance, the large coefficients of the 12-fold and 32-fold formulas of Table I. approach very near to those of the 11-term and 15-term formulas of Table B., and their ratios r and ρ are also nearly alike.

A similar process of repetitions has been applied to the 5-term formula of Table B., which in its exact form is

$$u'_0 = \frac{1}{195}[111u_0 + 56(u_1 + u_{-1}) - 14(u_2 + u_{-2})]. \quad (11)$$

(*Smithsonian Report* of 1871, p. 333.) The resultant coefficients are given in the first half of Table II. The second half contains similar results of repetitions of the formula, found at p. 289 of the *Report*,

$$u'_0 = \frac{1}{10}[4(u_0 + u_1 + u_{-1}) - (u_2 + u_{-2})]. \quad (12)$$

TABLE II.

	1	2	4	6	12	1	2	4	6	12
l_0	.570	.500	.416	.374	.310	.400	.500	.392	.344	.284
l_1	.287	.285	.285	.276	.253	.400	.240	.264	.261	.239
l_2	-.072	.001	.059	.089	.125	-.100	.080	.101	.115	.136
l_3		-.041	-.038	-.023	.015		-.080	-.037	-.008	.035
l_4		.005	-.019	-.029	-.028		.010	-.022	-.030	-.019
l_5			.003	-.004	-.021			-.008	-.016	-.026
l_6			.002	.003	-.004			.008	.004	-.011
l_7				.001	.003			-.002	.002	.000
l_8					.002				.001	.003
l_9									-.001	.001
r	.707	.645	.588	.558	.508	.707	.625	.563	.532	.486
ρ	.229	.122	.0622	.0411	.0195	.446	.239	.0829	.0365	.0134

It will be seen that although the coefficients l of the two original formulas (11) and (12) differ from each other very considerably, yet after 12 applications they approach pretty nearly to each other and also to those of the 12-fold formula of Table I. The idea is thus suggested, that when any 5-term formulas similar to (8), (11) and (12) are applied a large number of

times in succession, the resultant coefficients approximate to a common form, and that they resemble the coefficients of Table B. a good deal more closely than those of Tables A. or C. This may perhaps be regarded as an argument in favor of the use of the formulas of Table B. for purposes of adjustment. It is reinforced by the fact that when other formulas are repeatedly used, they take a similar outline. The 9-term formula of Table A., in exact numbers, is

$u'_0 = \frac{1}{288} [59u_0 + 54(u_1 + u_{-1}) + 39(u_2 + u_{-2}) + 14(u_3 + u_{-3}) - 21(u_4 + u_{-4})]$,
and if it is used four times in succession, the coefficients of the equivalent formulas are, to three places of decimals,

.202	.185	.141	.086	.035	— .001	— .018
— .018	— .011	— .004	.001	.002	.001	

These differ but little from the coefficients in the 17-term formula of Table B. Again, the 13-term formula of Table A., in exact numbers, is

$$u'_0 = \frac{1}{144} [25u_0 + 24(u_1 + u_{-1}) + 21(u_2 + u_{-2}) + 16(u_3 + u_{-3}) + 9(u_4 + u_{-4}) + 0(u_5 + u_{-5}) - 11(u_6 + u_{-6})],$$

and, after four successive applications of it, the resultant formula is found to have the following coefficients:

.138	.132	.117	.095	.069	.043	.021
	.004	— .008	— .013	— .013	— .010	— .006
	— .003	— .001	.001	.001	.001	.001.

There is no very great difference between these and the coefficients of the 25-term formula of Table B.

A comparison of the values of r and ρ in Tables I. and II. affords some rather unexpected results. Although the original formula (12) gives values to r and ρ as large as, or larger than, either of the original formulas (8) and (11) do, yet after twelve successive applications, it gives smaller values than the others do. In other words, formula (8), which was found by the method of least squares, and makes, when the conditions assumed are fulfilled, a more accurate single adjustment than any other 5-term formula, is not the one which gives the greatest accuracy when the adjustment is repeated several times, or even when it is repeated once only. Likewise formula (11), which under the same conditions makes the smoothest single adjustment of any, is not the one which gives greatest smoothness when the adjustment is repeated.

Other calculations have been made for the purpose of learning what the results will be when the 25-term formulas of Tables A., B. and C. are used in succession, that is, when any one of them is used twice, or when any two are used one after the other. Each of the complete resultant formulas includes 49 terms, with 25 different coefficients, and from 17 to 25 of these

are large enough to affect the third decimal figure. It is thought unnecessary to give any of them here in full, but the ratios r and ρ for all of them are shown in Table III. For instance, when the 25-term formulas of Tables A. and B. are used in succession, no matter which is used first, the error-ratio and irregularity ratio of the resulting adjustment will be

$$r = .281 \qquad \rho = .000191.$$

TABLE III.

	Values of r .			Values of ρ .		
	A	B	C	A	B	C
A	.266			.002649		
B	.281	.313		.000191	.000324	
C	.272	.291	.279	.000301	.000185	.000313

When the 25-term formula of Table C. is applied twice, the ratios for the result will be $r = .279$ $\rho = .000313$, and so on. As all the values of r in Table III. are larger than .250, it will obviously be difficult, if not impossible, ever to adjust a series by these methods so as to reduce the error-ratio below $\frac{1}{4}$. It diminishes but very slowly with each repetition, the ratio for the second application of A., for instance, being .266, while for the first application Table A. shows it to be .300, and we have

$$\frac{.266}{.300} = .887,$$

so that the probable error of the second adjustment is little less than nine tenths that of the first one.

We also see that the smallest value of ρ in Table III. is not, as might have been expected, that which results from two applications of the B formula. Smaller values are given not only by two applications of the C, but also by either A and B, A and C, or B and C.

SIGNS OF RESIDUALS.—In the ANALYST for Jan. 1878, p. 3, it was shown by me that when a series has been adjusted by a formula whose error-ratio r is small, a safe limit being $r < \frac{1}{2}$, the probability that the signs of any two adjacent residual errors will be alike is approximately

$$q = \frac{1-2r^2}{2-2r^2}, \qquad (13)$$

and as a consequence of this, that the most probable number M of signs falling in groups of only one or two signs each, in a periodic series, is given by

$$M = (p_1 + 2p_2)N \pm .6745 \sqrt{[p_1(1-p_1) + 4p_2(1-p_2)]N}, \quad \left. \begin{array}{l} p_1 = (1-q)^2, \\ p_2 = p_1 q, \end{array} \right\} \quad (14)$$

where N is the whole number of terms in the series. The probable number falling in groups of more than two is

$$M' = N - M.$$

The above will answer also for non-periodic series, if $N-\frac{1}{2}$ is put in place of N , in which case the first and last signs of the series, and as many adjacent ones as happen to be like them, are not counted. If we write

$$M' = aN \pm b \sqrt{N},$$

the values of a and b , for different values of r , will be as here given in Table IV.

TABLE IV.

r	a	b	a_1	b_1
.25	.450	.55	.467	.34
.30	.426	.56	.451	.34
.35	.396	.56	.430	.33
.40	.359	.57	.405	.33
.45	.314	.58	.373	.33
.50	.259	.58	.333	.32

The process by which (14) was obtained assumes that any four consecutive signs may be regarded as independent of each other, except as adjacent ones are connected by the relation (13). This is only approximately true. For the purpose of testing an adjustment, it may be better, and is simpler, to consider only the number of permanences of sign, that is, the number of cases where adjacent signs are alike, in passing along the series of residuals from one end to the other. The most probable number of permanences will be, in a periodic series,

$$P = qN \pm .6745 \sqrt{[q(1-q)N]}, \quad (15)$$

and the same for a non-periodic series, only putting $N-1$ in the place of N .

If (15) is written

$$P = a_1 N \pm b_1 \sqrt{N}$$

the values of a_1 and b_1 , for different values of r , will be found in Table IV. For example, in the observed series of signs at p. 5 of the ANALYST referred to, we have $r = .372$, $N = 70$ and by interpolation in Table IV., $a = .380$, $b = .57$, $a_1 = .419$, $b_1 = .33$, so that computation gives

$$M' = .380 \times 67.5 \pm .56 \sqrt{(67.5)} = 25.7 \pm 4.6,$$

$$P = .419 \times 69 \pm .33 \sqrt{(69)} = 28.9 \pm 2.7.$$

Observation by actual count in the series shows

$$M' = 22, \quad P = 25.$$

Both fall short of the computed numbers, but the deviation is on the safe side, and must be considered accidental. It is only when the observed values of M' and P exceed the computed values, by more than the probable errors, that we are to infer the existence of a danger that the adjusted series has been smoothed out too much, so as to deviate materially from its true law.

EQUATIONS OF THE THIRD DEGREE.

BY PROF. L. G. BARBOUR, RICHMOND, KY.

In the equation $X^3 \pm AX^2 \pm BX \pm C = Y$, A , B and C will be considered as representing real and not imaginary quantities. Let X be any abscissa and Y the ordinate corresponding to it. We shall then have a curve of which $X^3 \pm AX^2 \pm BX \pm C = Y$ is the equation. The origin can be so taken as to reduce the equation to the form $x^3 \pm px \pm q = y$. This will not alter the form of the curve at all. Again the constant q , the absolute term, can be omitted without changing the form of the curve. Hence we need consider only such equations as $x^3 \pm px = y$.

CASE I. $x^3 + px = y$. The curve will be of the general type seen in fig. 1.

1. It will pass through the origin.
2. It will have two infinite branches.
3. For every + value of x , there will be a corresponding + value of y ; and for every equal - value of x there will be an equal - value of y . Hence every chord passing through the origin is bisected there, and the origin may be called the centre of the curve.

4. Since $\frac{dy}{dx} = 3x^2 + p$, when $x = 0$, the trigonometrical tangent of the angle the curve tangent makes with the axis of abscissas = p . As the sign of this is +, the angle is less than 90° , so long as p is finite.

The tangent to the curve at other points than the origin will approach a vertical direction, i. e. a direction parallel to the axis of ordinates, whether x be + or -, because in either case x^2 will be +. Since $3x^2 + p$ increases with every increase of x , the curve is concave to the axis of ordinates both below and above the origin.

5. It is evident, even without a second differentiation, that the minimum value of $3x^2 + p$ is p ; $\therefore$ there is a point of inflexion at the origin.

Points of inflexion mark maximum and minimum values of angles where the trigonometrical tangent does not change its sign. The ordinary method, however, is very simple; $\frac{d^2y}{dx^2} = 6x = 0$, $\therefore x = 0$. As $6x$ is of the 1st degree there can be only one point of inflexion, and there must be one even though p were imaginary since p disappears at the 2nd differentiation.



6. As $p > 0$, the minimum angle of inclination is > 0 ; $\therefore$ the curve never becomes parallel to the axis of abscissas.

7. An indefinite number of curves may be drawn of this class.

CASE II. Let $p = 0$, then $x^3 = y$. (See fig. 2.)

1. As p has only one value here, only one curve can be described.

2. $\frac{dy}{dx} = 3x^2 = 0$ when $x = 0$; $\therefore$ the curve is parallel to the axis of abscissas at the origin. The point of inflexion is there, also. In most other particulars the curve resembles those of the 1st class.

CASE III. $x^3 - px = y$. (See fig. 3.)

1. $\frac{dy}{dx} = 3x^2 - p$. Therefore at the origin the curve makes with the axis of abscissas an angle whose tangent $= -p$. As x increases on the $+$ side or on the $-$, it will reach a point where $3x^2 = p$, or $3x^2 - p = 0$; $\therefore$ there are two points at which the curve is parallel to the axis of abscissas, one for $+x = \sqrt{\frac{1}{3}p}$, and the other for $-x = \sqrt{\frac{1}{3}p}$. For greater values of x , either positive or negative, $3x^2 - p$ is positive, and, as before, the curve is concave towards the axis of ordinates.

2. The point of inflexion is at the origin, making a maximum value of the tangent.

3. An indefinite number of lines of this class can be drawn.

TRANSFORMATION OF CO-ORDINATES.

1. The origin can be placed higher up the page by annexing $-q$ to the left hand member of an eq'n, or lower by annexing $+q$. Thus if the eq'n be $x^3 \pm px = y$ the origin is at the point of inflexion. It may be moved 6 spaces higher by making the eq'n $x^3 \pm px - 6 = y$, or 6 spaces lower by $x^3 \pm px + 6 = y$. The origin remains on the axis of ordinates.



2. The origin can be removed to the right by substituting $x' + a$ for x ; to the left, by substituting $x' - a$ for x . Or the same result can be obtained by Horner's method of detached coefficients, dividing by $+a$, or $-a$.

ROOTS OF THE THREE FORMS.

1. The form $x^3 + px = y$ has only one real root, and can have no more real roots when the origin is removed by any of the transformations. This

root is found by making $y = 0$. The curve at that point touches or cuts the axis of abscissas. The root may be found by Cardan's Rule.

2. The 2nd form, $x^3 = y = 0$, has three equal roots so long as the origin remains on the axis of abscissas.

$x^3 = 0$ has 3 roots, each $= 0$, $(x-a)^3 = 0$ has 3 roots, each $= +a$, and $(x+a)^3 = 0$ has 3 roots, each $= -a$. But $x^3 \pm q = 0$ has one real root and two imaginary ones; however the origin may be changed to right or left, the number of the real and the imaginary roots will be the same, one real and two imaginary.

In the former case $q = \mp a^3$; $\therefore$ the distance to the right or left of the origin at which the curve cuts the axis of abscissas is equal the cube root of the distance above or below the origin at which it cuts the axis of ordinates.

In $x^3 \pm q = 0$, $x = \sqrt[3]{\mp q}$ gives the one real root. The imaginary roots can be obtained by the ordinary methods.

3. The 3rd form, $x^3 - px = 0$, and its transformations, will always have at least one real root. It will have 3 real roots, 2 of them being equal, when the origin is removed from the point of inflexion to e or d , or to any pos'n on the line kef or cdl , these lines being parallel to the axis of abscissas and tangent respectively to the max. and min. points of the curve.

We have already found $Oa = \sqrt[3]{(\frac{1}{3}p)}$, $Oi = -\sqrt[3]{(\frac{1}{3}p)}$. If $q > Oe$, or $-q > -Od$, numerically, there will be only one real root. If $q < Oe$, or $-q < -Od$, numerically, there will be 3 real and unequal roots. If $q = 0$, one root will be 0, one will $= \sqrt[3]{(\frac{1}{3}p)}$, and one will $= -\sqrt[3]{(\frac{1}{3}p)}$.

Example. $x^3 - 12x = 0$. The roots are 0, $+\sqrt[3]{(12)}$, $-\sqrt[3]{(12)}$. It is required to ascertain what absolute term shall be added or subtracted so that there shall be two equal roots. Here $p = 12$, $\therefore \sqrt[3]{(\frac{1}{3}p)} = 2$. Substituting this in $x^3 - 12x$, we get $8 - 24 = -16$. $\therefore x^3 - 12x + 16 = 0$ has the roots $+2$, $+2$, and -4 . $x^3 - 12x - 16 = 0$ has the roots -2 , -2 , $+4$. In general $x^3 - px + \sqrt[3]{(\frac{4}{27}p^3)}$ or $x^3 - px + \frac{2}{3}p\sqrt[3]{(\frac{1}{3}p)}$ will have the roots $+\sqrt[3]{(\frac{1}{3}p)}$, $+\sqrt[3]{(\frac{1}{3}p)}$, and $-2\sqrt[3]{(\frac{1}{3}p)}$; and $x^3 - px - \sqrt[3]{(\frac{4}{27}p^3)}$ will have the roots $-\sqrt[3]{(\frac{1}{3}p)}$, $-\sqrt[3]{(\frac{1}{3}p)}$, $+2\sqrt[3]{(\frac{1}{3}p)}$. These roots are identical with those obtained by Cardan's Rule, which applies to all cases in which there is only one real root, and also to those in which there are two equal real roots. But it has no sway over the charmed area between the lines kef and cdl . Just here the "irreducible case" lies.

4. Our attention must now be directed to the curve $kcofl$. But as this curve is symmetrical, it will be sufficient in most respects to consider only one half of it, as kco .

There is no trouble in finding the roots when the origin is at O , or at d ; as we have seen. But suppose it is placed at n , any point between O and d .

Then we are to determine the lengths of the lines nT , ns and nt . Let $nT = r$, $ns = s$, $nt = t$. Then, by general theory of equations, $r+s-t=0$, $rs-rt-st=-p$ and $rst=q$. When, then, the equation is $x^3-px+q=0$ and there are three unequal roots, two of them will be +, and the other—.

It is readily seen that if the origin be removed to p in the figure, the curve will cut the axis of ordinates below the new origin; $\therefore q$ is negative. Therefore in the equation $x^3-px-q=0$, if there are three unequal real roots two of them are — and the other +.

If the axis of abscissas be removed from n upward, ns will gradually decrease to zero and nT increase to Ob ; therefore the ratio $s \div r$ has for one of its limits zero. If the axis be removed downward, ns will increase and nT will diminish until they both become dc ; therefore the other limit of the ratio $s \div r$ is unity. Let a denote this ratio. It may have any value between 0 and 1. $s = ar$, $r+s=t$; $\therefore t=r+ar$, $ar^2-r^2-ar^2-ar^2-a^2r^2=-p$.

$$\therefore r^2 = \frac{p}{1+a+a^2}, \quad ar^3+a^2r^3=q, \quad \therefore r^3 = \frac{q}{a+a^2}.$$

$$r^6 = \frac{p^3}{(1+a+a^2)^3} = \frac{q^2}{(a+a^2)^2}; \quad \therefore \frac{p^3}{q^2} = \frac{(1+a+a^2)^3}{(a+a^2)^2}.$$

We may now form a table taking the different values of a from .001 by intervals of one thousandth up to 1. As p and q are known, $p^3 \div q^2$ is known, and $\therefore (1+a+a^2)^3 \div (a+a^2)^2$ is known, and we can find the value of a just as from a given natural sine we find the corresponding arc. Then

$$r = \sqrt{\left(\frac{p}{1+a+a^2}\right)},$$

$$s = ar, \text{ and } t = r+ar. \quad \text{Or } r = \sqrt[3]{\left(\frac{q}{a+a^2}\right)}.$$

A better mode of procedure is to take the logarithms of both members. Thus $3 \log p - 2 \log q = 3 \log (1+a+a^2) - 2 \log (a+a^2)$.

Before illustrating by an example, it may be well to get the maximum and minimum values of $(1+a+a^2)^3 \div (a+a^2)^2$ or of

$$3 \log (1+a+a^2) - 2 \log (a+a^2).$$

We find $2a^3+3a^2-3a-2=0$, or $a^3+\frac{3}{2}a^2-\frac{3}{2}a-1=0$. Evidently one of the roots is 1. The others are $-\frac{1}{2}$ and -2 . Taking both the abscissas on the right of the axis ed , we interpret $a=1$ by $dc=dc$; $a=-\frac{1}{2}$ brings in the negative $ef=\frac{1}{2}ek$, and $a=-2$, $dl=2dc$. Confining ourselves to the case in which $a=1$, we reach the lowest point of the curve at c . Then $\frac{(1+a+a^2)^3}{(a+a^2)^2} = \frac{27}{4}$, which is the smallest admissible value also of $\frac{p^3}{q^2}$ when there are 3 real roots. This corresponds precisely with the results of Cardan's Rule.

If $a > 1$, which is contrary to the supposition, we merely exchange s for t and r for s . The substitution of $-\frac{1}{2}$ or -2 for a will give the same value of the fraction $\frac{2}{3}$.

It is plain then that our present method is applicable within the parallels kef and cdl , as Cardan's is without them, while they both give the same roots for the points f and c on those parallels.

Let us now take a common example, $x^3 - 7x + 7 = 0$. Here $-p = -7$, $+q = 7$, $\log(p^3 \div q^3) = \log 7 = .8450980$. Referring to the table we see that

$$\text{when } a = .801, \log[(1+a+a^2)^3 \div (a+a^2)^2] = .84526533.$$

$$\text{When } a = .802, \log \quad \quad \quad = .84508693.$$

therefore .802 is a pretty close approximation. But we can come nearer by a proportion:

$$\begin{array}{r} .84526533 \quad .84526533 \\ .84508693 \quad .84509800 \\ \hline 17840 : \quad 16733 :: .001 : .000937948. \\ \quad \quad \quad a = .801937948. \end{array}$$

To find $r = \sqrt{\left(\frac{p}{1+a+a^2}\right)}$; because $\log r = \frac{1}{2}(\log p - \log(1+a+a^2))$, it is necessary to square a ; but as a is too large when obtained by the process in full, we multiply as follows:

$$\begin{array}{r} .801937948 \\ .801937948 \\ \hline 640 \\ 3204 \\ 72171 \\ 561351 \\ 2405811 \\ 72174411 \\ 80193794 \\ \hline .64155035840 \\ \hline .64310447222 \\ \therefore 1+a+a^2 = 2.44504242. \end{array}$$

$$\begin{array}{r} \log p \quad \quad = .8450980 \\ \log(1+a+a^2) = .3882862 \\ \hline \end{array}$$

$$\therefore 2 \log r \quad = .4568118$$

$$\log r \quad \quad = .2284059; \therefore r = 1.692021484.$$

$$\begin{array}{r} \text{By Sturm and Horner} \quad \quad \quad r = 1.692021471, \\ \quad \quad \quad \text{error} \quad .000000013. \end{array}$$

This is more exact however than we can ordinarily be.

As a convenient test we may take $r^3 = q \div (a+a^2)$, $\therefore 3 \log r = \log q - \log(a+a^2)$; $\therefore r = 1.6920212$, not so close.

One marked advantage of the method is the ease of finding the 2nd root, s .

$$\text{Log } r = 0.2284059$$

$$" \quad a = \underline{1.9041407}$$

$$.1325466. \quad \therefore s = 1.356896250;$$

by Sturm and Horner

$$\underline{1.356895867}$$

$$\text{error } .000000383.$$

The 3rd root is negative and is equal to the numerical sum of the other two roots. The error is about .0000004. It should be observed that if a , as actually found, is too large, $r = p \div (1 + a + a^2)$ is too small; but then ar tends to right the second root, and not to increase the error; and vice versa.

For another example take $x^3 - 3x - 1 = 0$. $\text{Log } p^3 - \log q^2 = \log 27 = 1.4313638$, $\log q$ being 0. Two roots are negative.

$$a = .226 \text{ gives } 1.43345238;$$

$$a = .227 \text{ gives } 1.43039080.$$

By proportion we get $a = .22668219$;

$$1 + a + a^2 = 1.27806700.$$

$$\text{Log } p = .4771213,$$

$$\log(1 + a + a^2) = \underline{.1065536},$$

$$-1.532088886$$

$$.3705677 \div 2 = .18528385; \therefore t = -1.532088516$$

$$\text{error } .000000370.$$

As before, I compare with the value found by Sturm and Horner, which I have not verified but suppose to be correct. The log. of 1.532088886 is

$$.18528395,$$

$$\underline{.18528385},$$

$$\text{error } .00000010.$$

Value of $s = at$ is

$$-.347297000;$$

by Sturm and Horner

$$-.347296355,$$

$$\text{error } .000000645.$$

$$r = s + t$$

$$= +1.879385516;$$

by Sturm and Horner

$$+1.879385242,$$

$$\text{error } .000000274.$$

By comparing in this way with the results of Horner more particularly, we can learn how to improve this method practically. Since a as rigorously found is always too large, and I have been using Vega's 7-place logarithms, it will be observed that, in subtracting in the first example, .8450980 from .84526533 the last 3 is retained in the remainder 16733. The effect is to diminish a slightly. The abbreviation in squaring .801937948 makes a a little too small. Squaring in full gives

$$\log r = .228405763262.$$

Employed above

$$.2284059;$$

by Sturm and Horner

$$.228405896576.$$

TABLE.

α	$3 \log \alpha - 2 \log \beta$	α	$3 \log \alpha - 2 \log \beta$	α	$3 \log \alpha - 2 \log \beta$
.01	4.0044504	.34	1.1719586	.67	0.8807489
.02	3 4070508	.35	1 1553623	.68	0 8770654
.03	3 0597325	.36	1 1395263	.69	0 8735639
.04	2 8149164	.37	1 1244086	.70	0 8702383
.05	2 6263477	.38	1 1099713	.71	0 8670826
.06	2 4734207	.39	1 0961787	.72	0 8640913
.07	2 3451407	.40	1 0829978	.73	0 8612582
.08	2 2349420	.41	1 0703975	.74	0 8585783
.09	2 1385877	.42	1 0583499	.75	0 8560465
.10	2 0531836	.43	1 0468278	.76	0 8536580
.11	1 9766634	.44	1 0358067	.77	0 8514079
.12	1 9075000	.45	1 0252635	.78	0 8492921
.13	1 8445332	.46	1 0151758	.79	0 8473057
.14	1 7868588	.47	1 0055233	.80	0 8454444
.16	1 7337605	.48	0 9962871	.81	0 8437049
.16	1 6846586	.49	0 9874490	.82	0 8420827
.17	1 6390794	.50	0 9789914	.83	0 8405743
.18	1 5966287	.51	0 9708992	.84	0 8391764
.19	1 5569765	.52	0 9631565	.85	0 8378848
.20	1 5198427	.53	0 9557489	.86	0 8366967
.21	1 4849872	.54	0 9486628	.87	0 8356087
.22	1 4522036	.55	0 9418858	.88	0 8346182
.23	1 4213126	.56	0 9354055	.89	0 8337214
.24	1 3921466	.57	0 9292099	.90	0 8329157
.25	1 3645979	.58	0 9232884	.91	0 8321983
.26	1 3385138	.59	0 9176304	.92	0 8315668
.27	1 3137961	.60	0 9122259	.93	0 8310183
.28	1 2903471	.61	0 9070654	.94	0 8305504
.29	1 2680795	.62	0 9021398	.95	0 8301607
.30	1 2469152	.63	0 8974405	.96	0 8298465
.31	1 2267826	.64	0 8929597	.97	0 8296059
.32	1 2076173	.65	0 8886889	.98	0 8294366
.33	1 1893606	.66	0 8846212	.99	0 8293367
				1.00	0 8293038

This table has been carefully prepared by the aid of the 50th edition of the Bremiker—Vega tables. If any mistake should be found, please address me at Richmond Kentucky.

[In the heading of the above table, $\alpha = (1 + a + a^2)$ and $\beta = (a + a^2)$.]

PROOF OF THE THEOREM THAT EVERY EQUATION
HAS A ROOT.

BY JOHN MACNIE, A. M., NEW YORK.

Let $f(x)$ be a rational integral algebraic function of the n th degree, where n is an even number; if $f(x)$ does not vanish for any real value of x , it will for some value of x of the form $A + B\sqrt[n]{-1}$, where B is not zero.

In $f(x)$ putting $y + z\sqrt[n]{-1}$ for x , we have:

$$f(y + z\sqrt[n]{-1}) = f(y) + f_1(y)z\sqrt[n]{-1} + \frac{1}{2}f_2(y)z^2(\sqrt[n]{-1})^2 + \dots + \frac{1}{(n-1)!}f_{n-1}(y)z^{n-1}(\sqrt[n]{-1})^{n-1} - z^n. \quad [1]$$

Now $\sqrt[n]{-1}$, as is proved independently of this proposition, must be of the form $A + B\sqrt[n]{-1}$, equal, say, $a_1 + b_1\sqrt[n]{-1}$; its powers must also be of the same form, say, $a_2 + b_2\sqrt[n]{-1}$, $a_3 + b_3\sqrt[n]{-1}$, $\dots$, $a_{n-1} + b_{n-1}\sqrt[n]{-1}$, where $a_1, a_2, \dots, b_1, b_2, \dots$ are positive or negative, but not both zero in the same power of $\sqrt[n]{-1}$. Substituting these values for $\sqrt[n]{-1}$ and its powers in [1], we have:

$$\begin{aligned} f(y) + z\sqrt[n]{-1} &= f(y) + f_1(y)(a_1 + b_1\sqrt[n]{-1})z + \frac{1}{2}f_2(y)(a_2 + b_2\sqrt[n]{-1})z^2 + \\ &\dots + \frac{1}{(n-1)!}f_{n-1}(y)(a_{n-1} + b_{n-1}\sqrt[n]{-1})z^{n-1} - z^n \\ &= f(y) + \frac{a_1}{1}f_1(y)z + \frac{a_2}{2}f_2(y)z^2 + \dots + \frac{a_{n-1}}{(n-1)!}f_{n-1}(y)z^{n-1} - z^n \\ &+ \left[\frac{b_1}{1}f_1(y) + \frac{b_2}{2}f_2(y)z + \frac{b_3}{3!}f_3(y)z^2 + \dots + \frac{b_{n-1}}{(n-1)!}f_{n-1}(y)z^{n-2} \right] z\sqrt[n]{-1} \\ &= P + Qz\sqrt[n]{-1}; \text{ or} \end{aligned}$$

$$-P = z^n - \frac{a_{n-1}}{(n-1)!}f_{n-1}(y)z^{n-1} - \dots - \frac{a_2}{2}f_2(y)z^2 - \frac{a_1}{1}f_1(y)z - f(y). \quad [2]$$

$$Q = \frac{b_{n-1}}{(n-1)!}f_{n-1}(y)z^{n-2} + \frac{b_{n-2}}{(n-2)!}f_{n-2}(y)z^{n-3} + \dots + \frac{b_2}{2}f_2(y)z + \frac{b_1}{1}f_1(y). \quad [3]$$

What we have to prove is that there must be some values of y and z for which [2] and [3] will vanish simultaneously.

— P , being a function of even degree with its final term negative,—since, by hypothesis, $f(y)$ cannot be negative for any real value of y ,—must vanish for some real positive value of z , whatever real value, positive or negative, is given to y : but these values cannot include either infinity or zero, since neither the first nor the last coefficient can be zero. (See Note 1.) These positive values of z for which — P can be made to vanish, vary continuously with y . For suppose — $P = 0$ when $y = c, z = t$; then

$$-P(c, t) = t^n - \frac{a_{n-1}}{(n-1)!}f_{n-1}(c)t^{n-1} - \dots - \frac{a_1}{1}f_1(c)t - f(c) = 0.$$

When y becomes $c+h$ we have

$$-P(c+h, t) = -P(c, t) - h \left[\frac{a_{n-1}}{(n-1)!} (f_{n-2}(c) + \&c.) t^{n-1} \dots \right. \\ \left. + \frac{a_1}{1} (f_2(c) + \&c.) t + (f_1(c) + \&c.) \right]$$

As the difference between $-P(c, t)$ and $-P(c+h, t)$ may be made as small as we please by taking h small enough, $-P(c+h, z)$ will vanish for a value of z differing from t by an indefinitely small quantity, if h is indefinitely small. For, regarding $c+h$ as constant and z as variable, we know that, corresponding to an indefinitely small change in the value of z there will be an indefinitely small change in the value of the function $-P(c+h, z)$.

Q , again, is a function of even degree in z with its final term negative for every value of y that makes $f_1(y)$ negative, if b_1 is positive; or for every value of y that makes $f_1(y)$ positive, if b_1 is negative. But $f_1(y)$, being a function of odd degree in y , can become zero for at least one real value of y , and positive or negative for an indefinite number of such values. When we assign to y that value for which $f_1(y)$ vanishes, then Q will vanish for $z=0$; and when we assign to y that value for which $f_{n-1}(y)$ —a function of the first degree—vanishes, then Q will vanish for $z=\infty$, (See NOTE 1.) that is, by assigning appropriate values to y , Q may be made to vanish for positive values of z indefinitely small or great. That the values of z between these limits for which Q may be made to vanish vary continuously with y may be shown as in regard to $-P$.

Now when we assign to y in both $-P$ and Q that value (c_1) that makes $f_1(y)$ vanish, the value of z (t_1) for which $-P$ will vanish is greater than that (i. e. zero) for which Q will vanish; while if we assign to y , in both $-P$ and Q , that value (c_2) for which $f_{n-1}(y)$ vanishes, the value of z (t_2) for which $-P$ vanishes will be less than that (i. e. infinity) for which Q vanishes. If, therefore, we conceive y , in both $-P$ and Q , to pass through all values from c_1 to c_2 , y must reach some intermediate value (c_3) at which the value of z (t_3), causing $-P$ to vanish, ceases to be greater than that which causes Q to vanish; that is, $\mp P = 0$ and $Qz \vee -1 = 0$ simultaneously for $y = c_3$, $z = t_3$, or $f(x) = 0$ when $x = c_3 + t_3(a_1 \pm b_1 \vee -1) = (c_3 + a_1 t_3) \pm b_1 t_3 \vee -1$, a quantity of the form $A + B \vee -1$, in which A and B are real finite quantities since a_1, b_1, c_3, t_3 , are real and finite.

NOTE 1. The theorem, that the equation having for coefficients those of a proposed equation taken in reverse order has for roots the reciprocals of the roots, if any, of the proposed equation, is independent of the property of equations here discussed. From this, in connection with the obvious principle that for $f(z)$ to vanish for $z = 0$ it is necessary and sufficient that the

absolute term be zero, follows the principle here employed, that for $f(z)$ to vanish for $z = \infty$ it is necessary and sufficient that the coefficient of its highest power be evanescent.

NOTE 2. We have found that a pair of roots of $f(x) = 0$ may be expressed by $y_p + z_p(a_p \pm \beta_p \sqrt{-1})$, where we shall suppose $a_p \pm \beta_p \sqrt{-1}$ to denote the values of $\sqrt[n]{-1}$ given by the formula

$$\cos \frac{(2\lambda+1)\pi}{n} \pm \sin \frac{(2\lambda+1)\pi}{n} \sqrt{-1},$$

where $\lambda = p$, p being any integer from 0 to n inclusive. A pair is also expressed by $y_q + z_q(a_q \pm \beta_q \sqrt{-1})$, where $a_q \pm \beta_q \sqrt{-1}$ denotes the values of $\sqrt[n]{-1}$ given by the above trigonometrical formula when $\lambda = q$, q being any integer, except p , from 0 to n . The question arises: does $y_q + z_q(a_q \pm \beta_q \sqrt{-1})$ necessarily represent a different pair of roots from that represented by $y_p + z_p(a_p \pm \beta_p \sqrt{-1})$, by different here meaning not equal. This conclusion seems necessary; since, otherwise, there would be more than one expression for some pairs of roots, and none at all for others, there being but $\frac{1}{2}n$ pairs of values of $\sqrt[n]{-1}$. If we can assume this result we are led to an interesting generalization respecting the roots of all equations of, at least, even degree, which could then be expressed by the formula

$$x = y + z \left[\cos \frac{(2\lambda+1)\pi}{n} \pm \sin \frac{(2\lambda+1)\pi}{n} \sqrt{-1} \right].$$

In this when $y = 0$, $z = \sqrt[n]{A}$, we have the well known formula for the roots of $x^n + A = 0$, when n is even; for $z = 0$, the formula represents real roots; if $z_p \beta_p = z_q \beta_q$ and $y_p + z_p a_p = y_q + z_q a_q$ there are two pairs of equal imaginary roots, or one pair of equal real roots if $z = 0$. The subject seems worthy of further investigation.

A COLLECTION OF PROOFS OF THE RELATION

$$r' + r'' + r''' - r = 4R.$$

BY MARCUS BAKER, U. S. COAST SURVEY, WASHINGTON, D. C.

IF R is the radius of the circle circumscribed about a plane triangle, r the radius of the inscribed circle, and if r' , r'' , r''' be the radii of the escribed circles, then is the above relation true. The earliest appearance of this relation in print that I have seen is in J. H. Van Swinden's *Elemente der geometrie* von C. F. A. Jacobi: Jena, 1834; though the relation was doubtless found earlier than this. The collection here made is not the result of a special effort to get all the proofs known of it but rather to put together such as have been met with incidentally in mathematical reading.

The relation in the sixth proof

$$R+r = x+y+z$$

is older than the relation discussed in this article and was first given by Carnot in his Geometry of position, the german translation of which by Schumacher appeared in 1808. This incidentally answers in part the query by the editor at p. 122, Vol. IV, of the ANALYST. For, though Carnot does not prove that

$$2(R+r) = a \cot A + b \cot B + c \cot C$$

he does prove, and by the same method as the editor of the ANALYST, page 123, Vol. IV, that

$$R+r = x+y+z.$$

FIRST PROOF.— $K = sr = abc \div 4R = 2R^2 \sin A \sin B \sin C$,

$$\frac{r}{R} = \frac{2R \sin A \sin B \sin C}{\frac{1}{2}(a+b+c)};$$

and since $a = 2R \sin A$, $b = 2R \sin B$, $c = 2R \sin C$

$$\begin{aligned} \frac{r}{R} &= 2 \frac{\sin A \sin B \sin C}{\sin A + \sin B + \sin C} = \frac{\sin A \sin B \sin C}{2 \cos \frac{1}{2}A \cos \frac{1}{2}B \cos \frac{1}{2}C} \\ &= 4 \sin \frac{1}{2}A \sin \frac{1}{2}B \sin \frac{1}{2}C; \end{aligned} \quad (1)$$

$$\begin{aligned} \frac{r'}{R} &= 2 \frac{\sin A \sin B \sin C}{-\sin A + \sin B + \sin C} = \frac{\sin A \sin B \sin C}{2 \cos \frac{1}{2}A \sin \frac{1}{2}B \sin \frac{1}{2}C} \\ &= 4 \sin \frac{1}{2}A \cos \frac{1}{2}B \cos \frac{1}{2}C; \end{aligned} \quad (2)$$

$$\begin{aligned} \frac{r''}{R} &= 2 \frac{\sin A \sin B \sin C}{\sin A - \sin B + \sin C} = \frac{\sin A \sin B \sin C}{2 \sin \frac{1}{2}A \cos \frac{1}{2}B \sin \frac{1}{2}C} \\ &= 4 \cos \frac{1}{2}A \sin \frac{1}{2}B \cos \frac{1}{2}C; \end{aligned} \quad (3)$$

$$\begin{aligned} \frac{r'''}{R} &= 2 \frac{\sin A \sin B \sin C}{\sin A + \sin B - \sin C} = \frac{\sin A \sin B \sin C}{2 \sin \frac{1}{2}A \sin \frac{1}{2}B \cos \frac{1}{2}C} \\ &= 4 \cos \frac{1}{2}A \cos \frac{1}{2}B \sin \frac{1}{2}C. \end{aligned} \quad (4)$$

But $\sin \frac{1}{2}A \cos \frac{1}{2}B \cos \frac{1}{2}C + \cos \frac{1}{2}A \sin \frac{1}{2}B \cos \frac{1}{2}C + \cos \frac{1}{2}A \cos \frac{1}{2}B \sin \frac{1}{2}C - \sin \frac{1}{2}A \sin \frac{1}{2}B \sin \frac{1}{2}C = 1$; (5)

therefore subtracting (1) from the sum of (2), (3) and (4) and clearing of fractions, we have

$$r' + r'' + r''' - r = 4R.$$

SECOND PROOF.—Because $r' = s \tan \frac{1}{2}A$, $r'' = s \tan \frac{1}{2}B$, $r''' = s \tan \frac{1}{2}C$ and $r = s \tan \frac{1}{2}A \tan \frac{1}{2}B \tan \frac{1}{2}C$;

$$\begin{aligned} \therefore r' + r'' + r''' - r &= s(\tan \frac{1}{2}A + \tan \frac{1}{2}B + \tan \frac{1}{2}C - \tan \frac{1}{2}A \tan \frac{1}{2}B \tan \frac{1}{2}C), \\ &= \frac{s}{\cos \frac{1}{2}A \cos \frac{1}{2}B \cos \frac{1}{2}C}, \quad [\text{Chauv's Trig. p. 39 (177).}] \\ &= 4R. \quad [\text{Chauv's Trig. p. 76 (286).}] \end{aligned}$$

THIRD PROOF.—Let O' , O'' , O''' represent the centres of the escribed circles, then is $O'' O''' = 2R' \sin \frac{1}{2}(B+C) = 4R \cos \frac{1}{2}A$;

$$O'' O''' = (r'' + r''') \sec \frac{1}{2}A; \quad (6)$$

therefore $r'' + r''' = 4R \cos^2 \frac{1}{2}A$; (6)

similarly $r''' + r' = 4R \cos^2 \frac{1}{2}B$; (7)

$$r' + r'' = 4R \cos^2 \frac{1}{2}C. \quad (8)$$

Adding $r' + r'' + r''' = 2R(\cos^2 \frac{1}{2}A + \cos^2 \frac{1}{2}B + \cos^2 \frac{1}{2}C),$
 $= R(3 + \cos A + \cos B + \cos C),$
 $= 4R(1 + \sin \frac{1}{2}A \sin \frac{1}{2}B \sin \frac{1}{2}C).$

But $r = 4R \sin \frac{1}{2}A \sin \frac{1}{2}B \sin \frac{1}{2}C$, by (1), and therefore

$$r' + r'' + r''' - r = 4R.$$

FOURTH PROOF.—The circle which circumscribes the triangle ABC bisects OO' and therefore

$$\frac{1}{2}(r' - r) = R - R \cos A, \text{ or } r' - r = 4R \sin^2 \frac{1}{2}A; \quad (9)$$

similarly $r'' - r = 4R \sin^2 \frac{1}{2}B, \quad (10)$

$$r''' - r = 4R \sin^2 \frac{1}{2}C. \quad (11)$$

Adding $r' + r'' + r''' - 3r = 4R(\sin^2 \frac{1}{2}A + \sin^2 \frac{1}{2}B + \sin^2 \frac{1}{2}C),$
 $= 2R(3 - \cos A - \cos B - \cos C),$
 $= 2R(2 - 4\sin \frac{1}{2}A \sin \frac{1}{2}B \sin \frac{1}{2}C)$
 $= 2R[2 - (r \div R)],$
 $= 4R - 2r;$

$$r' + r'' + r''' - r = 4R.$$

FIFTH PROOF.—From the *third* and *fourth* proofs we have

$$r'' + r''' = 4R \cos^2 \frac{1}{2}A, \quad (6)$$

$$r' - r = 4R \sin^2 \frac{1}{2}A. \quad (9)$$

Adding $r' + r'' + r''' - r = 4R.$

SIXTH PROOF.—By Ptolemy's Theorem

$$cy + bz = aR, \quad (12)$$

$$az + cx = bR, \quad (13)$$

$$bx + ay = cR, \quad (14)$$

and $ax + by + cz = (a+b+c)r. \quad (15)$

Adding and rejecting the factor $a+b+c$,

$$r + R = x + y + z. \quad (16)$$

Again $ax + by + cz = (-a+b+c)r' \quad (17)$

$$= (a-b+c)r'' \quad (18)$$

$$= (a+b-c)r''' \quad (19)$$



Replacing (15) with (17), (18) and (19) respectively and subtracting from the sum of (12), (13) and (14) we obtain

$$r' - R = -x + y + z \quad (20), \quad r'' - R = x - y + z \quad (21), \quad r''' - R = x + y - z, \quad (22).$$

Subt. (16) from the sum of (20), (21) and (22), $r' + r'' + r''' - r = 4R.$

SEVENTH PROOF.—

$$\begin{aligned}
 R &= \frac{\frac{1}{2}a}{\sin A} = \frac{\frac{1}{2}b}{\sin B} = \frac{\frac{1}{2}c}{\sin C}, \\
 &= \frac{\frac{1}{2}a \cot A}{\cos A} = \frac{\frac{1}{2}b \cot B}{\cos B} = \frac{\frac{1}{2}c \cot C}{\cos C}, \\
 &= \frac{\frac{1}{2}(a \cot A + b \cot B + c \cot C)}{\cos A + \cos B + \cos C}, \\
 &= \frac{\frac{1}{2}(a \cot A + b \cot B + c \cot C)}{1 + 4\sin\frac{1}{2}A \sin\frac{1}{2}B \sin\frac{1}{2}C}, \\
 &= \frac{x+y+z}{1+(r \div R)}.
 \end{aligned}$$

$$\therefore r+R = x+y+z.$$

Similarly we obtain the rest and

thence $r' + r'' + r''' - r = 4R.$

EIGHTH PROOF.—[See Chauvenet's Trig., 7th ed., p.78 (298).]

$$x = R \cos A, y = R \cos B, z = R \cos C;$$

hence $r+R = x + y + z = R(\cos A + \cos B + \cos C),$

or $\frac{r}{R} = \cos A + \cos B + \cos C - 1. \quad (23)$

Similarly $\frac{r'}{R} = -\cos A + \cos B + \cos C + 1, \quad (24)$

$$\frac{r''}{R} = \cos A - \cos B + \cos C + 1, \quad (25)$$

$$\frac{r'''}{R} = \cos A + \cos B - \cos C + 1. \quad (26)$$

Subtracting (23) from the sum of (24), (25) and (26) we obtain

$$r' + r'' + r''' - r = 4R.$$

NINTH PROOF.— $K = sr = (s-a)r' = (s-b)r'' = (s-c)r'''$;

$$\begin{aligned}
 \therefore r' + r'' + r''' - r &= K \left(\frac{1}{s-a} + \frac{1}{s-b} + \frac{1}{s-c} - \frac{1}{s} \right) \\
 &= K \left(\frac{abc}{s(s-a)(s-b)(s-c)} \right), \\
 &= \frac{abc}{K} = 4R.
 \end{aligned}$$

TENTH PROOF.— $(s-a)r' + (s-b)r'' + (s-c)r''' = 3sr. \quad (27)$

or $ar' + br'' + cr''' + 2K = 2K', \quad (28)$

Adding (27) and (28)

$$s(r' + r'' + r''') = 4Rs + rs,$$

or $r' + r'' + r''' - r = 4R.$

ELEVENTH PROOF. — The circle which circumscribes the triangle ABC is the nine point circle of the triangle $O'O''O'''$ and consequently bisects $O''O'''$ and also OO' . (O denoting the intersection of the lines AO' , BO'' , CO''' .) Because M is the middle point of $O''O'''$

$$MP = \frac{1}{2}(r'' + r'''),$$

and because N is the middle point of OO'

$$NP = \frac{1}{2}(r' - r).$$

$$\begin{aligned} \text{Adding} \quad MP + NP = 2R &= \frac{1}{2}(r' + r'' + r''' - r), \\ \therefore 4R &= r' + r'' + r''' - r. \end{aligned}$$

A PROBLEM WITH SOLUTION.

BY A. S. HATHAWAY, CORNELL UNIV., ITHACA, N. Y.

LET $u', u'', \dots, v', v'', \dots$ be such functions of x that each vanishes when $x = a$; and also the derivatives of each, up to (but not including) those of $s', s'', \dots, t', t'', \dots$ order, respectively. It is then required to find the limit as x approaches a , of $\frac{u'u'' \dots}{v'v'' \dots}$.

We proceed to find the order and value of the first non-vanishing derivative of each product.

Since differentiation always produces expressions which are homogeneous with respect to the introduced $\frac{d}{dx}$, the $s' + s'' + \dots$ derivative of $u'u'' \dots$ contains the term $A \left(\frac{d}{dx}\right)^{s'} u' \left(\frac{d}{dx}\right)^{s''} u'' \dots$ which does not vanish; and besides, other terms of equal order, cannot be without a vanishing derivative, and are all zero, unless some other derivative of not greater than order $s' + s'' + \dots$ is infinite. This indeterminate case we will exclude; and then all terms of less order also vanish.

We find $A = \frac{(s' + s'' + \dots)!}{s'! s''! \dots}$, for in repeated differentiations of a product, the combinations of partial derivatives are the same as in the case of powers of a multinomial algebraic expression; as is evident on considering that

$$\frac{d}{dx} u'u'' \dots = \left(\frac{d}{dx} + \frac{d}{dx} + \dots\right) u'u''. \quad (\text{See NOTE.})$$

Similar results hold for $v'v'' \dots$, with the restriction that no derivative of v' or $v'' \dots$, of not greater order than $t' + t'' + \dots$ is infinite; and the coefficient of the term which does not vanish $= \frac{(t' + t'' + \dots)!}{t'! t''! \dots}$.

Therefore, using D for $\frac{d}{dx}$, and a' b for a or b ,

$$\lim \frac{u'u'' \dots}{v'v'' \dots} = 0' \lim \left(\frac{D'u' D''u'' \dots}{s'! s''! \dots} \div \frac{D'v' D''v'' \dots}{t'! t''! \dots} \right), \infty,$$

according as $s' + s'' + \dots >, =$ or $< t' + t'' + \dots$ (1)

A special case of this problem is to find, dropping the accents, limit $\frac{u^m}{v^n}$, where m and n are any quantities.

We have the identity $\lim \frac{u^m}{v^n} = \lim \left(\frac{u^t}{v^s} \right)^{\frac{n}{s}} \lim u^{\frac{ms - nt}{s}}$, and this is, if no derivative of u or v of not greater order than st is infinite,

$$0' \lim \left[\left(\frac{D^s u}{s!} \right)^m \div \left(\frac{D^t v}{t!} \right)^n \right], \infty, \text{ according as } ms >, = \text{ or } < nt. \quad (2)$$

NOTE.—We may find A inductively, using Leibnitz's Theorem. Thus, when $a = a$,

$$\begin{aligned} D^{s'} u' &= \frac{s'!}{s'!} D^{s'} u' \\ D^{s'+s''} u' u'' &= \frac{(s'+s'') \dots (s'+1)}{s'!} D^{s'} u' D^{s''} u'' = \frac{(s'+s'')!}{s'! s''!} D^{s'} u' D^{s''} u'' \\ D^{s'+s''+s'''} u' u'' u''' &= \frac{(s'+s''+s''') \dots (s'+s''+1)}{s''!} D^{s'+s''} u' u'' D^{s'''} u''' \\ &= \frac{(s'+s''+s''')!}{s'! s''! s'''!} D^{s'} u' D^{s''} u'' D^{s'''} u''' \end{aligned}$$

EXAMPLES.

$$\begin{aligned} \lim \frac{\sin^3(a-1)[2\cos \frac{1}{2}\pi a + \pi(a-1)][2\cos(a-1) - \cos^2(a-1) + \frac{1}{2}(a-1)^4]}{[(\log a)^2 - (a-1)^2 \div a][2\cos(1-a) + a^2 - 2a - 1][8\varepsilon^{a-1} - \varepsilon^{2a-2} - 2a(a+1) - 3]} \\ = (a=1) \frac{2}{3}\pi^3. \\ \frac{[a-1-\sin(a-1)]^{\frac{1}{2}}}{[1-\cos(a-1)]^{\frac{1}{4}}} = (a=1) [\frac{1}{2}\sqrt{2}]^{\frac{1}{2}}; \text{ Ex. 55, chap. 10, Tod. Dif. Cal.} \end{aligned}$$

DEMONSTRATION OF THE PROP. AT PAGE 8.

BY PROF. D. J. MC. ADAM, WASHINGTON, PA.

Let $B C D A$ be a semicircle, diameter AB , $ABCD$ being any inscribed trapezium, to prove that the sides AD , DC and CB may represent the reciprocals of the lines a , b , c if AB represent the reciprocal of r .

Draw AC . Then is

$$\begin{aligned} AB^2 &= BC^2 + AC^2 = BC^2 + CD^2 + AD^2 - 2AD \times CD \cos D \\ &= BC^2 + DC^2 + AD^2 + 2AD \times DC \sin CAB \\ &= BC^2 + DC^2 + AD^2 + 2 \frac{AD \times DC \times BC}{AB}. \end{aligned} \quad (1)$$

In a triangle with the given notation,

$$\begin{aligned} \frac{1}{a^2} &= \frac{\sin^2 \frac{1}{2}A}{r^2}; \quad \frac{1}{b^2} = \frac{\sin^2 \frac{1}{2}B}{r^2}; \quad \frac{1}{c^2} = \frac{\sin^2 \frac{1}{2}C}{r^2}; \\ \therefore \quad \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} &= \frac{1}{r^2} \left(\sin^2 \frac{1}{2}A + \sin^2 \frac{1}{2}B + \sin^2 \frac{1}{2}C \right) \\ &= \frac{1}{r^2} \left(1 - 2 \sin \frac{1}{2}A \sin \frac{1}{2}B \sin \frac{1}{2}C \right) \\ &= \frac{1}{r^2} \left(1 - \frac{2r^3}{abc} \right). \end{aligned} \quad (2)$$

Substituting values of $\sin \frac{1}{2}A$ &c.,

$$\frac{1}{r^2} = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} + \frac{2r}{abc}. \quad (3)$$

Comparing (1) with (3) we find $1 \div r^2$ corresponds to AB^2 , $1 \div a^2$ to BC^2 &c., hence the values of these reciprocals might be made the sides of a trapezium upon AB .

[A demonstration of the above prop. was also sent by Mr. E. B. Seitz.]

SOLUTIONS OF PROBLEMS IN NUMBER TWO.

SOLUTIONS of problems in No. 2 have been received as follows:

From Amateur, 199; R. J. Adecock, 200, 201; Marcus Baker, 196, 197; Prof. W. P. Casey, 196, 197, 200; Prof. P. E. Chase, 197, 200; Geo. M. Day, 196, 197; E. L. De Forest, 201; Capt. J. L. de Fremery, 196, 197; W. E. Heal, 196; H. Heaton, 196, 197, 198, 200; Prof. E. W. Hyde, 201; Chas. H. Kummell, 198, 201; Prof. J. H. Kershner, 196, 197; W. V. Mc. Knight, 196, 197; Prof. D. J. Mc. Adam, 196, 197, 200; Prof. H. T. J. Ludwick, 198; Artemas Martin, 196, 197, 198; K. S. Putnam, 196, 197; P. Richardson, 197; Prof. J. Scheffer, 196, 197, 199, 200; Prof. T. A. Smith, 196, 197; E. B. Seitz, 196, 197, 198, 200; Anna T. Snyder, 197; T. P. Stowell, 197; C. A. Van Velzer, 196, 197, 200.

196. "The sides of a triangle are respectively $x^2 + x + 1$, $2x + 1$ and $x^2 - 1$, x being any number greater than one; prove that the angle opposite the side $x^2 + x + 1$ is equal to 120° ."

Let y denote the angle opposite the side x^2+x+1 , then we have

$$(x^2+x+1)^2 = (2x+1)^2 + (x^2-1)^2 - 2(2x+1)(x^2-1)\cos y.$$

Expanding and reducing, we get

$$\cos y = -\frac{1}{2}; \therefore y = 120^\circ.$$

197. " AB is the diameter of a circle whose center is C , D is the middle point of CB , and is the center of a circle whose diameter is CB , and E is the middle p't of AC . EF is perp. to AB and intersects the circle C in F , and EG is tangent to the circle D at G . Prove that EFG is an equilateral triangle."

DEMONSTRATION BY ANNA T. SNYDER, ORANGE, INDIANA.

With center C and radius CD describe the semi-circle ED . Then, because the angle DGE is a right angle, G is a point in the semi-circle. Draw CG and CF . Then is $CG = CD = DG =$ radius of the circle $D = r$, say; and $CF = CB = ED = 2r =$ radius of the circle C .

Now because $CD = CG = DG = r$, the triangle CDG is equilateral; $\therefore \angle CDG = 60^\circ$, and the complementary angle $DEG = 30^\circ$; therefore $\angle GEF = 60^\circ$.

Because $ED = CF = 2r$, $CE = DG = r$, and the angles at E and G are right angles, $\therefore EG = EF$, and the triangle GEF is isosceles; but it has been shown that the vertical angle $GEF = 60^\circ$, hence the triangle is equiangular, and therefore equilateral.



[In the announcement of this proposition we should have stated that the required demonstration must be independent of trigonometrical functions.]

198. "Find the average distance between two points taken at random in the surface of a rectangle, sides a and b ."

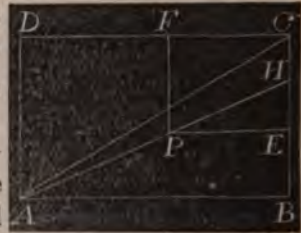
SOLUTION BY E. B. SEITZ, GREENVILLE OHIO.

From A draw AP to any point P in the triangle ABC .

Now if AP is the distance and the direction of the second point from the first, the area of the rectangle $PECF$ represents the number of ways the two points can be taken.

Let $AB = a$, $BC = b$, $AC = c$, $AP = x$, $AH = x'$, $\angle PAB = \theta$, $\angle CAB = \theta'$, area $PECF = u$, Δ = the requ'd average, Δ' , Δ'' = the average distance between the two points when P is in the triangles ABC , ACD respectively.

Then $x' = a \sec \theta$, $\theta' = \tan^{-1}(b \div a)$, $u = (a - x \cos \theta)(b - x \sin \theta)$; an element of the rectangle at P is $x d\theta dx$; for Δ' the limits of θ are 0 and θ' ; and of x , 0 and x' .



$$\therefore \Delta' = \int_0^{\theta'} \int_0^{x'} u x^2 d\theta dx \div \int_0^{\theta'} \int_0^{x'} u x d\theta dx = \frac{8}{a^2 b^2} \int_0^{\theta'} \int_0^{x'} u x^2 d\theta dx$$

$$= \frac{2a^2}{15b^2} \int_0^{\theta'} (5b - 3a \tan \theta) \sec^3 \theta d\theta = \frac{c}{3} + \frac{2a^3}{15b^2} - \frac{2c^3}{15b^2} + \frac{a^2}{3b} \log \left(\frac{b+c}{a} \right).$$

By symmetry we have

$$\Delta'' = \frac{c}{3} + \frac{2b^3}{15a^2} - \frac{2c^3}{15a^2} + \frac{b^2}{3a} \log \left(\frac{a+c}{b} \right).$$

$$\therefore \Delta = \frac{1}{2}(\Delta' + \Delta'') = \frac{1}{15} \left[\frac{a^3}{b^2} + \frac{b^3}{a^2} + c \left(3 - \frac{a^2}{b^2} - \frac{b^2}{a^2} \right) + \frac{5a^2}{2b} \log \left(\frac{b+c}{a} \right) + \frac{5b^2}{2a} \log \left(\frac{a+c}{b} \right) \right].$$

199. "Find an infinite series the sum of which is a square, the sum of n terms of which is a square, and the sum of the remaining terms, after the n terms are taken away, is a square."

SOLUTION BY THE PROPOSER.

Let S_n^2 = sum of n terms, and S_{x-n}^2 = sum of remaining terms. Then

$$S_x^2 = S_n^2 + S_{x-n}^2, \quad (1)$$

and the series is

$$S_1^2 + (S_2^2 - S_1^2) + (S_3^2 - S_2^2) + (S_4^2 - S_3^2) + \dots = S_x^2, \quad (2)$$

of which the general term is $\varphi(n) = S_n^2 - S_{n-1}^2$: if its first term follows the same law as the others, it would be

$$\varphi(1) = S_1^2 = S_1^2 - S_0^2; \therefore S_0^2 = 0. \quad (3)$$

In any series

$$1^2 = \left(\frac{n^2 - r^2}{n^2 + r^2} \right)^2 + \left(\frac{2rn}{n^2 + r^2} \right)^2; \quad (4)$$

$\therefore$ if $S_\infty = 1$, $S_n = 1$ when n is infinite. In (4) when $n = \infty$,

$$\frac{n^2 - r^2}{n^2 + r^2} = 1, \text{ and } \frac{2rn}{n^2 + r^2} = 0; \therefore S_n \text{ must be taken } = \frac{n^2 - r^2}{n^2 + r^2}.$$

But (3) is not satisfied by this value of S because it vanishes when $n = r$ and not when $n = 0$.

In (4) substitute $n+r$ for n , and it becomes

$$1^2 = \left(\frac{n^2 + 2rn}{(n+r)^2 + r^2} \right)^2 + \left(\frac{2r(n+r)}{(n+r)^2 + r^2} \right)^2,$$

in which the first term of the second member vanishes when $n = 0$;

$$\therefore S_n = \frac{n^2 + 2rn}{(n+r)^2 + r^2}, \therefore S_{x-n} = \frac{2r(n+r)}{(n+r)^2 + r^2}.$$

$$\varphi(n) = S_n^2 - S_{n-1}^2 = \left(\frac{n^2 + 2rn}{(n+r)^2 + r^2} \right)^2 - \left(\frac{(n-1)^2 + 2r(n-1)}{(n-1+r)^2 + r^2} \right)^2.$$

Substitute this in (2) and the required series is found.

$$\text{If } r = 1, \varphi(n) = \frac{4(2n+1)(n^2+n-1)(n^2+n+1)}{(n^2+1)^2[(n+1)^2+1]^2}.$$

Rejecting the factor 4 because it is a square, the series is

$$\frac{3.1.3}{2^2.5^2} + \frac{5.5.7}{5^2.10^2} + \frac{7.11.13}{10^2.17^2} + \frac{9.19.21}{17^2.26^2} + \dots + \frac{(2n+1)(n^2+n-1)(n^2+n+1)}{(n^2+1)^2[(n+1)^2+1]^2} +$$

The sum of this series is $\frac{1}{4}$, a square. The sum of n terms is

$$\frac{1}{4} \left(\frac{n^2 + 2n}{(n+1)^2 + 1} \right)^2,$$

a square; and the sum of the remaining terms is

$$\left(\frac{n+1}{(n+1)^2 + 1} \right)^2, \text{ a square.}$$

SOLUTION BY PROF. J. SCHEFFER, COLLEGE OF ST. JAMES, MD.

Putting $a \div (1-q) = x^2$, $a(1-q^n) \div (1-q) = y^2$, $x^2 - y^2 = z^2$, which equations express the three conditions required, we obtain from the third, putting $y = x-p$,

$$x = \frac{p^2 + z^2}{2p}, \quad y = \frac{z^2 - p^2}{2p}.$$

Substituting in the second, we get $1-q^n = z^2 \div x^2$, whence

$$q = \sqrt[n]{\left(\frac{z}{x}\right)^2} = \left(\frac{2pz}{p^2 + z^2}\right)^{\frac{2}{n}},$$

and from the first

$$a = \left(\frac{z^2 + p^2}{2p}\right)^2 \left[1 - \left(\frac{2pz}{p^2 + z^2}\right)^{\frac{2}{n}}\right],$$

in which expression z and p can be arbitrarily chosen.

200. "Prove that the locus of points, which have the sum of the squares of their distances from fixed points constant, is the surface of a sphere."

SOLUTION BY H. HEATON, SABULA, IOWA.

Taking the center of gravity of the fixed points as the origin, let $a_1, b_1, c_1; a_2, b_2, c_2, \dots, a_n, b_n, c_n$ be their co-ordinates, and x, y, z , the co-ordinates

of the points whose locus is required. Then we have

$$(x-a_1)^2+(y-b_1)^2+(z-c_1)^2+\dots(x-a_n)^2+(y-b_n)^2+(z-c_n)^2=C. \quad (1)$$

$$\therefore nx^2+ny^2+nz^2-2(a_1+a_2\dots a_n)x-2(b_1+b_2\dots b_n)y-2(c_1+c_2\dots c_n)z \\ +a_1^2+b_1^2+c_1^2\dots+a_n^2+b_n^2+c_n^2=C. \quad (2)$$

But since the center of gravity is the origin, $a_1+a_2\dots a_n$, $b_1+b_2\dots b_n$ and $c_1+c_2\dots c_n$, each equals nothing, $\therefore$

$$x^2+y^2+z^2=\frac{1}{n}\left(C-a_1^2-b_1^2-c_1^2\dots-a_n^2-b_n^2-c_n^2\right),$$

the equation of the surface of a sphere.

SOLUTION BY PROF. P. E. CHASE.

The fixed points are at the extremities of a hypotenuse, $\therefore$ the locus in any possible plane is a circle; and in all possible planes, a sphere of which the fixed hypotenuse is a diameter.

[Miss Ladd writes, "Question 200 is a well known proposition. It is given by Catalan, *Théorèmes et Problèmes de la Géométrie Élémentaire*, V., 18."]

$$201. \quad \text{"Given} \quad \begin{aligned} x &= a \cos A \pm r_1, \\ y &= a \sin A \pm r_2, \end{aligned}$$

where r_1 and r_2 are the probable errors of x and y . Required the probable errors of a and A ."

SOLUTION BY E. L. DE FOREST, WATERTOWN, CONN.

From the approximate relations

$$x = a \cos A \qquad y = a \sin A$$

$$\text{we have} \qquad a^2 = x^2 + y^2, \qquad \tan A = y \div x.$$

Actual errors may be regarded as finite differences, and if these are small they may be found in the same way as differentials. Hence,

$$\left. \begin{aligned} a \Delta a &= x \Delta x + y \Delta y \\ \Delta a &= \frac{1}{a} (x \Delta x + y \Delta y) \end{aligned} \right\} \begin{aligned} \frac{\Delta A}{\cos^2 A} &= \frac{x \Delta y - y \Delta x}{x^2} \\ \Delta A &= \frac{1}{a^2} (x \Delta y - y \Delta x) \end{aligned} \right\}$$

Since Δx and Δy are independent of each other, and their probable values are r_1 and r_2 , we have, by a well known rule,

$$(\text{Prob. Error of } a) = \pm \frac{1}{a} \sqrt{(x^2 r_1^2 + y^2 r_2^2)},$$

$$(\text{Prob. Error of } A) = \pm \frac{1}{a^2} \sqrt{(x^2 r_2^2 + y^2 r_1^2)},$$

and these may also be written

$$(P. E. \text{ of } a) = \pm \sqrt{r_1^2 \cos^2 A + r_2^2 \sin^2 A},$$

$$(P. E. \text{ of } A) = \pm \frac{1}{a} \sqrt{r_2^2 \cos^2 A + r_1^2 \sin^2 A}.$$

SOLUTION BY PROF. E. W. HYDE, CINCINNATI, OHIO.

If we have $z = f(z_1, z_2, z_3 \text{ \&c.})$ in which $z_1, z_2 \text{ \&c.}$ are independent observations, and if $r, r_1, r_2 \text{ \&c.}$ are the probable errors of the quantities $z, z_1, z_2 \text{ \&c.}$, then

$$r = \sqrt{\left\{ \left[r_1 \frac{df}{dz_1} \right]^2 + \left[r_2 \frac{df}{dz_2} \right]^2 + \left[r_3 \frac{df}{dz_3} \right]^2 + \&c. \right\}}.$$

From the given relations $x = a \cos A, y = a \sin A$ we find

$$a = \sqrt{x^2 + y^2} \pm r_a, \quad A = \tan^{-1} \frac{y}{x} \pm r_A;$$

$$[Prob. \text{ er. of } a] = r_a = \left\{ \left(r_1 \frac{da}{dx} \right)^2 + \left(r_2 \frac{da}{dy} \right)^2 \right\}^{\frac{1}{2}} = \left\{ \frac{r_1^2 x^2 + r_2^2 y^2}{x^2 + y^2} \right\}^{\frac{1}{2}},$$

$$[Prob. \text{ er. of } A] = r_A = \left\{ \left(r_1 \frac{dA}{dx} \right)^2 + \left(r_2 \frac{dA}{dy} \right)^2 \right\}^{\frac{1}{2}} = \left\{ \frac{r_1^2 y^2 + r_2^2 x^2}{x^2 + y^2} \right\}^{\frac{1}{2}}.$$

SOLUTION BY CHAS. H. KUMMELL, DETROIT, MICH.

$$\text{We have } a = \sqrt{(x \pm r_1)^2 + (y \pm r_2)^2} = \sqrt{x^2 + y^2} \pm \frac{xr_1 + yr_2}{\sqrt{x^2 + y^2}} + \dots$$

$$A = \tan^{-1} \frac{y \pm r_2}{x \pm r_1} = \tan^{-1} \frac{y}{x} \pm \frac{xr_2 - yr_1}{x^2 + y^2} + \dots$$

But in applying the method of least squares to non-linear functions, small quantities like probable error, mean error, corrections to observed quantities &c., are considered as differentials, i. e., they are considered small enough to omit their higher powers, therefore (see ANAL., Vol. III, p. 140)

$$r_a = \frac{1}{a} (xr_1 + yr_2),$$

$$r_A = \frac{1}{a^2} (xr_2 - yr_1).$$

SOLUTION OF 188 (SEE P. 57) BY P. RICHARDSON, BROOKLYN, N. Y.

Let AOC be the route of quickest travel from A to C , M be any point in OA produced, then is $MAOC$ the route of quickest travel from M to C .

Let $AO'C$ be any other route from A to C ; then the time in $AOC = AO \div 2m + OC \div m$ and the time in $AO'C = AO' \div 2m + O'C \div m$. By hypothesis $AO \div 2m + OC \div m < AO' \div 2m + O'C \div m$, and if we add the time in MA ($MA \div 2m$) to both sides of this inequality the inequality will still exist; $\therefore$ if AOC is the route of quickest travel from A to C , $MAOC$ is the route of quickest travel from M to C .

Take any point O in EF . Draw $OI = OC$, and through I draw HL perp. to OE produced.

Take $IM = OC$ and draw OK and KL each equal to OC . The rate of travel may be regarded as 2 and 1.

The time in KOC at rate 1 = OC at rate 1 + MO at rate 2, and time from K to C at rate 1 will be least when KOC is a straight line, and when it is least its half will be least likewise. The time from L to O at rate 2 will be least when LKO is a straight line, and at rate 2 it is equal half the time in KOC at rate 1. KO is common to KOC and LKO , $\therefore LKOC$ must be a straight line; $\therefore$ because $LO = 2OC$, $HO = 2OF$.

In the triangles OIH and OAE we have $OI : OH :: OA : OE$. Substituting CO for OI and $2OF$ for OH and multiplying extremes and means,

$$CO \times OE = 2OF \times OA. \quad (1)$$

Putting $AE = a$, $CF = b$, $EF = c$, $EO = x$ and $FO = c - x$, we get

$$x \sqrt{b^2 + (c - x)^2} = 2(c - x) \sqrt{x^2 + a^2}.$$

QUERY. "Which of the two following statemens is correct ?

"The point $x'y'z'$ is a multiple point of the order k , if all the differential coefficients of the order $k - 1$ vanish for that point." (Salmon, *Higher Plane Curves*, Art. 73.)

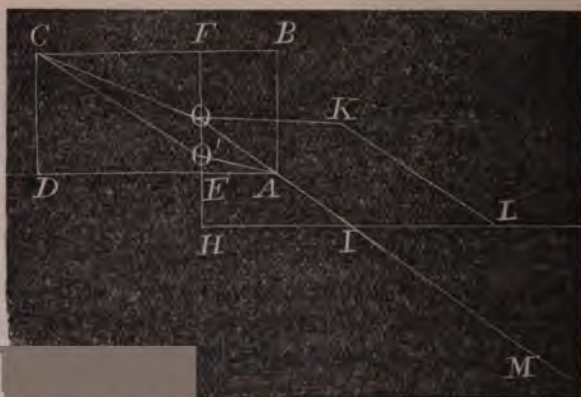
"Il s'ensuit que, pour un point multiple de l'ordre n , il est nécessaire, mais non suffisant, que les dérivées partielles de la fonction f des ordres inférieurs à n s'annulent toutes pour les co-ordonnées du point m ." (Serret, *Calcul. Différential*, Art. 16.)"

[Chas. H. Kummell writes, "Mr. Serret evidently means *real* multiple points, viz., the case where the equation of the n th degree for $\frac{dy}{dx}$ has more than one real root, in which case we have a *visible* multiple point."

A. S. Hathaway writes, "Salmon and Serret differ as to what is the order of a multiple point whose coordinates x_0, y_0 make the partial derivatives of f , below the n th order vanish.

Serret reckons by the number of *real* branches at the point. * * *

Salmon means by a multiple point of order n , a point, every line through which meets the curve there in n coincident points." * * *



PROBLEMS.

202. BY P. M. MC KAY. — If $r \div d$ reduces to a repetend, prove that $s \div d = R \div 9$, where s is the sum of the remainders occurring in the division, and R is the sum of the digits of the repetend.

203. BY L. REGAN, BOONSBORO, IOWA. — Two lines Am and An make a given angle at A , and D is a point given in position between Am and An . Draw a straight line BC through D so that $BD \times DC =$ a given quantity, B and C being points in the lines Am and An , respectively.

204. (SELECTED) BY PROF. M. L. COMSTOCK, GALESBURG, ILL. — Find the locus of the intersection of normals to the ellipse which are perpendicular to each other.

205. BY PROF. W. P. CASEY, SAN FRANCISCO, CAL. — The distance between the centres of two circles is given, and the radius of one, find the radius of the other, when the length of the endless band which passes around them is given.

206. BY R. J. ADCOCK, MONMOUTH, ILL. — Determine the position of a plane from the measured rectangular coordinates of n points, each measure being equally good.

207. BY E. B. SEITZ, GREENVILLE, OHIO. — If M, N, P, Q are four random points in the surface of a circle, find the chance that E , the intersection of the straight lines through M, N and P, Q , lies between M, N and between P, Q .

208. BY PROF. J. H. KERSHNER. — When the tangent, QT , to a fixed parabola is of constant length, and one extremity, T , moves along the axis to the origin, what is the curve described by the other extremity, Q ?

209. BY S., NEW YORK CITY. — C is the center of a circle of known radius, DB is a diameter produced and AB is an inflexible rod, of known length. One end, A , moves continually in the circumference of the circle while the other end, B , slides to and fro upon DB .

Required the equation of the curve described by any point P , in the rod.

210. BY ARTEMAS MARTIN, M. A., ERIE, PA. — Find such integral values of x, y and z as will satisfy the conditions

$$x^2 + axy + y^2 = \square,$$

$$x^2 + bxz + z^2 = \square,$$

$$y^2 + cyz + z^2 = \square.$$

QUERY. BY PROF. D. J. MC ADAM.—In works on Practical Geometry, the following method of inscribing a regular polygon of n sides is given:

Let AB be a diameter of a circle. With A and B as centers and radius AB describe arcs intersecting in D . Divide AB into n equal parts. Draw DE through the second point of division; the arc AE is one- n th part of the circumference. Is there a general demonstration published?

QUERY (BY REQUEST).—On page 59 of Prof. Peirce's "Linear Associative Algebra", under the head of "Quadruple Algebras", the algebra " g_4 " is said to have for its multiplication table the following table and to be also a "form of quaternions".

g_4	i	j	k	l
i	i	j	o	o
j	o	o	i	j
k	k	l	o	o
l	o	o	k	l

Explain by an example.

PUBLICATIONS RECEIVED.

Micrometrical Measurement of 517 Double Stars, at the Cincinnati Observatory, During the year 1877. ORMOND STONE A. M., Director. 74 pages, 8vo.

American Journal of Mathematics Pure and Applied. Volume I. Number 1. 104 pages, 4to. Baltimore. — The first Number of this periodical fully justifies the anticipations of its friends, based upon the high character and mathematical attainments of its editors and contributors. Subscription price \$5.00 per Vol. (4 numbers), or \$1.50 for a single number.

ERRATA.

On page 173, Vol. II, commencing in last line, cancel the sentence, "As to the double sign before these factors, the upper or lower is taken according as i''' is even or odd." And, on page 174, for $\pm$, read $+$.

On page 39, line 6, for $(m-2)_1$, read $-(m-2)_1$.

" " 51, line 26, for *Lauch*, read *Cauchy*.

" " 67, formula (7), insert bracket $()$ before $(u_2 + u_{-2})$ and before $(u_4 + u_{-4})$.

" " 79, line 17, for 16, read 15.

THE ANALYST.

VOL. V.

JULY, 1878.

No. 4.

EVALUATION OF ELLIPTIC FUNCTIONS OF THE SECOND AND THIRD SPECIES, ANALYST, VOL. V, PAGES 18 & 19.

BY CHAS. H. KUMMELL, ASSIST. U. S. LAKE SURV. DETROIT, MICH.

4. FOR the second species and ascending scale of moduli we assume the fundamental form :

$$\int_0^\phi d\varphi \sqrt{[a^2 - c^2 \sin^2 \varphi]} = \int_0^\phi a A(\varphi) d\varphi = a E_0^\phi \left(\frac{c}{a} \right). \quad (21)$$

From (11) follows:

$$\begin{aligned} \tan \varphi &= \frac{a \sin 2\varphi'}{c + a \cos 2\varphi'}; \\ \therefore \sin \varphi &= \frac{a \sin 2\varphi'}{\sqrt{[c^2 + 2ac \cos 2\varphi' + a^2]}} \\ &= \frac{a \sin \varphi' \cos \varphi'}{\sqrt{[a'^2 - c'^2 \sin^2 \varphi']}} \\ &= \frac{a \sin \varphi' \cos \varphi'}{a' A'(\varphi')}, \end{aligned} \quad (22)$$

$$\begin{aligned} a A(\varphi) &= \frac{a}{a' A'(\varphi')} \sqrt{(a'^2 - c'^2 \sin^2 \varphi' - c^2 \sin^2 \varphi' \cos^2 \varphi')} \\ &= \frac{a}{a' A'(\varphi')} (a' - c \sin^2 \varphi') \\ &= a' A'(\varphi') + \frac{\frac{1}{2}(a^2 - c^2)}{a' A'(\varphi')}, \end{aligned} \quad (23)$$

consequently regarding (12)

$$\begin{aligned} a A(\varphi) d\varphi &= a^2 A(\varphi)^2 \cdot \frac{d\varphi}{a A(\varphi)} \\ &= \left[a' A'(\varphi') + \frac{\frac{1}{2}(a^2 - c^2)}{a' A'(\varphi')} \right]^2 \frac{d\varphi'}{a' A'(\varphi')} \\ &= a' A'(\varphi') d\varphi' + \frac{1}{2}(a^2 - c^2) \frac{d\varphi'}{a' A'(\varphi')} + \frac{1}{16}(a^2 - c^2)^2 \frac{d\varphi'}{a'^3 A'(\varphi')^3}. \end{aligned} \quad (24)$$

Differentiating (22) we have

$$\begin{aligned} d\varphi \cos \varphi &= \frac{ad\varphi'}{a'^3 D'(\varphi')^3} \left[a'^2(1 - 2\sin^2 \varphi') + c'^2 \sin^4 \varphi' \right] \\ &= \frac{ad\varphi'}{a'^3 D'(\varphi')^3} \left[\frac{a'^4 D'(\varphi')^4}{c'^2} + a'^2 - \frac{a'^4}{c'^2} \right] \\ &= \frac{d\varphi'}{ca'^3 D'(\varphi')^3} \left[a'^4 D'(\varphi')^4 - \frac{1}{16}(a^2 - c^2)^2 \right], \end{aligned} \quad (25)$$

$$\therefore \frac{1}{16}(a^2 - c^2)^2 \frac{d\varphi'}{a'^3 D'(\varphi')^3} = a' D'(\varphi') d\varphi' - c \cos \varphi d\varphi. \quad (25')$$

Adding this to (24) and integrating we obtain

$$\int_0^\varphi a D(\varphi) d\varphi = 2 \int_0^{\varphi'} a' D'(\varphi') d\varphi' + a'(a-c) \int_0^{\varphi'} \frac{d\varphi'}{a' D'(\varphi')} - c \sin \varphi$$

or $a E_0^\varphi \left(\frac{c}{a} \right) = 2a' E_0^{\varphi'} \left(\frac{c'}{a'} \right) + a'(a-c) \cdot \frac{1}{a'} F_0^{\varphi'} \left(\frac{c'}{a'} \right) - c \sin \varphi. \quad (26)$

At the next higher step in the scale of moduli we have

$$2a' E_0^{\varphi'} \left(\frac{c'}{a'} \right) = 2^2 a'' E_0^{\varphi''} \left(\frac{c''}{a''} \right) + 2a''(a' - c') \cdot \frac{1}{a''} F_0^{\varphi''} \left(\frac{c''}{a''} \right) - 2c' \sin \varphi'. \quad (26')$$

Finally when

$$a^{(n)} = c^{(n)} = \left\| \frac{1}{2} \left\{ a + c \right\} \right\|^{\frac{1}{2}} \quad \text{we have}$$

$$2^n a^{(n)} E_0^{\varphi^{(n)}}(1) = 2^n a^n \sin \varphi^{(n)}. \quad (26'')$$

Adding (26), (26'), ... (26'') we have regarding (13)

$$\begin{aligned} a E_0^\varphi \left(\frac{c}{a} \right) &= 2^n a^{(n)} \sin \varphi^{(n)} + \sum_{r=0}^{n-1} [2^r a^{(r+1)} (a^{(r)} - c^{(r)})] \cdot \frac{1}{a^{(n)}} \log \tan \frac{1}{2} (\frac{1}{2}\pi + \varphi^{(n)}) \\ &\quad - \sum_{r=0}^{n-1} [2^r c^{(r)} \sin \varphi^{(r)}]. \end{aligned} \quad (27)$$

This is the length of an arc of an ellipse the equation of which is

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1.$$

between $x = 0$ and $x = a \sin \varphi$.

5. For the second species and descending scale we take the form:

$$\int_0^\varphi d\varphi \sqrt{[a^2 \cos^2 \varphi + b^2 \sin^2 \varphi]} = \int_0^\varphi a D(\varphi) d\varphi = a E_0^\varphi \sqrt{1 - \frac{b^2}{a^2}}. \quad (28)$$

Consistent with (15) we have also

$$\sin (2\varphi - \varphi_1) = \frac{a-b}{a+b} \sin \varphi_1 = \cos \varphi_1 (\sin 2\varphi - \tan \varphi_1 \cos 2\varphi); \quad (29)$$

for if we add and then subtract the identity

$$\sin \varphi_1 = \sin \varphi_1$$

and divide the difference by the sum (15) results.

Solving for $\cos 2\varphi$ we have

Formula (34) might have been also derived from (26) by regarding the following relations, which exist between the quantities with primes and those with subscripts if each series is continued in both directions. We have then the amplitudes

$\varphi^{(n)} = \varphi_{-n}; \dots \varphi' = \varphi_{-1}; \varphi = \varphi; \varphi_i = \varphi_1; \varphi_{ii} = \varphi_2; \dots \varphi_{(n)} = \varphi_n; (37)$
also the moduli

$$\frac{c^{(n)}}{a^{(n)}} = \left\{ 1 - \frac{b_{-n}^2}{a_{-n}^2} \right\}^{\frac{1}{2}}; \dots \frac{c'}{a'} = \left\{ 1 - \frac{b_{-1}^2}{a_{-1}^2} \right\}^{\frac{1}{2}}; \frac{c}{a} = \left\{ 1 - \frac{b^2}{a^2} \right\}^{\frac{1}{2}};$$

$$\frac{c_i}{a_i} = \left\{ 1 - \frac{b_1^2}{a_1^2} \right\}^{\frac{1}{2}}; \dots \frac{c_{(n)}}{a_{(n)}} = \left\{ 1 - \frac{b_n^2}{a_n^2} \right\}^{\frac{1}{2}}; (38)$$

whence

$$\Delta^{(n)}(\varphi) = \Delta_{-n}(\varphi); \dots \Delta'(\varphi) = \Delta_{-1}(\varphi); \Delta\varphi = \Delta\varphi; \Delta_i(\varphi) = \Delta_1(\varphi);$$

$$\dots \Delta_{(n)}(\varphi) = \Delta_n(\varphi). (39)$$

By (12) we have

$$\int_0^{\varphi^{(n)}} \frac{d\varphi}{a^{(n)} \Delta^{(n)}(\varphi)} = \int_0^{\varphi^{(n-1)}} \frac{d\varphi}{a^{(n-1)} \Delta^{(n-1)}(\varphi)} = \dots \int_0^{\varphi'} \frac{d\varphi}{a' \Delta'(\varphi)} = \int_0^{\varphi} \frac{d\varphi}{a \Delta(\varphi)}$$

$$= \int_0^{\psi_i} \frac{d\psi^*}{a_i \Delta_i(\psi)} = \dots \int_0^{\psi_{(n)}} \frac{d\psi}{a_{(n)} \Delta_{(n)}(\psi)}, (40)$$

while by (16) we have

$$\int_0^{\psi_{-n}} \frac{d\psi}{2^{-n} a_{-n} \Delta_{-n}(\psi)} = \int_0^{\psi_{-n+1}} \frac{d\psi}{2^{-n+1} a_{-n+1} \Delta_{-n+1}(\psi)} = \dots \int_0^{\psi_{-1}} \frac{d\psi}{2^{-1} a_{-1} \Delta_{-1}(\psi)}$$

$$= \int_0^{\psi} \frac{d\psi}{a \Delta(\psi)} = \int_0^{\psi_1} \frac{d\psi}{2 a_1 \Delta_1(\psi)} = \dots \int_0^{\psi_n} \frac{d\psi}{2^n a_n \Delta_n(\psi)}. (41)$$

We have then regarding (37) and (39)

$$a^{(n)} = 2^{-n} a_{-n}; a^{(n-1)} = 2^{-n+1} a_{-n+1}; \dots a' = 2^{-1} a_{-1}; a = a; a_i = 2 a_1;$$

$$\dots a_{(n)} = 2^n a_n. (42)$$

These relations enable us to derive (34) from (26) and vice versa.

We have by (26) for the next lower step in the modular scale

$$\int_0^{\psi_i} a_i \Delta_i(\psi) d\psi = 2 \int_0^{\psi} a \Delta(\psi) d\psi + \frac{1}{2} (a_i^2 - c_i^2) \int_0^{\psi} \frac{d\psi}{a \Delta(\psi)} - c_i \sin \psi_i,$$

$$\therefore \int_0^{\psi} a \Delta(\psi) d\psi = \frac{1}{2} \int_0^{\psi_i} a_i \Delta_i(\psi) d\psi - \frac{1}{4} (a_i^2 - c_i^2) \int_0^{\psi} \frac{d\psi}{a \Delta(\psi)} + \frac{1}{2} c_i \sin \psi_i.$$

But $\psi_i = \psi_1$; $a_i = 2a_1$; $\Delta_i(\psi) = \Delta_1(\psi)$; $a_i^2 - c_i^2 = 4b_1^2$; $c_i = 2\sqrt{a_1^2 - b_1^2}$

$$= a - b \text{ and } \int_0^{\psi} \frac{d\psi}{a \Delta(\psi)} = \int_0^{\psi_1} \frac{d\psi}{2a_1 \Delta_1(\psi)} \text{ by (41),}$$

$$\therefore \int_0^{\psi} a \Delta(\psi) d\psi = \int_0^{\psi_1} a_1 \Delta_1(\psi) d\psi - \frac{1}{2} b_1^2 \int_0^{\psi_1} \frac{d\psi}{a_1 \Delta_1(\psi)} + \frac{1}{2} (a - b) \sin \psi_1,$$

which is (34).

[*For want of sorts, ψ is here, and throughout the rest of this article, used for ϕ .—Com. P.]

These relations are also very useful for the inverse problem of determining the amplitude to a given integral. It is not my purpose however to show this now.

6. For the third species and ascending scale we take the fundamental form:

$$\int_0^\psi \frac{d\phi}{(p+r\sin^2\phi)\sqrt{[a^2-c^2\sin^2\phi]}} = \int_0^\psi \frac{d\phi}{aP(\phi)\Delta(\phi)} = \frac{1}{ap} \Pi_0^\psi\left(\frac{c}{a}, \frac{r}{p}\right). \quad (43)$$

We have, substituting (22)

$$\frac{1}{P(\phi)} = \frac{a'^2 - c'^2 \sin^2 \phi'}{a'^2 p + a(ar - cp) \sin^2 \phi' - a^2 r \sin^4 \phi'}. \quad (44)$$

Assume

$$a'^4 p^2 + aa'^2 p(ar - cp) \sin^2 \phi' - a^2 a'^2 pr \sin^4 \phi' = (a'^2 p + am' \sin^2 \phi')(a'^2 p + am' \sin^2 \phi') \quad (45)$$

whence

$$m' + m'' = ar - cp; \quad m' m'' = -a'^2 pr \quad (45)$$

and

$$m' - m'' = \sqrt{[(p+r)(ar^2 + cp^2)]}. \quad (46)$$

Now let

$$\begin{aligned} \frac{\frac{1}{p} - \frac{ac}{a'^2 p} \sin^2 \phi'}{\left(1 + \frac{am'}{a'^2 p} \sin^2 \phi'\right) \left(1 + \frac{am''}{a'^2 p} \sin^2 \phi'\right)} &= \frac{1}{p' + r' \sin^2 \phi'} + \frac{1}{p' + r' \sin^2 \phi'} \\ &= \frac{1}{p'} + \frac{1}{p'} + \frac{r' + r''}{p' p'} \sin^2 \phi' \div \left(1 + \frac{r'}{p'} \sin^2 \phi'\right) \left(1 + \frac{r''}{p'} \sin^2 \phi'\right). \end{aligned}$$

By comparison and regarding (45) we have

$$r' = \frac{am' p'}{a'^2 p} = -\frac{p'}{m'} ar; \quad r'' = \frac{am'' p''}{a'^2 p} = -\frac{p''}{m''} ar. \quad (47)$$

$$\frac{1}{p} = \frac{1}{p'} + \frac{1}{p''}; \quad -c = \frac{m'}{p'} + \frac{m''}{p''} \quad (48)$$

whence

$$p' = \frac{m' - m''}{m' + cp} p; \quad p'' = -\frac{m' - m''}{m'' + cp} p. \quad (49)$$

Thus p' , p'' and r' , r'' are known, and placing $P'(\phi) = p' + r' \sin^2 \phi$; $P''(\phi) = p'' + r'' \sin^2 \phi$ we have regarding (12) or (41)

$$\int_0^\psi \frac{d\phi}{aP(\phi)\Delta(\phi)} = \int_0^{\psi'} \frac{d\phi}{a'P'(\phi)\Delta'(\phi)} + \int_0^{\psi''} \frac{d\phi}{a'P''(\phi)\Delta'(\phi)} \quad (50)$$

At the next higher step in the modular scale we have besides (11') the formulæ

$$m'' + m''' = a'r' - c'p', \quad m'' + m''' = a'r'' - c'p'', \quad (45')$$

$$m'' - m''' = \sqrt{[(p' + r')(a'^2 r' + c'^2 p')]}, \quad m'' - m''' = \sqrt{[(p'' + r'')(a'^2 r'' + c'^2 p'')]}, \quad (46')$$

$$p'' = \frac{m'' - m'''}{m'' + c'p'} p'; \quad p''' = \frac{m'' - m'''}{m''' + c'p'} p'; \quad p'' = \frac{m'' - m'''}{m'' + c'p''} p''; \quad p''' = \frac{m'' - m'''}{m''' + c'p''} p'' \quad \dots (49')$$

$$r'' = -\frac{p''}{m'} a' r'; \quad r^{(1)} = -\frac{p^{(1)}}{m''} a' r'; \quad r^{(2)} = -\frac{p^{(2)}}{m^{(1)}} a' r'; \quad r^{(3)} = -\frac{p^{(3)}}{m^{(2)}} a' r'; \quad (47')$$

and if $P''(\psi) = p'' + r'' \sin^2 \psi$; $P^{(1)}(\psi) = p^{(1)} + r^{(1)} \sin^2 \psi$; $P^{(2)}(\psi) = p^{(2)} + r^{(2)} \sin^2 \psi$; $P^{(3)}(\psi) = p^{(3)} + r^{(3)} \sin^2 \psi$ we have

$$\int_0^\psi \frac{d\psi}{a P(\psi) \Delta(\psi)} = \int_0^{\psi'} \frac{d\psi}{a' P'(\psi) \Delta'(\psi)} + \int_0^{\psi''} \frac{d\psi}{a'' P''(\psi) \Delta''(\psi)} + \int_0^{\psi^{(1)}} \frac{d\psi}{a^{(1)} P^{(1)}(\psi) \Delta^{(1)}(\psi)} + \int_0^{\psi^{(2)}} \frac{d\psi}{a^{(2)} P^{(2)}(\psi) \Delta^{(2)}(\psi)} + \int_0^{\psi^{(3)}} \frac{d\psi}{a^{(3)} P^{(3)}(\psi) \Delta^{(3)}(\psi)} \quad (51)$$

Pursuing the ascending scale of moduli until $a^{(n)} = c^{(n)} = \left\| \frac{1}{2} \left(a + \frac{1}{c} \right) \right\|$ and consequently $\Delta^{(n)}(\psi) = \cos \psi$, we have finally 2^n integrals of the form:

$$\begin{aligned} \int_0^{\psi^{(n)}} \frac{d\psi}{a^{(n)} P^{(n)}(\psi) \cos \psi} &= \frac{r^{(n)}}{a^{(n)}(p^{(n)} + r^{(n)})} \int_0^{\psi^{(n)}} \frac{d\psi \cos \psi}{P^{(n)}(\psi)} + \frac{1}{a^{(n)}(p^{(n)} + r^{(n)})} \int_0^{\psi^{(n)}} \frac{d\psi}{\cos \psi} \\ &= \frac{1/r^{(n)}}{a^{(n)}(p^{(n)} + r^{(n)}) \sqrt{p^{(n)}}} \tan^{-1} \left\{ \sqrt{\left(\frac{r^{(n)}}{p^{(n)}} \right) \sin \psi^{(n)}} \right\} \\ &\quad + \frac{1}{a^{(n)}(p^{(n)} + r^{(n)})} \log \tan \frac{1}{2} \left(\frac{1}{2} \pi + \psi^{(n)} \right). \end{aligned} \quad (51)$$

These reductions are very troublesome if $p', p'', \dots, r', r'', \dots$ are complex quantities. In this case and if a problem leads to the form

$$\int_0^\psi f(\sin^2 \psi) \frac{d\psi}{a \Delta(\psi)}$$

(f denoting a rational function of $\sin^2 \psi$) which might be expressed by means of some of the three species of elliptic functions and elementary forms, it is perhaps better to express $f(\sin^2 \psi)$ successively in terms of $\sin^2 \psi', \sin^2 \psi'', \dots, \sin^2 \psi^{(n)}$ so that if $f'(\sin^2 \psi), f''(\sin^2 \psi), \dots$ are the resulting forms we have

$$\begin{aligned} \int_0^\psi f(\sin^2 \psi) \frac{d\psi}{a \Delta(\psi)} &= \int_0^{\psi'} f'(\sin^2 \psi) \frac{d\psi}{a' \Delta'(\psi)} = \int_0^{\psi''} f''(\sin^2 \psi) \frac{d\psi}{a'' \Delta''(\psi)} \dots \\ &= \int_0^{\psi^{(n)}} f^{(n)}(\sin^2 \psi) \frac{d\psi}{a^{(n)} \cos \psi} = \frac{1}{a^{(n)}} \int_0^{\sin \psi^{(n)}} \frac{dx}{f(x^2) \sqrt{1-x^2}}. \end{aligned} \quad (52)$$

7. For the third species and descending scale we take the form:

$$\begin{aligned} \int_0^\psi \frac{d\psi}{(p \cos^2 \psi + q \sin^2 \psi) \sqrt{[a^2 \cos^2 \psi + b^2 \sin^2 \psi]}} &= \int_0^\psi \frac{d\psi}{a P(\psi) \Delta(\psi)} \\ &= \Pi_0^\psi \left\{ \sqrt{\left(1 - \frac{b^2}{a^2} \right)}, \frac{p}{q} - 1 \right\}. \end{aligned} \quad (53)$$

Substituting (31) we have regarding (16) or (41)

$$\frac{d\psi}{a P(\psi) \Delta(\psi)} = \frac{1}{p + \frac{1}{2}(q-p) \left[1 + \frac{a-b}{a+b} \sin^2 \phi_1 - \cos \phi_1 \Delta_1(\phi_1) \right]} \frac{d\phi_1}{2a_1 \Delta_1(\phi_1)}$$

$$\begin{aligned}
 &= \frac{p + \frac{1}{2}(q-p) \left[1 + \frac{a-b}{a+b} \sin^2 \phi_1 + \cos^2 \phi_1 \Delta_1(\phi) \right]}{pq + \frac{q-p}{(a+b)^2} (a^2 q - b^2 p) \sin^2 \phi_1} \cdot \frac{d\phi_1}{2a_1 \Delta_1(\phi_1)} \\
 &= \frac{a^2 q^2 - b^2 p^2}{4(a^2 q - b^2 p)} \cdot \frac{1}{pq \cos^2 \phi_1 + \left(\frac{aq + bp}{a+b} \right)^2 \sin^2 \phi_1} \cdot \frac{d\phi_1}{a_1 \Delta_1(\phi_1)} \\
 &+ \frac{a^2 - b^2}{4(a^2 q - b^2 p)} \cdot \frac{d\phi_1}{a_1 \Delta_1(\phi_1)} + \frac{q-p}{4a_1} \cdot \frac{d\phi_1 \cos \phi_1}{pq \cos^2 \phi_1 + \left(\frac{aq + bp}{a+b} \right)^2 \sin^2 \phi_1}. \quad (54)
 \end{aligned}$$

Let

$$m_1 = \frac{a^2 q^2 - b^2 p^2}{a^2 q - b^2 p}, \quad (55)$$

$$p_1 = \frac{q}{m_1} p, \quad (56)$$

$$q_1 = \frac{(aq + bp)^2}{4a_1^2 m_1}. \quad (57)$$

We have then if $P_1(\phi_1) = p_1 \cos^2 \phi_1 + q_1 \sin^2 \phi_1$

$$\int_0^\phi \frac{d\phi}{aP(\phi)\Delta(\phi)} = \frac{1}{2} \left\{ \int_0^{\phi_1} \frac{d\phi}{a_1 P_1(\phi)\Delta_1(\phi)} + \frac{a^2 - b^2}{a^2 q - b^2 p} \int_0^{\phi_1} \frac{d\phi}{a_1 \Delta_1(\phi)} + \frac{q-p}{a_1 m_1} \int_0^{\phi_1} \frac{d\phi \cos \phi}{P_1(\phi)} \right\};$$

or if we put

$$q-p = r; \quad q_1 - p_1 = r_1 \quad (58)$$

we have $r_1 = \frac{a^2 q - b^2 p}{4a_1^2 m_1} r$; hence $\frac{a^2 - b^2}{a^2 q - b^2 p} = \frac{(a-b)r}{2a_1 m_1 r_1}$ and

$$\int_0^\phi \frac{d\phi}{aP(\phi)\Delta(\phi)} = \frac{1}{2} \left\{ \int_0^{\phi_1} \frac{d\phi}{a_1 P_1(\phi)\Delta_1(\phi)} + \frac{(a-b)r}{2a_1 m_1 r_1} \int_0^{\phi_1} \frac{d\phi}{a_1 \Delta_1(\phi)} + \frac{r}{a_1 m_1} \int_0^{\phi_1} \frac{d\phi \cos \phi}{P_1(\phi)} \right\}. \quad (59)$$

For the next lower step in the modular scale we have

$$\frac{1}{4} \int_0^{\phi_1} \frac{d\phi}{a_1 P_1(\phi)\Delta_1(\phi)} = \frac{1}{4^2} \left\{ \int_0^{\phi_2} \frac{d\phi}{a_2 P_2(\phi)\Delta_2(\phi)} + \frac{(a_1 - b_1)r_1}{2a_2 m_2 r_2} \int_0^{\phi_2} \frac{d\phi}{a_2 \Delta_2(\phi)} + \frac{r_1}{a_2 m_2} \int_0^{\phi_2} \frac{d\phi \cos \phi}{P_2(\phi)} \right\}, \quad (59_1)$$

where we have besides (17)

$$m_2 = \frac{a_1^2 q_1^2 - b_1^2 p_1^2}{a_1^2 q_1 - b_1^2 p_1}, \quad (55_1)$$

$$p_2 = \frac{q_1}{m_2} p_1, \quad (56_1)$$

$$q_2 = \frac{(a_1 q_1 + b_1 p_1)^2}{4a_2^2 m_2}, \quad (57_1)$$

$$r_2 = \frac{a_1^2 q_1 - b_1^2 p_1}{4a_2^2 m_2} r_1 = q_2 - p_2. \quad (58_1)$$

Descending the modular scale until $a_n = b_n = \|\frac{1}{2}(a \pm b)^{\frac{1}{2}}\|$ we have finally

$$\frac{1}{4^n} \int_0^{\psi_n} \frac{d\phi}{a_n P_n(\phi)} = \frac{1}{4^n a_n \sqrt{(p_n q_n)}} \tan^{-1} \left\{ \sqrt{\left(\frac{q_n}{p_n}\right)} \tan \phi_n \right\}. \quad (59)$$

Adding (59), (59₁), ... (59_n) we obtain

$$\begin{aligned} \int_0^{\psi} \frac{d\phi}{a P(\phi) \Delta(\phi)} &= \frac{1}{4^n a_n \sqrt{(p_n q_n)}} \tan^{-1} \left\{ \left(\frac{q_n}{p_n}\right)^{\frac{1}{2}} \tan \phi_n \right\} + \sum_1^n \left\{ \frac{(a_{s-1} - b_{s-1}) r_{s-1}}{2^{2s+1} a_s m_s r_s} \right. \\ &\quad \times \left. \int_0^{\psi_s} \frac{d\phi}{a_s \Delta_s(\phi)} \right\} + \sum_1^n \left\{ \frac{r_{s-1}}{2^{2s} a_s m_s} \int_0^{\psi_s} \frac{d\phi \cos \phi}{P_s(\phi)} \right\}. \quad (60) \end{aligned}$$

But

$$\int_0^{\psi_s} \frac{d\phi}{a_s \Delta_s(\phi)} = \frac{\phi_s}{2^{n-s} a_n} \quad (61)$$

and

$$\int_0^{\psi_s} \frac{d\phi \cos \phi}{P_s(\phi)} = \frac{1}{\sqrt{(p_s r_s)}} \tan^{-1} \left\{ \left(\frac{r_s}{p_s}\right)^{\frac{1}{2}} \sin \phi_s \right\}, \quad (62)$$

consequently

$$\begin{aligned} \int_0^{\psi} \frac{d\phi}{a P(\phi) \Delta(\phi)} &= \frac{\tan^{-1} \left[\sqrt{(q_n \div p_n)} \tan \phi_n \right]}{2^{2n} a_n \sqrt{(p_n q_n)}} + \sum_1^n \left\{ \frac{(a_{s-1} - b_{s-1}) r_{s-1}}{2^{n+s+1} a_s m_s r_s} \right\} \frac{\phi_s}{a_n} \\ &\quad + \sum_1^n \left\{ \frac{r_{s-1}}{2^{2s} a_s m_s \sqrt{(p_s r_s)}} \tan^{-1} \left[\left(\frac{r_s}{p_s}\right)^{\frac{1}{2}} \sin \phi_s \right] \right\}. \quad (63) \end{aligned}$$

It is hardly necessary to state that if p_s has the opposite sign to q_s or the $\tan^{-1}$ must be replaced by logarithms.

If $\phi = \frac{1}{2}\pi$ then $\phi_1 = \pi$; $\phi_2 = 2\pi$; ... $\phi_n = 2^{n-1}\pi$; hence

$$\int_0^{\frac{\pi}{2}} \frac{d\phi}{a P(\phi) \Delta(\phi)} = \frac{\pi}{2^{n+1} a_n \sqrt{(p_n q_n)}} + \sum_1^n \left\{ \frac{(a_{s-1} - b_{s-1}) r_{s-1}}{2^{n+s+2} a_s m_s r_s} \right\} \frac{\pi}{a_n} \quad (64)$$

for

$$\int_0^{2^{s-1}\pi} \frac{d\phi \cos \phi}{P_s(\phi)} = 0 = \int_0^{\pi} \frac{d\phi \cos \phi}{P_s(\phi)},$$

because the elements of this integral from 0 to $\frac{1}{2}\pi$ are equal and opposite in sign to those from $\frac{1}{2}\pi$ to π .

Equation (59) could be inverted by means of the relations (38), ... (42). I have not yet succeeded however in reducing the relation between the p , q and r to a convenient form. The reduction by ascending the modular scale as made under No. 6 seems to be preferable.

THE SECULAR ACCELERATION OF THE MOON.

BY G. W. HILL.

IN the Philosophical Transactions for 1853, Prof. J. C. Adams, of Cambridge University, England, showed that the values of the secular acceleration of the mean motion of the moon, obtained by Plana and Damoiseau, were erroneous, for the simple reason that these authors had, inadvertently, made the solar eccentricity constant throughout a certain portion of the investigation. This statement of Prof. Adams gave rise to an animated and prolonged controversy, the history of which will, no doubt, always possess much interest.

I propose to obtain here the coefficient of the term in the moon's mean motion involving the square of the solar eccentricity, supposed variable, to quantities of the order of the square of the sun's disturbing force, when the lunar eccentricity and inclination of orbit are neglected. The method employed has no novelty, having been used before by Mr. Donkin. But, at the end of the investigation, I have found that it is possible to do without an explicit development of R in a periodic series, and thus the treatment is, to a considerable degree, abbreviated.

Let ζ denote the mean longitude of the moon as affected by this secular inequality, and n_0 the mean motion at a given epoch taken as the origin of time, we propose to prove that, in the equation

$$\frac{d\zeta}{dt} = n = n_0[1 + H(e'^2_0 - e'^2)],$$

the true value of H is

$$\frac{3}{2} \left(\frac{n'}{n_0} \right)^2 - \frac{3771}{64} \left(\frac{n'}{n_0} \right)^4.$$

Employing the method of variation of the elements, we have, for determining the four elements n , ζ , e and ω of the lunar orbit, these equations

$$\begin{aligned} \frac{dn}{dt} &= -\frac{3}{\mu a^2} \frac{dR}{d\zeta}, & \frac{de}{dt} &= -\frac{na}{\mu e} \frac{dR}{d\omega}, \\ \frac{d\zeta}{dt} &= n - 2\frac{na^2}{\mu} \frac{dR}{da} + \frac{1}{2} \frac{nae}{\mu} \frac{dR}{de}, & \frac{d\omega}{dt} &= \frac{na}{\mu e} \frac{dR}{de}. \end{aligned}$$

In writing them, all terms, multiplied by higher powers of e than the first, have been neglected, as they are not needed in obtaining H to the degree of accuracy we propose. It may be noted that R is taken with such a sign

that $\frac{dR}{dx}$ denotes the force tending to increase x .

Since we need not retain any terms multiplied by the ratio $\frac{a}{a'}$, the value of R is

$$R = \frac{1}{4}n'^2r^2\frac{a'^3}{r'^3}[1+3\cos 2(\lambda-\lambda')],$$

where λ and λ' are the true longitudes of the moon and sun. The constant part of R is evidently the same as that of $\frac{1}{4}n'^2a^2\frac{a'^3}{r'^3}$ when we reject e^2 , that is, it is equal to $\frac{1}{4}n'^2a^2(1+\frac{3}{2}e'^2)$.

Considering first those terms in R which are independent of e (we need those multiplied by e only when taking account of the effects produced by the variations δe and $\delta\omega$), we see that the only terms in R which produce terms in $\frac{dn}{dt}$, and, consequently, can give rise to terms independent of sines

or cosines of arguments in $\frac{d\zeta}{dt}$, have arguments of the form $2\zeta+\phi$, where ϕ denotes an angle depending on the sun's mean motion. Hence, denoting any one of these terms of R by $n'^2a^2A\cos(2\zeta+\phi)$, where A is independent of the lunar elements, but will generally contain e'^2 , and regard being had to this term alone, the equations determining the elements become

$$\begin{aligned}\frac{dn}{dt} &= 6n'^2A\sin(2\zeta+\phi), \\ \frac{d\zeta}{dt} &= n - 4\frac{n'^2}{n}A\cos(2\zeta+\phi),\end{aligned}$$

where μ has been eliminated by using the equation $\mu = n^2a^3$. Integrating these, and considering ϕ as constant, since its variability affects only the terms in H multiplied by $\frac{n'^5}{n_0^6}$, but regard being had to the variability of

A through e'^2 , where we may consider $\frac{d.e'^2}{dt}$ as constant, we obtain

$$\begin{aligned}\delta n &= -3\frac{n'^2}{n}A\cos(2\zeta+\phi) + \frac{3}{2}\frac{n'^2}{n^2}\frac{dA}{d.e'^2}\frac{d.e'^2}{dt}\sin(2\zeta+\phi), \\ \delta\zeta &= -\frac{7}{2}\frac{n'^2}{n^2}A\sin(2\zeta+\phi) - \frac{5}{2}\frac{n'^2}{n^3}\frac{dA}{d.e'^2}\frac{d.e'^2}{dt}\cos(2\zeta+\phi).\end{aligned}$$

This being only a first approximation in which we have had regard only to quantities of the order of n'^2 , we proceed to a second approximation. And first, in the expression for $\frac{dn}{dt}$, we substitute for ζ , $\zeta+\delta\zeta$; and we find, for that part of the increment which is independent of the sines or cosines of arguments, the expression

$$\partial \cdot \frac{dn}{dt} = -15 \frac{n'^4}{n^3} A \frac{dA}{d.e'^2} \frac{d.e'^2}{dt}.$$

Integrating this and putting $e'^2 - e_0'^2 = \delta.e'^2$,

$$\delta n = -\frac{15}{2} \frac{n'^4}{n^3} \frac{d.A^2}{d.e'^2} \delta.e'^2.$$

Again, in the expression for $\frac{d\zeta}{dt}$, increasing n and ζ by their variations δn and $\delta \zeta$,

$$\partial \cdot \frac{d\zeta}{dt} = -\frac{15}{2} \frac{n'^4}{n^3} \frac{d.A^2}{d.e'^2} \delta.e'^2 - 6 \frac{n'^4}{n^3} A^2 - 14 \frac{n'^4}{n^3} A^2.$$

Now the constant part of this value of $\partial \cdot \frac{d\zeta}{dt}$ goes to form part of the constant n_0 , hence, desiring to retain only the varying part, we may write $\frac{d.A^2}{d.e'^2} \delta.e'^2$ for A^2 , and thus obtain

$$\partial \cdot \frac{d\zeta}{dt} = -\frac{55}{2} \frac{n'^4}{n^3} \frac{d.A^2}{d.e'^2} \delta.e'^2. \quad (1)$$

In the next place let us consider the terms in R multiplied by e ; they are all of the form

$$n'^2 a^2 e A \cos(\omega + k\zeta + \phi),$$

where A and ϕ possess the same quality as before, and k may be -3 , -1 or 1 . Representing $\omega + k\zeta + \phi$ by θ , the equations determining the elements are, regard being had to this term alone,

$$\begin{aligned} \frac{dn}{dt} &= 3kn'^2 A e \sin \theta, & \frac{de}{dt} &= \frac{n'^2}{n} A \sin \theta, \\ \frac{d\zeta}{dt} &= n - \frac{7}{2} \frac{n'^2}{n} A e \cos \theta, & \frac{d\theta}{dt} &= \frac{n'^2}{n} A \frac{1}{e} \cos \theta. \end{aligned}$$

In the last equation we have written only the term divided by e , since this alone can produce terms in $\partial \cdot \frac{d\zeta}{dt}$ of the kind we seek. Integrating the last two as we integrated in the former case, we obtain

$$\begin{aligned} \delta e &= -\frac{1}{k} \frac{n'^2}{n^2} A \cos \theta + \frac{1}{k^2} \frac{n'^2}{n^3} \frac{d.A^2}{d.e'^2} \frac{d.e'^2}{dt} \sin \theta, \\ \delta \theta &= \frac{1}{ke} \frac{n'^2}{n^2} A \sin \theta + \frac{1}{k^2 e} \frac{n'^2}{n^3} \frac{d.A^2}{d.e'^2} \frac{d.e'^2}{dt} \cos \theta. \end{aligned}$$

Augmenting, in the expression for $\frac{dn}{dt}$, e and θ by these quantities, we obtain, regard being had only to the terms which are independent of sines or cosines of angles,

$$\delta \cdot \frac{dn}{dt} = \frac{3}{2k} \frac{n'^4}{n^3} \frac{d.A^2}{d.e'^2} \frac{d.e'^2}{dt}.$$

Integrating this we have

$$\delta n = \frac{3}{2k} \frac{n'^4}{n^3} \frac{d.A^2}{d.e'^2} \delta.e'^2.$$

Increasing in the expression for $\frac{d\zeta}{dt}$, the elements n , e and θ by their variations δn , δe and $\delta \theta$, and preserving only the terms independent of the sines or cosines of angles, we get

$$\delta \cdot \frac{d\zeta}{dt} = \frac{3}{2k} \frac{n'^4}{n^3} \frac{d.A^2}{d.e'^2} \delta.e'^2 + \frac{7}{2k} \frac{n'^4}{n^3} A^2.$$

In like manner as before, rejecting the constant part of this which coalesces with n_0 , we obtain

$$\delta \cdot \frac{d\zeta}{dt} = \frac{5}{k} \frac{n'^4}{n^3} \frac{d.A^2}{d.e'^2} \delta.e'^2. \quad (2)$$

When formulæ (1) and (2) are applied to all the terms of R to which each is applicable and the results added we shall have the complete value of $\delta \cdot \frac{d\zeta}{dt}$, since it is plain that the combination of two different terms in R will

always produce terms in $\delta \cdot \frac{d\zeta}{dt}$ involving the sines or cosines of angles.

Denoting the mean anomalies of the moon and sun by ξ and ξ' , and the mean angular distance of these bodies by τ , the part of aR which is independent of e may be written

$$aR = A_0 + A_1 \cos 2\tau + A_2 \cos \xi' + A_3 \cos (2\tau - \xi') + A_4 \cos (2\tau + \xi').$$

Formula (1), applied to this series, gives

$$\delta \cdot \frac{d\zeta}{dt} = -\frac{55}{2} n \frac{d.(A_1^2 + A_3^2 + A_4^2)}{d.e'^2} \delta.e'^2.$$

We can obtain the terms in R multiplied by e from the series just given by using the equation

$$\begin{aligned} \frac{dR}{de} &= r \frac{dR}{dr} \frac{d.\log r}{de} + \frac{dR}{d\lambda} \frac{d\lambda}{de} \\ &= -2R \cos \xi + 2 \frac{dR}{d\tau} \sin \xi. \end{aligned}$$

Whence

$$\begin{aligned} \frac{d.aR}{de} &= -2A_0 \cos \xi - 3A_1 \cos(2\tau - \xi) + A_1 \cos(2\tau + \xi) \\ &\quad - A_2 \cos(\xi - \xi') - A_2 \cos(\xi + \xi') \\ &\quad - 3A_3 \cos(2\tau - \xi' - \xi) + A_3 \cos(2\tau - \xi' + \xi) \\ &\quad - 3A_4 \cos(2\tau + \xi' - \xi) + A_4 \cos(2\tau + \xi' + \xi). \end{aligned}$$

Applying formula (2) to this series gives

$$\begin{aligned}\partial \cdot \frac{d\zeta}{dt} &= n \frac{d}{d.e'^2} [-5(4A_0^2 + A_2^2 + A_2^2) - \frac{5}{3}(A_1^2 + A_3^2 + A_4^2) \\ &\quad + 5(9A_1^2 + 9A_3^2 + 9A_4^2)] \partial \cdot e'^2 \\ &= n \frac{d}{d.e'^2} \left[\frac{130}{3}(A_1^2 + A_3^2 + A_4^2) - 10(2A_0^2 + A_2^2) \right] \partial \cdot e'^2.\end{aligned}$$

Adding this to the expression given by formula (1),

$$\partial \cdot \frac{d\zeta}{dt} = n \frac{d}{d.e'^2} \left[\frac{95}{6}(A_1^2 + A_3^2 + A_4^2) - 10(2A_0^2 + A_2^2) \right] \partial \cdot e'^2.$$

But, denoting the constant term of $a^2 R^2$ by K , we have

$$K = A_0^2 + \frac{1}{2}(A_1^2 + A_2^2 + A_3^2 + A_4^2).$$

Or

$$A_1^2 + A_3^2 + A_4^2 = 2K - (2A_0^2 + A_2^2),$$

and

$$\partial \cdot \frac{d\zeta}{dt} = n \frac{d}{d.e'^2} \left[\frac{25}{3}K - \frac{155}{6}(2A_0^2 + A_2^2) \right] \partial \cdot e'^2.$$

But we have

$$a^2 R^2 = \frac{1}{16} \frac{n'^4}{n^4} \frac{a'^6}{r^6} \left[\frac{1}{2} + 6 \cos 2(\lambda - \lambda') + \frac{3}{2} \cos 4(\lambda - \lambda') \right],$$

and hence K is equal to the constant term of $\frac{11}{32} \frac{n'^4}{n^4} \frac{a'^6}{r^6}$. In consequence,

denoting the constant term of $\frac{a'^6}{r^6}$ by L , we shall have

$$\partial \cdot \frac{d\zeta}{dt} = n \frac{d}{d.e'^2} \left[\frac{1045}{96} \frac{n'^4}{n^4} L - \frac{155}{6}(2A_0^2 + A_2^2) \right] \partial \cdot e'^2.$$

Also we evidently have

$$\frac{1}{4} \frac{n'^2}{n^2} \frac{a'^3}{r'^3} = A_0 + A_2 \cos \xi',$$

and thus

$$\frac{1}{8} \frac{n'^4}{n^4} L = 2A_0^2 + A_2^2.$$

Substituting this value

$$\partial \cdot \frac{d\zeta}{dt} = n \frac{d}{d.e'^2} \left[\frac{245}{32} \frac{n'^4}{n^4} L \right] \partial \cdot e'^2.$$

But the constant term of $\frac{a'^6}{r^6}$ is known to be $1 + \frac{15}{2}e'^2$; hence, in fine,

$$\partial \cdot \frac{d\zeta}{dt} = \frac{3675}{64} \frac{n'^4}{n^3} \partial \cdot e'^2.$$

To obtain $\frac{d\zeta}{dt}$ we must add to n both this term and that which arises, in

the first approximation, from the term $-\frac{2na^2 dR}{\mu da}$ in the differential equation

for $\frac{d\zeta}{dt}$, which is therefore equal to the constant term of $-\frac{n'^2}{n} \frac{a'^2}{r'^2}$, that is, to

$$-\frac{n'^2}{n} \left(1 + \frac{3}{2} e'^2\right). \text{ Thus}$$

$$\frac{d\zeta}{dt} = n - \frac{n'^2}{n} \left(1 + \frac{3}{2} e'^2\right) - \left(\frac{3}{2} \frac{n'^2}{n} - \frac{3675}{64} \frac{n'^4}{n^3}\right) \delta \cdot e'^2.$$

We could have added to the first two terms of this equation a term $B \frac{n'^4}{n^3} e'^2$, where B is a numerical coefficient, equal to the aggregate of the constants we have virtually neglected whenever we wrote $\delta \cdot e'^2$ for e'^2 , but it will be easily seen that this would not change the final result. We evidently have

$$n_0 = n - \frac{n'^2}{n} \left(1 + \frac{3}{2} e'^2\right).$$

From which, to a sufficient degree of approximation,

$$n = n_0 + \frac{n'^2}{n_0}.$$

Substituting this value of n , we get

$$\frac{d\zeta}{dt} = n_0 \left[1 + \left(\frac{3}{2} \frac{n'^2}{n_0^2} - \frac{3771}{64} \frac{n'^4}{n_0^4} \right) (e'^2 - e^2) \right].$$

DISCUSSION OF THE GENERAL EQUATION OF THE THIRD DEGREE.

BY JOHN BORDEN, CHICAGO, ILL.

$$x^3 + Ax^2 + Bx + C = y, \tag{1}$$

$$3x^2 + 2Ax + B = \frac{dy}{dx}, \tag{2}$$

$$6x + 2A = \frac{d^2y}{dx^2}. \tag{3}$$

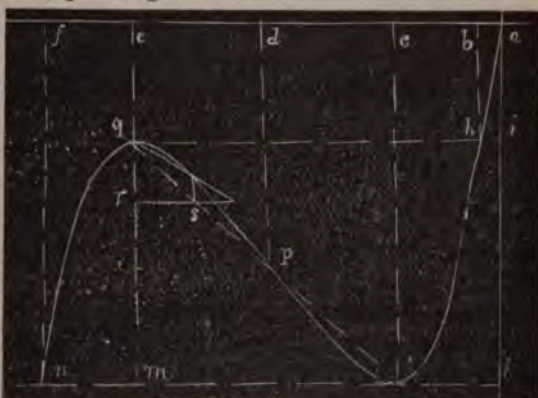
$$\text{If } \frac{dy}{dx} = 0, \text{ then, in (2),} \quad x = -\frac{A}{3} \pm \sqrt{\left(-\frac{B}{3} + \frac{A^2}{9}\right)}; \tag{4}$$

$$\text{and if } \frac{d^2y}{dx^2} = 0, \text{ then, in (3),} \quad x = -\frac{A}{3}. \tag{5}$$

First Case; A and B positive and $B < \frac{1}{3}A^2$.—The form of the curve is as in the Fig.; its locus being taken for $C = 0$.

1. The curve has an inflexion at p ; $ad = -\frac{1}{3}A$. And from p to the right the curve is concave to the line cl , from p to the left it is concave to the line qm . Revolve the left hand branch on the point p and it will coincide with the right hand branch, q falling on l .

2. The form of the curve is not dependent on the term C . Move the curve up and down so that the points q and l follow the lines em & cl and C will vary. For C positive all the roots are negative; and for C negative, one of the roots is positive. When q coincides with e a case of equal roots occurs, the third root being ab . When l coincides with c another case of equal roots occurs, the third root being af . When cl and eq are both negative the only real root refers to the curve from h upward. When cl and eq are both positive the only real root refers to the curve from n downward.



3. The values of x , in (4), are ac and ae . Substitute these values in (1) and cl and eq are found for any assumed value of C . If the results give the same sign there is only one real root, if different signs, there are three real roots. If the values of x in (4) are real, and when substituted in (1) give contrary signs, there are three real roots, otherwise only one real root.

4. To find ab , substitute ae in (1) and make C such as to reduce y to zero; the third root is ab , so, to find af , substitute ac in (1) and make C such as to reduce y to zero, and the third root is af . Or deduct twice ae from A and the result is ab , so also, deduct twice ac from A and the result is af .

5. When all the roots are real, one lies between ab and ac , one between ac and ae and one between ae and af .

To find the middle root, suppose C in (1) to have such value that rs will coincide with the axis of x , then $ae - rs$ would be the first approximation, and $rs = (ml \times qr) \div qm$, all of which are known, for qr is what eq becomes in such case. The line rs , if above the point of inflexion, is too little as the figure shows. The tangent to the curve at the point where the ordinate at s cuts the curve makes an angle with rs whose tangent is found by substituting $ae - rs$ for x in (2). The vertical line of the triangle is the value of y in (1) when $ae - rs$ is substituted for x in (1). Hence, by proportion

we reach a formula for the base of the triangle which is the same as Horner's rule. This, if added to rs , evidently makes it too much when rs is above the point of inflexion. An approximation for the real root between ac and ab , and for the one between ae and af can be found by proportion. The curve is distorted in the figure to show the curvature, it in fact approaches more nearly to a straight line.

If C is positive and cl and eq are both negative, the only real root lies between zero and ab , which can be found in the same manner. If C is negative and cl and eq are both negative, the only real root lies between zero and $-C \div B$. If C is positive and cl and eq are both positive, then the only real root is greater than af .

To find the other limit of this root, substitute af for x in (2) and we thus find the tangent of the angle which the tangent to the curve at the point n makes with the axis of abscissas. The curve in such cases cuts the axis of x to the right of the point where this tangent cuts it; and by preportion we find, when the tangent cuts that axis, the vertical of the triangle whose base we seek $= C - qm$.

Secod Case; A negative, B positiv, and $B < \frac{1}{3}A^2$.—In this case substitute $-x$ for x , and we have the previous case.

Third Case; B negative.—In this case the axis of y lies between cl and eq .

Fourth Case; B positive and $B > \frac{1}{3}A^2$.—In this case the values of x in (2) are imaginary. The points q and l have met at the point of inflexion and the curve from p to the right is concave to the line dp , and from p to the left it is concave to the extension of that line below the point p . There can be only one real root, whose locus can be fixed in the manner above indicated.

The higher equations can be treated in the same way. Take the biquadratic

$$x^4 + Ax^3 + Bx^2 + Cx + D = y, \quad (6)$$

$$4x^3 + 3Ax^2 + 2Bx + C = \frac{dy}{dx}, \quad (7)$$

$$12x^2 + 6Ax + 2B = \frac{d^2y}{dx^2}. \quad (8)$$

If (7), for $\frac{dy}{dx} = 0$, has only one real root, then (6) cannot have more than two real roots. But if $\frac{dy}{dx} = 0$ gives three real values for x in (7) which when substituted in (6) gives consecutively contrary signs, then equation (6) has four roots for $y = 0$.

PROPOSITION IN TRANSVERSALS.

BY PROF. E. W. HYDE, CINCINNATI UNIVERSITY, CIN., OHIO.

LET any transversal OY be drawn to the triangle ABC , cutting its sides in P_1, P_2 and P_3 : from any point, O , on the transversal, lay off distances OP', OP'' and OP''' such that $OP_1 \times OP' = OP_2 \times OP'' = OP_3 \times OP''' = \overline{OD}^2 = c^2$, say: also lay off distances $OQ',$ &c., in the opposite direction from O , such that $OP_1 \times OQ' = OP_2 \times OQ'' = OP_3 \times OQ''' = c^2$: then $P'A, P''B$ and $P'''C$ will meet in a point O' , and $Q'A, Q''B$ and $Q'''C$ will also meet in a point, O'' .

The distance OD may be of any length.

Through O , draw OX perpendicular to OY and take these lines for the axes of reference. The equations of BC, CA and AB will be

$$y = m_1x + b_1, \quad (1)$$

$$y = m_2x + b_2, \quad (2)$$

$$y = m_3x + b_3, \quad (3)$$

so that $b_1 = OP_1, b_2 = OP_2$ and $b_3 = OP_3$. The equations of lines through A, B and C will be

$$y - m_2x - b_2 = k_1(y - m_3x - b_3), \quad (4)$$

$$y - m_3x - b_3 = k_2(y - m_1x - b_1), \quad (5)$$

$$y - m_1x - b_1 = k_3(y - m_2x - b_2). \quad (6)$$

Now if these lines are to be AP', BP'' &c. of the figure, we must have their intercepts on Y equal respectively to $\frac{-c^2}{b_1}, \frac{-c^2}{b_2}$ and $\frac{c^2}{b_3}$; thus from (4) we shall have

$$\frac{b_2 - k_1 b_3}{1 - k_1} = \frac{-c^2}{b_1}, \text{ whence } k_1 = \frac{c^2 + b_1 b_2}{c^2 + b_3 b_1}.$$

To obtain the lines AQ', BQ'' &c., we should have to make the intercepts equal to $\frac{c^2}{b_1}, \frac{c^2}{b_2}$ and $\frac{c^2}{b_3}$ which will give $k_1 = \frac{c^2 - b_1 b_2}{c^2 - b_1 b_3}$. If we write



$$k_1 = \frac{\pm c^2 - b_1 b_2}{\pm c^2 - b_3 b_1}$$

we shall cover both cases at once. Substituting this value of k_1 and values of k_2 and k_3 similarly obtained, in (4), (5) and (6), we have

$$[\pm c^2(m_1 - m_2) + b_3(m_2 b_1 - m_1 b_2)]x - b_3(b_1 - b_2)y \pm c^2(b_1 - b_2) = 0, \quad (7)$$

$$[\pm c^2(m_2 - m_3) + b_1(m_3 b_2 - m_2 b_3)]x - b_1(b_2 - b_3)y \pm c^2(b_2 - b_3) = 0, \quad (8)$$

$$[\pm c^2(m_3 - m_1) + b_2(m_1 b_3 - m_3 b_1)]x - b_2(b_3 - b_1)y \pm c^2(b_3 - b_1) = 0. \quad (9)$$

If (7), (8) and (9) be added they vanish identically, showing that the three lines represented by them pass through a single point. Thus the proposition is proved.

The following is a special case of this proposition.

Let the triangle ABC be inscribed in a conic: take any point P as a pole, and find its polar, say L : from the points P_1, P_2, P_3 , where the sides of ABC cut L , draw lines through P , and find their poles, which will of course fall on L : call these poles P', P'' and P''' : then $P'A, P''B$ and $P'''C$ meet in a point.

To prove this, draw the diameter through P , cutting L in O ; then we have only to show that $OP_1 \times OP' = OP_2 \times OP'' = OP_3 \times OP''' = a$ constant.

L is parallel to the diameter conjugate to OP ; referring the conic to this pair of conjugate diameters, let its equation be

$$\frac{x^2}{a_1^2} + \frac{y^2}{\pm b_1^2} = 1; \quad (10)$$

then that of its polar for the point (x_1, y_1) is

$$\frac{xx_1}{a_1^2} + \frac{yy_1}{\pm b_1^2} = 1. \quad (11)$$

For the point $P, y_1 = 0$ and $x_1 = a$, say; then

$$x = \frac{a_1^2}{a}. \quad (12)$$

Take any line through P , as

$$y = m(x - a). \quad (13)$$

To find the pole of this line compare (13) with (11), the conditions that they shall represent the same line are

$$m = -\frac{\pm b_1^2 x_1}{a_1^2 y_1} \text{ and } -ma = \frac{\pm b_1^2}{y_1}.$$

These conditions give

$$x_1 = \frac{a_1^2}{a}, \quad y_1 = \frac{\pm b_1^2}{-ma}.$$

Now let us find the portion of L intercepted between the line of (13) and OP ; $\therefore$ in (13) let $x = a_1^2 \div a$, and call the intercept y_2 , therefore

$$y_2 = m \left(\frac{a_1^2}{a} - a \right) = m \frac{a_1^2 - a^2}{a}.$$

Multiply together the values of y_1 and y_2 , therefore

$$y_1 y_2 = -\frac{\pm b_1^2}{a^2} (a_1^2 - a^2), \text{ a constant.}$$

Hence $OP_1 \times OP' = OP_2 \times OP'' = OP_3 \times OP''' = \text{a constant.}$ This proves the proposition for the hyperbola and ellipse, and it may likewise be easily proved for the parabola. If P be a focus, of course L is a directrix.

If we eliminate c between any two of equations (7), (8), (9), we shall find the equation of the locus of O' and O'' when c varies. Solving for $\pm c^2$, and placing for brevity $b_1 - b_2 = \beta_3$, $b_2 - b_3 = \beta_1$, $b_3 - b_1 = \beta_2$, $m_1 - m_2 = \mu_3$, &c., and $m_2 b_1 - m_1 b_2 = \delta_3$ &c., we have

$$\pm c^2 = \frac{b_3(\beta_3 y - \delta_3 x)}{\beta_3 + \mu_3 x} = \frac{b_1(\beta_1 y - \delta_1 x)}{\beta_1 + \mu_1 x} = \frac{b_2(\beta_2 y - \delta_2 x)}{\beta_2 + \mu_2 x}.$$

Taking either pair of these values of $\pm c^2$ we shall arrive at the same result after expansion and collection of terms, viz.;

$$(m_1 b_1 \delta_1 + m_2 b_2 \delta_2 + m_3 b_3 \delta_3) x^2 - (\mu_1 b_2 b_3 + \mu_2 b_3 b_1 + \mu_3 b_1 b_2) xy + (b_1^2 \delta_1 + b_2^2 \delta_2 + b_3^2 \delta_3) x + \beta_1 \beta_2 \beta_3 y = 0. \quad (14)$$

This equation is perfectly symmetrical with respect to the three sides of the triangle, as it should be; and it represents a hyperbola. If we substitute A, B, C and D for the coefficients, so that the equation becomes

$$Ax^2 + Bxy + Cx + Dy = 0,$$

we find for the equations of the asymptotes

$$x = -\frac{D}{B}, \quad y = -\frac{A}{B}x + \frac{AD - BC}{B^2};$$

hence the coordinates of the center are

$$x = -\frac{D}{B} \text{ and } y = \frac{2AD - BC}{B^2}.$$

If the co-ordinates of the vertices of the triangle ABC which are respectively

$$x_1 = -\frac{\beta_1}{\mu_1}, \quad y_1 = -\frac{\delta_1}{\mu_1}, \quad x_2 = -\frac{\beta_2}{\mu_2}, \quad y_2 = -\frac{\delta_2}{\mu_2}, \text{ etc.,}$$

be substituted in the equation of the hyperbola, they will each be found to satisfy it; hence the hyperbola passes through the vertices of the triangle.

ON SOME PROPERTIES OF FOUR CIRCLES INSCRIBED IN ONE AND CIRCUMSCRIBED ABOUT ANOTHER.

BY CHRISTINE LADD, UNION SPRINGS, N. Y.

LET the circles A, B, C, D be tangent to the circle R internally and to the circle I externally.

I. The circles A, B, C, D have a common axis of similitude; for, R and I being a pair of tangent circles, their radical axis is coincident with the axis of similitude of any three, and therefore all of the circles A, B, C, D . (Salmon's *Conic Sections*, page 109, note.)

If E, F, G, H are the points of contact of these circles with each other, then HE and GF intersect in S' , the external centre of similitude of the circles A and C , and HG & EF intersect in S'' , the external centre of similitude of D and B .

II. The circles A, B, C, D have a common radical centre; for the perpendiculars to the sides of the quadrilateral $ABCD$ at the points E, F, G, H intersect, each two, on the line RI (Salmon, loc. cit.) and therefore all four in the same point, O . The common orthogonal circle, O , is then inscribed in $ABCD$, touching it in the points E, F, G, H .

III. Since tangents to A and D , at the points H and F respectively, meet on the radical axis of A and D (namely, in the point O), it follows that H and F are antihomologous points on the circles D and A , and hence the line HF passes through S''' . In the same way it may be shown that GE passes through S'' . This is the essential quality of the quadrilateral $ABCD$ and $EFGH$.



IV. The diagonals of $ABCD$ and $EFGH$ meet in a point, V , the polar of $S'S''$ and therefore on the line RI (Booth, *New Geometrical Methods*, Vol. II., Art. 245). This includes Problem 187 of the ANALYST.

V. If $KLMN$ is the quadrilateral formed by joining the points of contact on the circle R , its sides and diagonals pass through the points already determined on the line $S'S''$; for N and K being centres of similitude of the circles A , D and R , NK passes through the external centre of similitude of A and D . The same is true for the corresponding circumscribed quadrilateral, $K'L'M'N'$, and the diagonals of these two quadrilaterals meet in a point, Q , which with the points $S'S''$ forms a triangle self-conjugate with respect to the circle R . In the same way it may be shown that $S'VS''$ is self-conjugate with respect to the circle I .

VI. The points S'' , S' , S''' , S^{iv} form an harmonic range (Catalan, *Théorèmes et Problèmes*, III., 49).

VII. Since $BG = BL$ and $CG = CM$, L' is the centre of an exscribed circle to the triangle BRC and that circle is tangent to CB at G .

VIII. The circles about L' , M' , N' , K' as centres, and tangent to the sides of $ABCD$, form another set of four circles circumscribed about one, the circle O , and inscriptible in another, since the quadrilaterals $L'M'N'K'$ and $LMNK$ have the diagonals of each passing through the intersections of sides of the other, and are corresponding circumscribed and inscribed quadrilaterals to the circle R (Proposition III).

The points A , B , C , D are evidently on an ellipse to which the points R and I are foci. They are concyclic only when O and V (and therefore R and I) coincide, for only then is VS'' , or HF , perpendicular to EG .

IX. To inscribe in a given circle four circles tangent to each other and to another circle:—

Take any triangle, as $S'QS^{iv}$, self-conjugate with respect to the given circle, R . At K , L , M , N , the points where its sides cut the circle, R , draw tangents to R , and with their intersections as centres describe arcs orthogonal to R . From the points where the diagonals of $KLMN$ meet $S'S^{iv}$ draw tangents to these arcs; their intersections will be the centres of the required circles. The proof of this construction readily follows from the properties already obtained.

CUBIC EQUATIONS.

BY HENRY HEATON, SABULA, IOWA.

If a and b be supposed to be either positive or negative, it is well known that every cubic equation may be reduced to the form

$$x^3 - 3ax = 2b. \quad (1)$$

Putting $x = \sqrt[3]{a}y$ and dividing by $\sqrt[3]{a}$ we get

$$y^3 - 3y = \frac{2b}{\sqrt[3]{a}} = 2c. \quad (2)$$

Putting $y = e^{\theta}\sqrt[3]{-1} + e^{-\theta}\sqrt[3]{-1} = 2\cos\theta$, this becomes

$$e^{3\theta}\sqrt[3]{-1} + e^{-3\theta}\sqrt[3]{-1} = 2\cos 3\theta = 2c. \quad (3)$$

$\therefore \theta = \frac{1}{3}\cos^{-1}c$, $y = 2\cos(\frac{1}{3}\cos^{-1}c)$, and $x = 2\sqrt[3]{a}\cos(\frac{1}{3}\cos^{-1}c)$,
a general expression for the roots of equation (1).

If we restrict $\cos^{-1}c$ to any one of its values the three roots are

$$\begin{aligned} x_1 &= 2\sqrt[3]{a}.\cos(\frac{1}{3}\cos^{-1}c), \\ x_2 &= 2\sqrt[3]{a}.\cos(\frac{2}{3}\pi + \frac{1}{3}\cos^{-1}c), \\ x_3 &= 2\sqrt[3]{a}.\cos(\frac{4}{3}\pi + \frac{1}{3}\cos^{-1}c). \end{aligned}$$

If $a^3 > b^2$, c is less than unity and the three roots are real, and their values may be readily found from a table of cosines.

If $a^3 < b^2$, c is greater than unity and consequently $\cos^{-1}c$ is imaginary, yet the value of x_1 is real and might readily be found from such a table as the one given on p. 37, Vol. II, ANALYST, if it were sufficiently extended. Put $n = \cos(\frac{1}{3}\cos^{-1}c)$. Then is

$$\begin{aligned} x_1 &= 2n\sqrt[3]{a}, \\ x_2 &= -2\sqrt[3]{a}.\left\{ \frac{1}{2}n + \frac{1}{2}\sqrt[3]{3(1-n^2)} \right\}, \\ x_3 &= -2\sqrt[3]{a}.\left\{ \frac{1}{2}n - \frac{1}{2}\sqrt[3]{3(1-n^2)} \right\}. \end{aligned}$$

Since n is greater than unity, x_2 and x_3 are imaginary.

If a be negative c is imaginary and the expressions for the roots seem to be, but we shall see that one of them is real.

Because $\cos^{-1}c = \frac{1}{2}\pi - \sin^{-1}c$,

$$\begin{aligned} x_1 &= 2\sqrt[3]{a}.\cos(\frac{1}{6}\pi - \frac{1}{3}\sin^{-1}c) = 2\sqrt[3]{a}.\left[\frac{1}{2}\sqrt[3]{(3)}.\cos(\frac{1}{3}\sin^{-1}c) + \frac{1}{2}\sin(\frac{1}{3}\sin^{-1}c)\right], \\ x_2 &= 2\sqrt[3]{a}.\cos(\frac{5}{6}\pi - \frac{1}{3}\sin^{-1}c) = -2\sqrt[3]{a}.\left[\frac{1}{2}\sqrt[3]{(3)}.\cos(\frac{1}{3}\sin^{-1}c) - \frac{1}{2}\sin(\frac{1}{3}\sin^{-1}c)\right], \\ x_3 &= 2\sqrt[3]{a}.\cos(\frac{3}{2}\pi - \frac{1}{3}\sin^{-1}c) = -2\sqrt[3]{a}.\sin(\frac{1}{3}\sin^{-1}c). \end{aligned}$$

The expression for x_3 is real and its value might readily be obtained from the table above referred to.

SOLUTION OF AN INDETERMINATE PROBLEM.

BY DR. DAVID S. HART, STONINGTON, CONN.

To find general values of x and y to solve the problem

$$x^2 - Ay^2 = -1.$$

Let A be a non-quadrade number = the sum of two squares = $r^2 + s^2$,

then we shall have

$$x^2 - (r^2 + s^2)y^2 = -1,$$

and by transposition,

$$x^2 - r^2y^2 = s^2y^2 - 1 = (sy-1)(sy+1);$$

whence

$$x^2 = r^2y^2 + (sy-1)(sy+1) = \square; \therefore x = [r^2y^2 + (sy-1)(sy+1)]^{\frac{1}{2}}.$$

Let $r^2y^2 + (sy-1)(sy+1) = [ry - (m \div n)(sy-1)]^2$. Reducing this we get

$$y = \frac{m^2 + n^2}{sm^2 - 2nrm - sn^2}.$$

Here, in order to have y integral, put $sm^2 - 2nrm - sn^2 = \pm 1$; whence, transposing and dividing by s , we have

$$m^2 - \frac{2nr}{s}m = \frac{sn^2 \pm 1}{s},$$

and by quadratics

$$m = \frac{rn \pm \sqrt{[(r^2 + s^2)n^2 \pm s]}}{s}. \therefore y = \pm (m^2 + n^2).$$

In the general value of m , r and s may represent any numbers one of which is even, and the other any odd number except 1, and n can be found by trial, or, if large, by the solution of the formula $P^2 - (r^2 + s^2)n^2 = \pm s$. (See Barlow's Theory of Numbers, prop. XI, p. 302.)

Having found the values of x , y as above, we can find x , y in the equation $x^2 - Ay^2 = 1$ by putting $x = 2p^2 + 1$ and $y = 2pq$, p , q , being the values of x , y , already found. (See Vol. XX, p. 64 of Reprint from Ed. Times.)

Let $r = 2$, $s = 3$, $n = 1$, then $m = 2$, using the plus signs. Whence, to solve $x^2 - 13y^2 = -1$, $x = 18$, $y = 5$; to solve $x^2 - 13y^2 = 1$, $x = 649$, $y = 180$.

Let $r = 2$, $s = 5$, $n = 2$, then $m = 3$, using the plus signs. Whence, to solve $x^2 - 29y^2 = -1$, $x = 70$, $y = 13$; to solve $x^2 - 29y^2 = 1$, $x = 9801$, $y = 1820$.

Let $r = 8$, $s = 3$, $n = 2$, then $m = 11$, using the second minus sign. To solve $x^2 - 73y^2 = -1$, $x = 1068$, $y = 125$; to solve $x^2 - 73y^2 = 1$, $x = 2281249$ and $y = 267000$.

Let $r = 6$, $s = 5$, then $m = \frac{1}{5}[6n \pm \sqrt{61n^2 + 5}]$, and $P^2 - 61n^2 = 5$. Developing $\sqrt{61}$ into a continued fraction, we find the partial quotients, 7, 1, 4, 3, 1, 2. Here the complete quotient corresponding to 2 has 5 for its denominator, in an even place, and the converging fractions are $\frac{7}{1}$, $\frac{8}{1}$, $\frac{39}{5}$, $\frac{125}{16}$, $\frac{164}{21}$, $\frac{453}{58}$; $\therefore n = 58$, $P = 453$, $m = -21$, using the minus sign. Hence, to solve $x^2 - 61y^2 = -1$, $x = 29718$, $y = 3805$; and to solve $x^2 - 61y^2 = 1$, $x = 1766319049$ and $y = 226153980$.

SOLUTION OF PROBLEMS IN NUMBER THREE.

SOLUTIONS of problems in No. 3 have been received as follows:

From Marcus Baker, 203, 209; Geo. M. Day, 204, 209; Prof. W. W. Hendrickson, 204; H. Heaton, 202, 207, 208; W. E. Heal, 208, 209, 210; Prof. J. H. Kershner, 202, 203, 204, 205, 208, 209; Prof. H. T. J. Ludwick, 204; Prof. D. J. Mc Adam, 205, 209; W. V. Mc Knight, 202, 203, 205, 209; Artemas Martin, 203, 204, 205, 208, 209, 210; P. Richardson, 209; E. B. Seitz, 202, 203, 204, 205, 207, 208, 209; Prof. J. Scheffer, 202, 203, 205, 206, 208, 209.

By an oversight, in No. 3, we failed to credit Prof. Kershner for a solution of problem 200. His solution was substantially the same as the published solution.

The query by Prof. Mc Adam was answered, in nearly the same way, by the querist, and by Prof. Scheffer, Mr. Mc Knight, and Mr. Seitz.

No answer to the second query on p. 96 has been received.

202. "If $r \div d$ reduces to a repetend, prove that $s \div d = R \div 9$, where s is the sum of the remainders occurring in the division, and R is the sum of the digits of the repetend."

SOLUTION BY PROF. J. SCHEFFER.

Denoting the successive figures of the circulator by $x, y, z, \dots u$, and the successive remainders by $m_1, m_2, m_3, \dots m_n$, we obviously have the relation:

$$\frac{r}{d} = \frac{x + (m_1 \div d)}{10},$$

$$\frac{m_1}{d} = \frac{y + (m_2 \div d)}{10},$$

$$\frac{m_2}{d} = \frac{z + (m_3 \div d)}{10},$$

$$\vdots$$

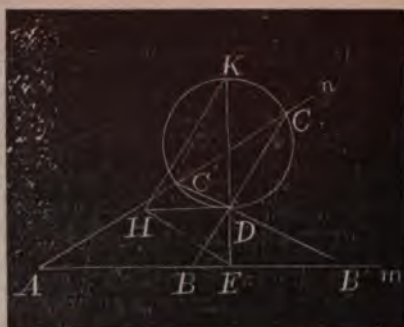
$$\frac{m_n}{d} = \frac{u + (r \div d)}{10}.$$

Adding we have $\frac{s}{d} = \frac{R + (s \div d)}{10}$; whence $\frac{9s}{d} = R$, or $\frac{s}{d} = \frac{R}{9}$.

203. "Two lines Am and An make a given angle at A , and D is a point given in position between Am and An . Draw a straight line BC through D so that $BD \times DC =$ a given quantity, B and C being points in the lines Am and An , respectively."

SOLUTION BY E. B. SEITZ, GREENVILLE, OHIO.

Draw DE perpendicular to Am ; from D draw DH parallel to Am , and let the square of the length of DH be equal to the given quantity. Draw HK perpendicular to HE , meeting ED produced in K ; on DK as diameter describe a circle, cutting Am in C and C' . Through D draw BC and $B'C'$, either of which is the line required.



For, in the similar right angled triangles BDE , CDK and EDH , HDK , we have $BD \times DC = ED \times DK = HD^2 =$ the given quantity. The proof for $B'C'$ is similar.

204. "Find the locus of the intersection of normals to the ellipse which are perpendicular to each other."

SOLUTION BY PROF. W. W. HENDRICKSON, ANNAPOLIS, MD.

The equation to the normal is

$$y = mx - \frac{m(a^2 - b^2)}{\sqrt{(b^2 m^2 + a^2)}}, \quad \text{whence}$$

$$\left. \begin{aligned} m^4 - \frac{2y}{x} m^3 + \frac{b^2 y^2 + a^2 x^2 - (a^2 - b^2)^2}{b^2 x^2} m^2 - \frac{2a^2 y}{b^2 x} m + \frac{a^2 y^2}{b^2 x^2} &= 0, \\ \text{or } m^4 + Pm^3 + Qm^2 + Rm + S &= 0. \end{aligned} \right\} \quad (1)$$

Since the product of two roots of this equation is -1 , one of its quadratic factors may be assumed as $m^2 + Am - 1$, and the other as $m^2 + Bm + C$; then the equation

$$m^4 + (A+B)m^3 + (AB+C-1)m^2 + (AC-B)m - C = 0$$

is identical with (1), hence

$$C = -S, \quad AC - B = R, \quad A + B = P \quad (2)$$

and

$$AB + C - 1 = Q. \quad (3)$$

Substituting in (3) the values of A , B and C , obtained from (2), we find

$$(Q + S + 1)(S - 1)^2 + (P + R)(R + PS) = 0;$$

and, replacing P , Q , R and S by their values, we have, after reduction,

$$(a^2 + b^2)(x^2 + y^2)(a^2 y^2 + b^2 x^2)^2 = (a^2 - b^2)^2 (a^2 y^2 - b^2 x^2)^2.$$

205. "The distance between the centres of two circles is given, and the radius of one, find the radius of the other, when the length of the endless band which passes around them is given."

SOLUTION BY E. B. SEITZ.

Let a equal the distance between the centers of the circles, b equal the length of the endless band, r , the radius of the given circle, x , the unknown radius, and θ , the angle formed by the line joining the centers and the radius of the given circle drawn to the point of contact of the common external tangent. Then we have

$$2a \sin \theta + 2r(\pi - \theta) + 2(r - a \cos \theta) \theta = b, \quad (1)$$

$$x = r - a \cos \theta. \quad (2)$$

From (1)

$$\sin \theta - \theta \cos \theta = (b - 2\pi r) \div 2a. \quad (3)$$

From (3) by means of a table of natural sines and cosines we can find, by the method of approximation, the value of θ , and then from (2) we can find x .

206. "Determine the position of a plane from the measured rectangular coordinates of n points, each measure being equally good."

SOLUTION BY PROF. J. SCHEFFER, COLLEGE OF ST. JAMES, MD.

We denote the co-ordinates of the n given points by $x', y', z'; x'', y'', z''; x''', y''', z''';$ etc.: and the equation of the plane by

$$Ax + By + Cz + D = 0,$$

in which the coefficients A, B, C and D are to be determined by means of the method of least squares. By substituting the above values we have

$$Ax' + By' + Cz' + D = 0,$$

$$Ax'' + By'' + Cz'' + D = 0,$$

$$Ax''' + By''' + Cz''' + D = 0,$$

$$\dots \dots \dots$$

Now we multiply the first of these equations by x' , the second by x'' , the third by x''' , etc.; and add up. Then we again multiply the first by y' , the second by y'' , the third by y''' , etc., and add up. Again multiply the first by z' , the second by z'' , the third by z''' , etc., and add up; and again multiply the first, second, third, etc. by 1, and add up. Whence we get, denoting for the sake of brevity,

$$x'^2 + x'^2 + x''^2 + \dots \text{ by } S(x^2),$$

$$x'y' + x'y'' + x''y''' + \dots \text{ by } S(xy),$$

$$x'z' + x'z'' + x''z''' + \dots \text{ by } S(xz),$$

$$y'z' + y'z'' + y''z''' + \dots \text{ by } S(yz),$$

$$x' + x'' + x''' + \dots \text{ by } S(x),$$

$$y' + y'' + y''' + \dots \text{ by } S(y),$$

$$z' + z'' + z''' + \dots \text{ by } S(z), \text{ the final equations,}$$

$$\begin{aligned} A.S(x^2) + B.S(xy) + C.S(xz) + D.S(x) &= 0, \\ A.S(xy) + B.S(y^2) + C.S(yz) + D.S(y) &= 0, \\ A.S(xz) + B.S(yz) + C.S(z^2) + D.S(z) &= 0, \\ A.S(x) + B.S(y) + C.S(z) + D.n &= 0; \end{aligned}$$

whence, by elimination, we can derive the values of A , B , C and D .

207. "If M , N , P , Q are four random points in the surface of a circle, find the chance that E , the intersection of the straight lines through M , N and P , Q , lies between M , N and between P , Q ."

SOLUTION BY HENRY HEATON, SABULA, IOWA.

Put p = the required probability, and q = the chance that the point N will fall within the triangle MPQ . Then $4q$ = the chance that one of the four points will fall within the triangle formed by the other three.

If no one of the points fall within the triangle formed by the other three, either MN will intersect PQ , MP will intersect NQ or MQ will intersect PN , either of which events is equally probable. Hence $p = \frac{1}{3}(1-4q)$.

In the figure put $CM = x$, $MQ = y$, $MP = z$, $\angle CMQ = \theta$, and $\angle CMP = \phi$. Then the area of the triangle $MPQ = \frac{1}{2}yz \sin(\phi - \theta)$, and



$$\begin{aligned} q &= \frac{\int_0^1 \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \int_0^{2x \cos \theta} \int_0^{2x \cos \phi} \frac{1}{2}zy \sin(\phi - \theta) x dx d\theta d\phi y dy dz}{\int_0^1 \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \int_0^{2x \cos \theta} \int_0^{2x \cos \phi} \pi x dx d\theta d\phi y dy dz} \\ &= \frac{6}{\pi^2} \int_0^1 \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \int_0^{2x \cos \theta} \int_0^{2x \cos \phi} zy \sin(\phi - \theta) x dx d\theta d\phi y dy dz \\ &= \frac{128}{3\pi^3} \int_0^1 \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} x^7 \cos^3 \theta \cos^3 \phi \sin(\phi - \theta) dx d\theta d\phi \\ &= \frac{8}{3\pi^3} \int_0^1 \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} x^7 (6\cos^4 \theta - 2\cos^6 \theta + 6\theta \cos^3 \theta \sin \theta - 3\pi \cos^3 \theta \sin \theta) dx d\theta \\ &= \frac{35}{6\pi^2} \int_0^1 x^7 dx = \frac{35}{48\pi^2}. \quad \text{Hence } p = \frac{1}{3} - \frac{35}{36\pi^2}. \end{aligned}$$

208. "When the tangent, QT , to a fixed parabola is of constant length, and one extremity, T , moves along the axis to the origin, what is the curve described by the other extremity, Q ?"

SOLUTION BY PROF. J. H. KERSHNER, MERCERSBERG, PA.

Let the rectangular coordinates of Q be (x_1, y_1) , of P , (x_2, y_2) , and let l be the length of QT , the given tangent.

By the similar triangles TPR_2 , TQR_1 , $TR_2 : PR_2 :: TR_1 : QR_1$, or

$$2x_2 : y_2 :: l^2 - y_1^2 : y_1^2.$$

Putting $z^2 = l^2 - y_1^2$, squaring, and substituting $2px_2$ for y_2 , we get

$$2x_2 : p :: z^2 : y_1^2, \text{ or } 2z - x_1 : p :: z^2 : y_1^2.$$

$$\therefore xy^2 + pz^2 = 2zy^2,$$

is the equation required, and the locus is one represented by an equation of the 6th degree.

209. " C is the center of a circle of known radius, DB is a diameter produced and AB is an inflexible rod, of known length. One end, A , moves continually in the circumference of the circle while the other end, B , slides to and fro upon DB . Required the equation of the curve described by any point P , in the rod."

SOLUTION BY E. B. S. EITZ.

Draw AN and PM perpendicular to DB , and put $AC = r$, $AB = a$, $BP = b$, $CM = x$, $PM = y$, and $\angle ABC = \theta$. Then

$$x = (a-b) \cos \theta \pm \sqrt{(r^2 - a^2 \sin^2 \theta)}, \quad (1)$$

$$y = b \sin \theta. \quad (2)$$

Substituting the values of $\sin \theta$ and $\cos \theta$ from (2) in (1) we have

$$bx = (a-b) \sqrt{(b^2 - y^2)} \pm \sqrt{(b^2 r^2 - a^2 y^2)},$$

the required equation, the double sign being taken $+$ when A is in the first or fourth quadrant from E , and $-$ when A is in the second or third quadrant. The curve is closed; its area is $\pi br^2 \div a$.

[Mr. Baker, who obtains a similar equation to the above, remarks: "It is evident from this equation without further change that the curve is *not* a conic section. * * * It is readily traced by a simple mechanical device, and the curve so traced resembles a longitudinal section of a hen's egg."]



210. "Find such integral values of x, y and z as will satisfy the conditions

$$x^2 + axy + y^2 = \square,$$

$$x^2 + bxz + z^2 = \square,$$

$$y^2 + cyz + z^2 = \square."$$

SOLUTION BY ARTEMAS MARTIN, M. A., ERIE, PA.

Assume $x = an^2 - 2mn$, $y = m^2 - n^2$, $z = cn^2 + 2mn$; then

$$x^2 + axy + y^2 = (m^2 - amn + n^2)^2,$$

$$y^2 + cyz + z^2 = (m^2 + cmn + n^2)^2.$$

But the second condition is not satisfied by the assumed values of x , y and z .

Let us put $an - 2m = r$, (1)

$$cn + 2m = s; \quad (2)$$

then $x = nr$, $z = ns$, $x^2 + bxz + z^2 = n^2(r^2 + brs + s^2)$.

Assume now $r = t(p^2 - q^2)$, (3)

$$s = t(bq^2 + 2pq), \quad (4)$$

and we have $r^2 + brs + s^2 = t^2(p^2 + bpq + q^2)^2$, and consequently

$$x^2 + bxz + z^2 = n^2 t^2 (p^2 + bpq + q^2)^2.$$

From (1), (2), (3) and (4) we have

$$an - 2m = t(p^2 - q^2), \quad (5)$$

$$cn + 2m = t(bq^2 + 2pq); \quad (6)$$

which give

$$m = \frac{at(bq^2 + 2pq) - ct(p^2 - q^2)}{2(a+c)} = a(bq^2 + 2pq) - c(p^2 - q^2),$$

$$n = \frac{2t(bq^2 + 2pq) + 2t(p^2 - q^2)}{2(a+c)} = 2(bq^2 + 2pq) + 2(p^2 - q^2),$$

if we take $t = 2(a+c)$; where p and q may be any numbers that will give x , y , z all positive or all negative. See *Laybourn's Mathematical Repository*, New Series, Vol. 3, Part I, pp. 151—164, for elaborate general solutions to this problem.

QUERY. "In works on Practical Geometry, the following method of inscribing a regular polygon of n sides is given:

Let AB be a diameter of a circle. With A and B as centers and radius AB describe arcs intersecting in D . Divide AB into n equal parts. Draw DE through the second point of division; the arc AE is one- n th part of the circumference. Is there a general demonstration published?"

ANSWER BY E. B. SEITZ.

The method is strictly correct only for regular polygons of 3, 4 and 6 sides, which is shown as follows:

Let C be the center of the circle, F the second point of division; draw EM perpendicular to DC produced. Let $AC=r$, $CM=x$, and $\angle ACE=\phi$.

Then $CD = r\sqrt{3}$, $AF = 4r \div n$, $FC = r - (4r \div n)$, and $r\sqrt{3} : x + r\sqrt{3} :: r - (4r \div n) : EM = (n-4)(x\sqrt{3} + 3r) \div 3n$; therefore

$$x^2 + \frac{(n-4)^2(x\sqrt{3} + 3r)^2}{9n^2} = r^2, \quad \text{whence}$$

$$x = \frac{nr \sqrt{[3(n^2 + 16n - 32)] - (n-4)^2 r \sqrt{3}}}{4(n^2 - 2n + 4)},$$

and $\varphi = \sin^{-1} \left(\frac{x}{r} \right).$

The following table shows the values of φ from $n=3$ to $n=10$, together with their errors.

n .	φ .	Error.
3	120° 00' 00''	6' 00''
4	90 00 00	0 00
5	71 57 12	2 48
6	60 00 00	0 00
7	51 31 05	5 22
8	45 11 15	11 15
9	40 16 39	16 39
10	36 21 21	21 21.

[By request of Mr. Adcock, who contends that the published solution of problem 195 is erroneous, we insert the following solution of that problem by Mr. Adcock. From a hasty examination of both solutions we have not been able to perceive any conflict between them: Mr. Eastwood has determined the normal pressure at both ends of the rod, while Mr. Adcock finds the locus of the upper end.]

Let l = length of the rod RD resting on the horizontal plane HBR at R and against the vertical plane HBK at D . $HB = x$, $HR = y$, $BD = z$, then $x^2 + y^2 + z^2 = l^2$ (1)

Through R and in the vertical plane RBD draw RFK making the angle $CRF = \tan^{-1}(1 \div m)$, and through G , the centre of gravity of the rod, draw the vertical line CGF , join FD , draw FL parallel to RB , and FF' parallel to RH . Then, from the figure, $RC = CB = FL = \frac{1}{2}\sqrt{(x^2 + y^2)}$, $CG = \frac{1}{2}BD = \frac{1}{2}z$, $CF = \sqrt{(x^2 + y^2) \div 2m}$, $RF = FK = (x^2 + y^2)^{\frac{1}{2}} \times (1 + m^2)^{\frac{1}{2}} \div 2m$, $2FG = KD = \sqrt{(x^2 + y^2)} \div m - z$, $FD = \sqrt{[(FC - BD)^2 + BC^2]} = \sqrt{\left\{ [\sqrt{(x^2 + y^2) \div 2m} - z]^2 + \frac{1}{4}(x^2 + y^2) \right\}}$,



$$FF' = \frac{1}{2}z, F'D = \sqrt{(FD^2 + FF'^2)} = \sqrt{\left\{ \left[\sqrt{(x^2 + y^2) \div 2m - z} \right]^2 + \frac{1}{4}x^2 \right\}}.$$

Now if $2FG = DK$ represent the weight, w , of the rod, FR will be the resultant of pressure FC and friction RC at R , and FF' and $F'D$ will represent the pressure and friction at D . Hence

$$\frac{F'D}{FF'} = \frac{\sqrt{\left\{ \left[\sqrt{(x^2 + y^2) \div 2m - z} \right]^2 + \frac{1}{4}x^2 \right\}}}{\frac{1}{2}z} = m'. \quad (2)$$

Eliminating y from (2) by (1),

$$\frac{\left[\sqrt{(l^2 - z^2) \div 2m - z} \right]^2 + \frac{1}{4}x^2}{l^2 - x^2 - z^2} = 4m'^2,$$

which is the equation of the locus of the upper end of the rod in all its positions of equilibrium on the supposition that the lower end is always in the axis of y .

PROBLEMS.

211. BY W. E. HEAL.—Prove that every number is either a triangular number or is the sum of two, or of three triangular numbers.

212. BY E. B. OPDYCKE, PULASKI, OHIO.—One half of a circular tract of land is cut off by an arc of a circle whose center is in the circumference of the circular tract. Find the radius with which the arc is described.

213. BY GEO. H. HARVILL, BONNER, LA.—Upon the three sides of any triangle construct equilateral triangles and join their centers by right lines. Prove that the triangle so formed is equilateral.

214. BY G. SHAW, KEMBLE, ONT., CANADA.—Prove that

$$\frac{1}{\tan A + 1} \frac{1}{\tan A + \&c.} = \frac{1}{2} [\sqrt{(\sec^2 A + 3)} - \tan A].$$

215. BY PROF. BEMAN.—A harbor A is so situated with reference to two headlands B and C , that the angle BAC is a right angle. A ship sails in a course making an angle of 55° with AB , to D , when $DB = DC$; she then sails forward on the same course 15 ms. to E , when BEC is a straight line. Required AB , AC , DB , EB and EC .

216. BY PROF. SCHEFFER. Through three given points to describe the maximum ellipse.

217. BY PROF. W. W. HENDRICKSON.—The hypotenuse of a right triangle is fixed, and squares are described upon the other two sides; it is required to find the equation to the locus of the intersection of two straight

lines, drawn from either extremity of the hyp. to the most distant corner of the square upon the opposite side. Find also the area of the curve.

218. (BY REQUEST)—Required the separate rates of dividend of two insolvent estates connected actually as follows:

John Doe's Estate. Direct liabilities \$33,425.61, also endorsements for Richard Roe \$34,949.16, less primary dividend to be paid from Roe's estate. Net proceeds to be divided \$12,395.76.

Richard Roe's Estate. Direct liabilities \$46,212.00, also endorsements for John Doe \$9,500.00, less primary dividend to be paid from Doe's estate. Net proceeds to be divided \$9,493.25, augmented by dividend on \$4,796.65 account due from John Doe's estate.

219. BY PROF. KERSHNER.—What is the locus generated by the vertex of a right angle, including sides given, sliding between the branches of a given parabola?

220. BY ARTEMAS MARTIN.—Through a given point in the surface of a circle, a chord is drawn at random. If another chord be drawn at right angles to the first, what is the probability that they will intersect?

QUERY. BY GEO. M. DAY, LOCKPORT, N. Y.—It is stated by Prof. Jevons in his "Principles of Science" that Prof. Clifford has found a function which approaches infinity as the variable approaches a certain limit; yet at the limit the function is finite. What is the function?

PUBLICATIONS RECEIVED.

The Observatory, (May and June Nos.) A Monthly Review of Astronomy. Edited by W. H. M. CHRISTIE, M. A. Taylor and Francis, London, England.

The Electric Constitution of Our Solar System. By JACOB ENNIS. 8vo. 19 pp. Philadelphia. 1878.

ERRATA.

On page 93, line 10, the brace, which includes the numerator and denominator of the last term, should only include the numerator; and on same page, lines 10 and 11 from bottom should read as on page 92, lines 2 and 3 from bottom.

On page 102, line 3 from bottom, after =, insert $1 \div ap$.

" " 112, " 16, for when, read where.

" " " " 17, for =, read being.

" " " last line, for four roots, read, four real roots.

" " 116, for e, f, g , on wood-cut, read E, F, G .

THE ANALYST.

VOL. V.

SEPTEMBER, 1878.

No. 5.

ON THE LIMIT OF REPEATED ADJUSTMENTS.

BY E. L. DE FOREST, M. A., WATERTOWN, CONN.

IT has been shown by me (ANALYST, May 1878, p. 70) that when a series is adjusted by a given formula several times in succession, the process is precisely equivalent to a single adjustment made by a certain resultant formula, and that the coefficients in such resultant formulas, considered as terms in a series or ordinates to a curve, follow outlines which appear to resemble each other, no matter what the outline of the original formula may have been.

We will now investigate the form of the curve followed by such coefficients at the limit, when the number of repetitions becomes very large, or infinite.

Proposition. If a series is adjusted k times in succession by the formula

$$u'_0 = \lambda_0 u_0 + \lambda_1(u_1 + u_{-1}) + \lambda_2(u_2 + u_{-2}) + \dots + \lambda_m(u_m + u_{-m}), \quad (1)$$

and if the equivalent or resultant formula is denoted by

$$u_0^{(k)} = l_0 u_0 + l_1(u_1 + u_{-1}) + l_2(u_2 + u_{-2}) + \dots + l_{km}(u_{km} + u_{-km}), \quad (2)$$

then any coefficient l_i and the $2m$ other coefficients nearest to it will be connected by the relation

$$\frac{\lambda_1(l_{i+1} - l_{i-1}) + 2\lambda_2(l_{i+2} - l_{i-2}) + \dots + m\lambda_m(l_{i+m} - l_{i-m})}{\lambda_0 l_i + \lambda_1(l_{i+1} + l_{i-1}) + \lambda_2(l_{i+2} + l_{i-2}) + \dots + \lambda_m(l_{i+m} + l_{i-m})} = \frac{-i}{k+1}. \quad (3)$$

This property has been reached inductively, by proceeding from some very simple cases to more complex ones, and I am unable to offer any general proof of it, but it has been tested in such a variety of ways that there can be no doubt of its generality. Take for example

$$u'_0 = \frac{1}{21}[7u_0 + 6(u_1 + u_{-1}) + 3(u_2 + u_{-2}) - 2(u_3 + u_{-3})].$$

By the method given in the ANALYST referred to, it is found that three successive applications of this formula are exactly equivalent to

$$u_0''' = \frac{1}{9261} [2617u_0 + 2232(u_1 + u_{-1}) + 1278(u_2 + u_{-2}) + 276(u_3 + u_{-3}) \\ - 171(u_4 + u_{-4}) - 234(u_5 + u_{-5}) - 105(u_6 + u_{-6}) + 18(u_7 + u_{-7}) \\ + 36(u_8 + u_{-8}) - 8(u_9 + u_{-9})].$$

Now take, for instance, $i = 2$, and (3) becomes

$$\frac{6(276 - 2232) + 2 \times 3(-171 - 2617) + 3 \times -2(-234 - 2232)}{7 \times 1278 + 6(276 + 2232) + 3(-171 + 2617) - 2(-234 + 2232)} = \frac{-2}{3+1},$$

in which both members reduce to the same value $-\frac{1}{2}$. By such trials, any one can satisfy himself that formula (3) is true, no matter what the values of i , k , λ_0 , λ_1 , &c. may be. If any l beyond l_{km} should be included, its value is zero, and any negative sub-index of l is the same as positive. The resultant coefficients shown in Tables I. and II. of the ANALYST article cited, will not afford exact tests of the present proposition, because they are only correct to three places of decimals.

Now let Δ_1 , Δ_2 , &c. denote the finite differences of the series l_i , l_{i+1} , &c., so that we have

$$\begin{aligned} l_{i+1} - l_i &= \Delta_1, & l_{i+3} - 3l_{i+2} + 3l_{i+1} - l_i &= \Delta_3, \\ l_{i+2} - 2l_{i+1} + l_i &= \Delta_2, & l_{i+4} - 4l_{i+3} + 6l_{i+2} - 4l_{i+1} + l_i &= \Delta_4, \\ l_{i+5} - 5l_{i+4} + 10l_{i+3} - 10l_{i+2} + 5l_{i+1} - l_i &= \Delta_5, \end{aligned}$$

and for convenience let differences higher than Δ_5 be neglected, so that Δ_5 is regarded as constant for any six consecutive coefficients l . Thus we get

$$\begin{aligned} l_{i+1} &= l_i + \Delta_1, & l_{i-1} &= l_i - \Delta_1 + \Delta_2 - \Delta_3 + \Delta_4 - \Delta_5, \\ l_{i+2} &= l_i + 2\Delta_1 + \Delta_2, & l_{i-2} &= l_i - 2\Delta_1 + 3\Delta_2 - 4\Delta_3 + 5\Delta_4 - 6\Delta_5, \\ l_{i+3} &= l_i + 3\Delta_1 + 3\Delta_2 + \Delta_3, & l_{i-3} &= l_i - 3\Delta_1 + 6\Delta_2 - 10\Delta_3 + 15\Delta_4 - 21\Delta_5, \\ l_{i+4} &= l_i + 4\Delta_1 + 6\Delta_2 + 4\Delta_3 + \Delta_4, & l_{i-4} &= l_i - 4\Delta_1 + 10\Delta_2 - 20\Delta_3 + 35\Delta_4 - 56\Delta_5, \\ &\text{\&c.} & &\text{\&c.} \end{aligned}$$

By subtracting and adding, we find in general

$$\begin{aligned} l_{i+n} - l_{i-n} &= 2n\Delta_1 - n\Delta_2 + \frac{1}{3}n(n^2 + 2)\Delta_3 - \frac{1}{2}n(n^2 + 1)\Delta_4 + \frac{1}{6}n(n^4 + 35n^2 + 24)\Delta_5, \\ l_{i+n} + l_{i-n} &= 2l_i + n^2(\Delta_2 - \Delta_3) + \frac{1}{2}n^2(n^2 + 11)\Delta_4 - \frac{1}{6}n^2(n^2 + 5)\Delta_5. \end{aligned}$$

These expressions would remain the same if differences higher than Δ_5 had been taken into account, only terms containing Δ_6 , Δ_7 , &c., would be added.

Substitute the above in (3), and write, for brevity,

$$\left. \begin{aligned} \lambda_0 + 2(\lambda_1 + \lambda_2 + \dots + \lambda_m) &= b \\ 1^2\lambda_1 + 2^2\lambda_2 + \dots + m^2\lambda_m &= b_2 \\ 1^4\lambda_1 + 2^4\lambda_2 + \dots + m^4\lambda_m &= b_4 \\ 1^6\lambda_1 + 2^6\lambda_2 + \dots + m^6\lambda_m &= b_6 \\ \text{\&c.} & \text{\&c.} \end{aligned} \right\} \quad (4)$$

then (3) becomes

$$\begin{aligned} &\frac{b_2(2\Delta_1 - \Delta_2) + \frac{1}{3}(2b_2 + b_4)\Delta_3 - \frac{1}{2}(b_2 + b_4)\Delta_4 + \frac{1}{6}(24b_2 + 35b_4 + b_6)\Delta_5 - \text{\&c.}}{b_4 + b_2(\Delta_2 - \Delta_3) + \frac{1}{2}(11b_2 + b_4)\Delta_4 - \frac{1}{6}(5b_2 + b_4)\Delta_5 + \text{\&c.}} \\ &= \frac{-i}{k+1}. \end{aligned} \quad (5)$$

But if (1) is an adjustment formula of such nature as to give exact results when applied to a series of the first order, that is, one whose first differences are constant, we shall have $b = 1$. If it is exact in the case of a series of the third order, we shall have both $b = 1$ and $b_2 = 0$. If exact for a series of the fifth order, then $b = 1$, $b_2 = 0$ and $b_4 = 0$; and so on for higher orders. This has been shown by Schiaparelli in his essay on empirical curves, *Sul modo di ricavare*, etc., as remarked by me in the *Smithsonian Reports* of 1871 and 1873, pp. 335 and 342. It can also be proved somewhat differently, as follows. Equidistant ordinates to the curve

$$y = A_0 + A_1x + A_2x^2 + A_3x^3 + \&c.,$$

form a series of an order equal to the degree of the curve, and terms in such series can always be expressed by such an equation, the origin being taken at any term we please, while the constant interval between the terms may be the unit of x . Thus we have $y_0 = A_0$, and

$$\begin{aligned} y_1 &= A_0 + A_1 + A_2 + \&c., & y_{-1} &= A_0 - A_1 + A_2 - A_3 + \&c., \\ y_2 &= A_0 + 2A_1 + 2^2A_2 + 2^3A_3 + \&c., & y_{-2} &= A_0 - 2A_1 + 2^2A_2 - 2^3A_3 + \&c., \\ y_3 &= A_0 + 3A_1 + 3^2A_2 + 3^3A_3 + \&c., & y_{-3} &= A_0 - 3A_1 + 3^2A_2 - 3^3A_3 + \&c., \\ & & & \&c., & \&c., \end{aligned}$$

and the adjustment formula

$$y'_0 = \lambda_0 y_0 + \lambda_1 (y_1 + y_{-1}) + \lambda_2 (y_2 + y_{-2}) + \dots + \lambda_m (y_m + y_{-m})$$

becomes by substitution

$$\begin{aligned} y'_0 &= A_0 [\lambda_0 + 2(\lambda_1 + \lambda_2 + \dots + \lambda_m)] + 2A_2 (1^2\lambda_1 + 2^2\lambda_2 + \dots + m^2\lambda_m) \\ &\quad + 2A_4 (1^4\lambda_1 + 2^4\lambda_2 + \dots + m^4\lambda_m) + 2A_6 (1^6\lambda_1 + 2^6\lambda_2 + \dots + m^6\lambda_m) + \&c., \end{aligned}$$

which by the notation of (4) is equivalent to

$$y'_0 = A_0 b + 2(A_2 b_2 + A_4 b_4 + A_6 b_6 + \&c.).$$

In the case of a series of the first order, A_2, A_4 &c. will be zero, and we cannot have $y'_0 = y_0 = A_0$ without having $b = 1$. For a series of the second or third order, A_4, A_6 , &c. will be zero, and since A_0 and A_2 may be supposed to have any values, we cannot have $y'_0 = y_0$ without having both $b = 1$ and $b_2 = 0$. Likewise for a series of the fourth or fifth order, we must have $b = 1$, $b_2 = 0$, $b_4 = 0$; and so on for series of higher orders.

Reverting now to equation (5), let us see what becomes of it in each of the foregoing cases at the limit, when the number of repetitions of the adjusting process is very great, so that k is a very large number, or infinite. The successive values of l become consecutive ordinates to the limiting curve, so that $l_i = y$, $\Delta_1 = dy$, $\Delta_2 = d^2y$, $\Delta_3 = d^3y$, &c.

Differentials of higher orders are to be neglected in comparison with those of a lower order, and all differentials are neglected in comparison with y .

Taking l_0 as the axis of Y , the abscissa corresponding to any y will be $x = idx$, which becomes $x = idx$ at the limit. Thus in the first case, with only the condition $b = 1$, (5) reduces to

$$\frac{2b_2 dy}{y} = \frac{-x}{(k+1)dx}, \quad \therefore \frac{dy}{dx} = \frac{-\frac{1}{2}xy}{b_2(k+1)(dx)^2}. \quad (6)$$

With the two conditions $b = 1$, and $b_2 = 0$, (5) becomes

$$\frac{\frac{1}{3}b_4 d^3y}{y} = \frac{-x}{(k+1)dx}, \quad \therefore \frac{d^3y}{dx^3} = \frac{-3xy}{b_4(k+1)(dx)^4}, \quad (7)$$

and with the three conditions $b = 1$, $b_2 = 0$, $b_4 = 0$, (5) is

$$\frac{\frac{1}{60}b_6 d^5y}{y} = \frac{-x}{(k+1)dx}, \quad \therefore \frac{d^5y}{dx^5} = \frac{-60xy}{b_6(k+1)(dx)^6}. \quad (8)$$

It is to be presumed that if orders of differences higher than the fifth were taken into account, four or more of the conditions would give differential equations of the seventh and higher orders similar to the above. We may say then in general, that the limiting curve sought will be characterized by the linear differential equation

$$\frac{d^n y}{dx^n} = axy, \quad (9)$$

where n is an odd number, and has the values 1, 3, 5, &c., according as the original adjustment formula is exact in the case of series of the first, third, fifth, &c. orders. The simplest case, $n = 1$, gives

$$\frac{dy}{y} = axdx, \quad \therefore y = Ce^{\frac{1}{2}ax^2}.$$

The constant of integration C can be determined by the condition $b = 1$, for this is satisfied not only by the original formula (1), but also by the resultant formula for any number of repetitions. (ANALYST, p. 66.) At the limit, the condition becomes

$$\int_{-\infty}^{\infty} ydx = 1, \quad \therefore C = \sqrt{\frac{-a}{2\pi}}.$$

Formula (6) shows that a is, in general, essentially negative. Writing $a = -2h^2$, we get

$$h = \frac{1}{2dx \sqrt{[b_2(k+1)]}}, \quad y = \frac{h}{\sqrt{\pi}} e^{-h^2 x^2}, \quad (10)$$

the well known equation of the probability curve, which is thus seen to be the limiting form of the coefficients, for adjustment formulas which satisfy the single condition $b = 1$. Such, for instance, are the simple mean

$$u'_0 = \frac{1}{2m+1} \left\{ u_0 + (u_1 + u_{-1}) + (u_2 + u_{-2}) + \dots + (u_m + u_{-m}) \right\},$$

and this other which has some times been used,

$$u'_0 = \frac{1}{25} \left\{ 5u_0 + 4(u_1 + u_{-1}) + 3(u_2 + u_{-2}) + 2(u_3 + u_{-3}) + (u_4 + u_{-4}) \right\}.$$

All such formulas give exact results only when applied to arithmetical progressions, that is, to series which follow not a curve, but a straight line.

Since the differentials of x and y bear to each other nearly the same relations as their finite differences do, provided these are small, it follows that the equation (10) expresses pretty nearly the outline of the resultant formula for any finite, but tolerably large, number of repetitions, and the larger the number, the more accurate is the representation. Take for example sixteen applications of

$$u'_0 = \frac{1}{3}(u_0 + u_1 + u_{-1}), \quad (11)$$

for which $b_2 = 1^2\lambda_1 = \frac{1}{3}$ and $k+1 = 17$. For dx put $\Delta x = 1$. Then

$$h = .21004, \quad \log y = 1.07374 - .019160x^2,$$

and Table I. shows the values of y for successive values of x .

TABLE I. ($h = .21004$.)

x	0	1	2	3	4	5	6	7	8	9	10	11	&c.
y	.119	.113	.099	.080	.059	.039	.024	.014	.007	.003	.001	.001	&c.

They come quite near to the true coefficients .121, .115, .101, .081, .059, .039, .023, .012, .006, .003, .001, .000, &c., as found by repeated use of formula (7), ANALYST, p. 67.

Instead of y , it would have been more strictly correct to use the finite difference of the area, having $\Delta x = 1$ for its base and y for its middle ordinate; but the numerical value of such a portion of the area is very nearly the same as that of y .

As is well known, all probability curves are similar figures in this sense, that any increase of the probable error gives a proportionate increase to all the abscissas, while all the ordinates are diminished in the same ratio, so that the whole area of the curve remains constantly unity. The probable error varies inversely as h and $\sqrt{-a}$, and is shown by (10) to vary directly as $1/b_2$ and $1/(k+1)$. Hence we can find approximate values of y for any number of repetitions of any formula, from similar values which have already been found for some other number of repetitions of some other formula. Take, for example, eight applications of

$$u'_0 = \frac{1}{5}(u_0 + u_1 + u_{-1} + u_2 + u_{-2}), \quad (12)$$

where $k+1 = 9$ and $b_2 = 1^2\lambda_1 + 2^2\lambda_2 = \frac{1}{5} + \frac{4}{5} = 1$, so that $h = \frac{1}{5}$. Let each x in Table I. be multiplied by the number $\frac{1}{5} \div .21004 = .7935$, and let each corresponding y be found by interpolation in that table, thus;

x	0	.79	1.59	2.38	3.17	3.97	4.76	5.55	6.35	7.14	7.93	8.73	9.52	10.3
y	.119	.114	.105	.092	.076	.058	.044	.031	.020	.013	.007	.004	.002	.001

Multiplying each y here by .7935 we get

.094 .091 .083 .073 .060 .046 .035 .025 .016 .010 .006 .003 .002 .001,
which is not far from the true series

.098 .095 .087 .075 .061 .047 .034 .023 .014 .008 .004 .002 .001 .000,
and the approximation would be closer if k were greater. If k were infinite all the terms of such a series would be infinitesimals, but the relation between them would be of the form (10).

We come now to the case of chief interest, that in which the adjustment formula satisfies the two conditions $b = 1$ and $b_2 = 0$, so that it gives exact results when applied to a series of the third or any lower order, and is adapted to the adjustment of series which follow not only straight lines, but curved ones. Here (7) gives for the differential equation of the limiting curve

$$a = \frac{-3}{b_4(k+1)(dx)^4}, \quad \frac{d^3y}{dx^3} = axy. \quad (13)$$

The integration of equations of this kind, under the general form (9), is treated of in various text-books; see for instance Price's *Calculus*, Vol. II., pp. 652—6; but I am not aware that it has ever been discussed under the peculiar conditions imposed by our present problem. The simplest way of getting the value of y , and perhaps the only way adapted for numerical computation, appears to be by means of a series, using the method of indeterminate coefficients. Since the curve is symmetrical on both sides of the middle ordinate or axis of Y , we take only even powers of x , and assume

$$y = A + Bx^2 + Cx^4 + Dx^6 + Ex^8 + \&c.,$$

which gives

$$\frac{d^3y}{dx^3} = 2.3.4Cx + 4.5.6Dx^3 + 6.7.8Ex^5 + \&c.,$$

$$axy = Aax + Bax^3 + Cax^5 + \&c.$$

These are equal to each other, so that

$$(2.3.4C - Aa)x + (4.5.6D - Ba)x^3 + (6.7.8E - Ca)x^5 + (8.9.10F - Da)x^7 + \&c. = 0.$$

The coefficients of each power of x being separately zero, give

$$\begin{aligned} C &= \frac{Aa}{2.3.4}, & D &= \frac{Ba}{4.5.6}, \\ E &= \frac{Ca}{6.7.8} = \frac{Aa^2}{2.3.4.6.7.8}, & F &= \frac{Da}{8.9.10} = \frac{Ba^2}{4.5.6.8.9.10}, \\ &\&c., & &\&c., \end{aligned}$$

and the equation of the curve is

$$y = A \left[1 + \frac{ax^4}{2.3.4} + \frac{(ax^4)^2}{2.3.4.6.7.8} + \frac{(ax^4)^3}{2.3.4.6.7.8.10.11.12} + \&c. \right] \\ + Bx^2 \left[1 + \frac{ax^4}{4.5.6} + \frac{(ax^4)^2}{4.5.6.8.9.10} + \frac{(ax^4)^3}{4.5.6.8.9.10.12.13.14} + \&c. \right]. \quad (14)$$

We may write it thus

$y = A[1 + H_1(ax^4) + H_2(ax^4)^2 + \&c.] + Bx^2[1 + K_1(ax^4) + K_2(ax^4)^2 + \&c.]$,
and exhibit in Table II. the logarithms of H_n and K_n for the first 14 terms of each series.

TABLE II.

n	$\log H_n$	$\log K_n$	n	$\log H_n$	$\log K_n$
1	2.6197887	3.9208187	8	28.7599903	30.9337755
2	4.0934493	5.0634861	9	32.1281410	34.2294878
3	8.9728755	9.7242336	10	37.3552326	39.3913946
4	11.4465362	12.0343922	11	42.4550623	44.4319716
5	15.6114802	16.0687204	12	47.4389654	49.3615644
6	19.5271184	21.8755956	13	52.3164218	54.1888914
7	23.2336233	25.4889183	14	57.0954772	60.9214004

At first sight, equation (14) seems to contain three arbitrary constants, but really there is only one, for the two conditions $b = 1$ and $b_2 = 0$ are satisfied by the resultant formula as much as by the original one, and can be used to find A and B in terms of a , as the single condition $b = 1$ was used to find the constant of integration for the probability curve in terms of h . At the limit, (4) shows that the two conditions become

$$\int_0^\infty y dx = \frac{1}{2}, \quad \int_0^\infty x^2 y dx = 0.$$

But it is not necessary to integrate up to $x = \infty$, for we have seen that for very moderate values of x the value of y becomes so small as to be hardly sensible. (ANALYST, p. 68.)

Let x_1 be the finite limit employed, and integration gives

$$\int_0^{x_1} y dx = Ax_1 \left[1 + \frac{1}{3} H_1(ax_1^4) + \frac{1}{3} H_2(ax_1^4)^2 + \frac{1}{15} H_3(ax_1^4)^3 + \&c. \right] \\ + Bx_1^3 \left[\frac{1}{3} + \frac{1}{15} K_1(ax_1^4) + \frac{1}{15} K_2(ax_1^4)^2 + \&c. \right], \quad (15)$$

$$\frac{1}{x_1^3} \int_0^{x_1} x^2 y dx = A \left[\frac{1}{3} + \frac{1}{15} H_1(ax_1^4) + \frac{1}{15} H_2(ax_1^4)^2 + \&c. \right] \\ + Bx_1^2 \left[\frac{1}{3} + \frac{1}{15} K_1(ax_1^4) + \frac{1}{15} K_2(ax_1^4)^2 + \&c. \right]. \quad (16)$$

Denoting the sums of the series by S_1, S_2, S_3, S_4 , we have

$$Ax_1S_1 + Bx_1^3S_2 = \frac{1}{2}, \quad AS_3 + Bx_1^2S_4 = 0,$$

and consequently

$$A = \frac{S_4}{2x_1(S_1S_4 - S_2S_3)}, \quad B = -\left(\frac{S_3}{S_4}\right)\frac{A}{x_1^2}. \quad (11)$$

With any given value of a , assuming some sufficiently large value of a we can compute S_1, S_2 , &c., and consequently A and B . Any increase of x_1 , up to infinity, ought to give sensibly the same values of A and B . Since S_1, S_2 , &c. are functions of ax_1^4 , they remain constant so long as ax_1^4 is constant, though a may diminish and x_1^4 increase. Take $ax_1^4 =$

$$\therefore x_1 = \left(\frac{c}{a}\right)^{\frac{1}{4}},$$

and (17) becomes

$$A = \frac{S_4 a^{\frac{1}{4}}}{2(S_1S_4 - S_2S_3)c^{\frac{1}{4}}}, \quad B = -\left(\frac{S_3}{S_4}\right)A\sqrt{a}. \quad (12)$$

It thus appears that A and B vary as $a^{\frac{1}{4}}$ and $a^{\frac{3}{4}}$ respectively. Let us assume a sufficiently large value of c , and compute A and B in terms of c . This practical difficulty arises, that the expression for A contains in the denominator the difference of two large numbers, S_1S_4 and S_2S_3 , and these are so nearly equal, that if computed by logarithms to seven figures, the difference contains only one or two figures, and the value of A is not found correctly enough. Another computation, therefore, has been made without using logarithms, taking $c = 10000$, and the work was carried to twelve figures. It gave

$$A = .40803a^{\frac{1}{4}}, \quad B = -.3379891A\sqrt{a}.$$

This value of B is the same as found by logarithms with smaller value of c , while A is a little different, so that the ratio of A to B is known more accurately than the absolute value of A . Putting $a = .001$, and calculating the area of the curve up to a certain point, $x = 36$, and allowing for the area of the rest by the help of the 32-fold coefficients (ANALYST, p. 68) carried to five figures, I found for the total area .99896 instead of the proper value unity. The above value of A , therefore, ought to be increased in the ratio of .99896 to 1, so that the values finally adopted are

$$A = .40845a^{\frac{1}{4}}, \quad B = -.3379891A\sqrt{a}. \quad (13)$$

The ordinates to the curve, as computed by (14) and (19), are shown in Table III. to four places of decimals.

From what we know of resultant formulas, as discussed in the ANALYST already cited, it is evident that the present curve consists of an infinite number of undulations, lying alternately above and below the axis of x .

TABLE III. ($a = \frac{1}{1000}$.)

x	y	x	y	x	y	x	y
0	.0726	10	.0196	20	— .0068	30	.0010
1	.0719	11	.0133	21	— .0059	31	.0010
2	.0696	12	.0077	22	— .0048	32	.0009
3	.0659	13	.0029	23	— .0037	33	.0008
4	.0610	14	— .0009	24	— .0026	34	.0006
5	.0550	15	— .0038	25	— .0016	35	.0004
6	.0483	16	— .0058	26	— .0007	36	.0002
7	.0412	17	— .0070	27	— .0000	37	&c.
8	.0338	18	— .0075	28	.0005	38	
9	.0265	19	— .0074	29	.0008	39	

Only three of these appear in our table, because as x increases, y becomes so very small as to be hardly sensible, and is only computed with great labor, since (14) gives it as the difference between two large numbers. If plotted on paper, the curve beyond these limits would not be distinguishable by the eye from the axis of X . It presents striking analogies to the probability curve. Since it has only one arbitrary constant a , and its area is always unity, the curves for different values of a are similar figures in the same sense as all probability curves are similar. The dimensions in the direction of x vary inversely as those in the direction of y , and since the latter vary as A and therefore as $a^{\frac{1}{4}}$, it follows from (13) that the x dimensions vary directly as the fourth root of $-b_4$, and also as the fourth root of $k+1$.

Table III. can be used as Table I. was, to compute approximately the coefficients for any given number of repetitions of any given adjustment formula. Take for example the 32-fold use of the formula (ANALYST, p.67),

$$u'_0 = \frac{1}{35}[17u_0 + 12(u_1 + u_{-1}) - 3(u_2 + u_{-2})],$$

where we have from (4),

$$b_4 = 1^4\lambda_1 + 2^4\lambda_2 = \frac{1^2}{35} + 16 \times \frac{-3}{35} = -\frac{36}{35},$$

while $k=32$ and for dx we put $\Delta x = 1$, so that (13) gives $a = \frac{35}{396}$. When $x=0$, then $y=A$, and we have seen that A varies as the fourth root of a , so that any coefficient in the formula now sought may be found by taking the corresponding y from Table III., and multiplying it by the ratio

$$\left(\frac{35}{396} \div \frac{1}{1000}\right)^{\frac{1}{4}} = 3.066.$$

The abscissas, found by multiplying this ratio into 0, 1, 2, 3, &c., are

0, 3.07, 6.13, 9.20, 12.26, 15.33, 18.40, 21.46 &c.

The corresponding ordinates, by interpolation in Table III., are

.0726, .0656, .0474, .0251, .0064, —.0045, —.0075, —.0054 &c.

and multiplying these by 3.066, we have

.223, .201, .145, .077, .020, —.014, —.023 —.017, —.006, .001, .003
.002, .000, &c

which approach pretty nearly to the true coefficients for 32 applications
(ANALYST, p. 68),

.230, .205, .145, .072, .014, —.017, —.022, —.014, —.004, .002, .001, .000, &c

and the agreement would be closer if k were a greater number. Taking
 $k = 64$, we get .188 for the first coefficient, the true value being .192. If
 k were infinite, all the coefficients would be infinitesimals, but the relation
between them would be of the form (14).

By a process quite similar to the foregoing, we can discuss the third case
in which the adjustment formula satisfies the three conditions

$$b = 1, \quad b_2 = 0, \quad b_4 = 0,$$

and gives exact results when applied to series of the fifth or any lower order.
Such a formula, for instance, is the one at p. 292 of the *Smithsonian*
Report of 1871,

$$u'_0 = \frac{1}{36}[15(u_0 + u_1 + u_{-1}) - 6(u_2 + u_{-2}) + (u_3 + u_{-3})].$$

For an infinite number of repetitions, the differential equation of the limiting
curve is given by (8),

$$a = \frac{-60}{b_6(k+1)(dx)^6}, \quad \frac{d^5y}{dx^5} = axy, \quad (2)$$

and by the method of indeterminate coefficients we find

$$\begin{aligned} y = & A \left(1 + \frac{ax^6}{2.3.4.5.6} + \frac{(ax^6)^2}{2.3.4.5.6.8.9.10.11.12} \right. \\ & \left. + \frac{(ax^6)^3}{2.3.4.5.6.8.9.10.11.12.14.15.16.17.18} + \&c \right) \\ & + Bx^2 \left(1 + \frac{ax^6}{4.5.6.7.8} + \frac{(ax^6)^2}{4.5.6.7.8.10.11.12.13.14} + \&c \right) \\ & + Cx^4 \left(1 + \frac{ax^6}{6.7.8.9.10} + \frac{(ax^6)^2}{6.7.8.9.10.12.13.14.15.16} + \&c \right). \end{aligned} \quad (2)$$

There is only one arbitrary constant here, for A , B and C are given
terms of a by the three conditions named, which as (4) shows, become
at the limit

$$\int_0^\infty y dx = \frac{1}{2}, \quad \int_0^\infty x^2 y dx = 0, \quad \int_0^\infty x^4 y dx = 0.$$

Considering, as in the previous case, that it is not necessary to integrate
up to infinity, we suppose x_1 to be a sufficiently large finite limit, and give
after integration,

$$\begin{aligned}\frac{1}{x_1} \int_0^{x_1} y dx &= AS_1 + Bx_1^2 S_2 + Cx_1^4 S_3 = \frac{1}{2x_1}, \\ \frac{1}{x_1^3} \int_0^{x_1} x^2 y dx &= AS_4 + Bx_1^2 S_5 + Cx_1^4 S_6 = 0, \\ \frac{1}{x_1^5} \int_0^{x_1} x^4 y dx &= AS_7 + Bx_1^2 S_8 + Cx_1^4 S_9 = 0,\end{aligned}$$

where S_1, S_2 , &c., are the sums of series involving powers of ax_1^6 . These three equations determine A, B, C as follows.

$$\left. \begin{aligned} A &= \frac{S_6(S_8 S_3 - S_5 S_9)}{2x_1[(S_1 S_6 - S_3 S_4)(S_6 S_8 - S_5 S_9) - (S_6 S_7 - S_4 S_9)(S_2 S_6 - S_3 S_5)]}, \\ B &= -\frac{A(S_6 S_7 - S_4 S_9)}{x_1^2(S_6 S_8 - S_5 S_9)}, \quad C = -\frac{A(S_4)}{x_1^4(S_6)} - \frac{B(S_5)}{x_1^2(S_6)}. \end{aligned} \right\} \quad (22)$$

Assuming a suitably large number $c = ax_1^6$, we can compute the numerical values of S_1, S_2 , &c., and so find A, B and C in terms of x_1 or of a . The above formulas show that A, B and C vary inversely as x_1, x_1^3 and x_1^5 respectively, that is to say, they vary directly as $(-a)^{\frac{1}{6}}, (-a)^{\frac{1}{2}}$ and $(-a)^{\frac{5}{6}}$, respectively, since ax_1^6 is supposed constant, and a as found by (20) is essentially negative. We therefore have

$$A = h_1(-a)^{\frac{1}{6}}, \quad B = h_2 A(-a)^{\frac{1}{2}}, \quad C = h_3 A(-a)^{\frac{5}{6}}. \quad (23)$$

The analogy with (19) is apparent. Since $y=A$ when $x=0$, the y dimensions of the curve vary as $(-a)^{\frac{1}{6}}$, so that the x dimensions vary directly as the sixth root of b_6 and also as the sixth root of $k+1$. I have not taken the trouble to compute the numerical values of h_1, h_2 and h_3 , and to construct the curve for a particular case, but there seems to be nothing impossible about it.

Analogy leads us to presume also, that a similar method might suffice to determine the limiting curve for repeated adjustments with formulas of higher orders, satisfying four or more of the conditions

$$b = 1, \quad b_2 = 0, \quad b_4 = 0, \quad b_6 = 0, \quad \&c.$$

All such curves will resemble each other in having but one arbitrary constant or parameter, while the area of the curve is always unity.

It may be noticed that the limiting value of $2b_2$,

$$2b_2 = 2 \int_0^\infty x^2 y dx,$$

is the definite integral which, in the probability curve, expresses the square of the mean error, or the mean of the squares of all the errors. Likewise the limit of $2b_4$, in the same curve,

$$2b_4 = 2 \int_0^\infty x^4 y dx,$$

would express the mean of the fourth powers of the errors; and so on.

NOTE. Throughout the foregoing investigation it has been assumed that the signs of b_2 , b_4 , and b_6 in formulas (6), (7) and (8) will be +, — and + respectively, so that the signs of the constant a will be —, + and — respectively. This is true for any adjustment formula likely to be used in practice, but others can be constructed so that b_2 , b_4 and b_6 will have signs contrary to the above. In such cases, the successive coefficients of the resultant formula tend to diverge from each other as k increases, and in passing to the limit from (5) we cannot neglect differences of a higher order in comparison with those of a lower one. Take for example

$$u'_0 = \frac{1}{3}[3u_0 + (u_1 + u_{-1})\sqrt{3}(u_2 + u_{-2})],$$

for which $b_2 = -1$. The following table shows a few of the coefficients of the resultant formulas for 2, 4, 8 and 16 applications, as found by formula (7). (ANALYST, p. 67.) It is evident that, at the limit, some of the coefficients will become infinite and positive, others infinite and negative, and the excess of the sum of the positive over the sum of the negative ones will be unity. The series will be discontinuous, and there can be no limiting curve. Thus in (10), when b_2 is negative, h becomes imaginary; and in (13) and (19), if b_4 is +, then a becomes —, and A and B are imaginary.

	1	2	4	8	16
l_0	1.00	1.44	3.22	21.1	1297
l_1	.33	.44	.99	5.9	341
l_2	— .33	— .55	— 1.73	— 14.5	— 1014
l_3		— .22	— 1.04	— 10.3	— 765
l_4		.11	.43	4.8	433
l_5			.35	6.7	717
l_6			— .07	— .4	— 4
&c.			&c.	&c.	&c.

NOTE ON THE CORRESPONDENCE OF MATERIAL FORMS WITH MATHEMATICAL RELATIONS.

BY THE EDITOR.

LIFE, sensation and consciousness are conditions found only in connection with definite material forms which constitute a part of the universe of matter; and the names used to denote the sensations experienced by the conscious individual are a representation of the external world. All knowledge is, therefore, but experience of impressions made by the world

of matter; and, therefore, all mathematical relations should have their analogue in the external world.

We have heretofore (see *Popular Science Monthly* for February 1874, p. 494) attempted to represent, by mathematical forms, the actual condition of the universe of matter, as here quoted: "There is, therefore, as I conceive, absolutely no limit to the division of matter, physically as well as mathematically; but our organization is such that, of the infinite series of terms in which it manifests itself, we can know, experimentally, only two; viz., the stellar universe, constituting the first order, of which the stars and the planets are the units; and, secondly, the chemical molecules, which constitute the second order.

"According to this view, the material universe might be represented in orders by the following series:

$$d^{-m}x, \dots d^{-3}x, d^{-2}x, d^{-1}x, d^0x, dx, d^2x, d^3x, \dots d^{n-1}x, d^nx,$$

in which x is the unknown quantity, called matter, and m and n are both infinitely great. In this series, d^0x , or simply x , would represent all tangible matter; and dx , which is the next term, descending, would represent the chemical molecule."

Whether the mathematical forms here presented are correct representations of the natural divisions of unorganized matter, or not, each individual will decide for himself, as no demonstration is, perhaps, possible; but that chemical combinations, and vegetable, and probably animal, organizations are represented in thought by algebraical forms, is, not only probable, but, a demonstrated fact; so that, in the language of Professor Sylvester, "Chemistry is the counterpart of a province of algebra, as probably the whole universe of fact is, or must be, of the universe of thought." (See *American Journal of Mathematics*, Vol. I, p. 83.)

TO FIND THE EARTH'S DISTANCE FROM THE SUN AT ANY GIVEN TIME.

BY ARTEMAS MARTIN, M. A., ERIE, PA.

LET AEP represent the orbit of the Earth, and let S represent the Sun in one of the foci.

Let a and b be the semi axes of the orbit, E the position of the Earth n days from the perihelion, $r =$ the radius vector SE , $\varphi = PSE$, the angular distance of the Earth from the perihelion, and $c =$ the area daily passed over by the radius vector.

The polar equation of the orbit, if $e^2 = (a^2 - b^2) \div a^2$, is

$$r = \frac{a(1-e^2)}{1+e \cos \varphi} \quad (1)$$

The differential of a polar area is $\frac{1}{2}r^2\varphi$;

$$\int \frac{a^2(1-e^2)^2 d\varphi}{2(1+e \cos \varphi)^2} = nc;$$

or
$$\int \frac{d\varphi}{(1+e \cos \varphi)^2} = \frac{2nc}{a^2(1-e^2)^2}.$$

To integrate $\frac{d\varphi}{(1+e \cos \varphi)^2}$, assume $\cos \varphi = \frac{\cos \theta - e}{1 - e \cos \theta}$; (2)

then

$$\frac{d\varphi}{(1+e \cos \varphi)^2} = \frac{(1-e \cos \theta)d\theta}{(1-e^2)^{\frac{3}{2}}},$$

$$\int \frac{(1-e \cos \theta)d\theta}{(1-e^2)^{\frac{3}{2}}} = \frac{\theta - e \sin \theta}{(1-e^2)^{\frac{3}{2}}} + C.$$

When $\varphi = 0$, $\theta = 0$ and $C = 0$; $\therefore$

$$\frac{\theta - e \sin \theta}{(1-e^2)^{\frac{3}{2}}} = \frac{2nc}{a^2(1-e^2)^2};$$

or

$$\theta - e \sin \theta = \frac{2nc}{a^2(1-e^2)^{\frac{5}{2}}}.$$

$$e = \frac{ab\pi}{T} = \frac{a^2\pi\sqrt{1-e^2}}{T},$$

where T is the periodic time. $\therefore$

$$\theta - e \sin \theta = \frac{2n\pi}{T}.$$

But $\theta = \sin \theta + \frac{1}{6}\sin^3 \theta + \frac{3}{40}\sin^5 \theta + \frac{5}{112}\sin^7 \theta + \&c.$, therefore by substitution and dividing by $1-e$,

$$\sin \theta + \frac{1}{6(1-e)} \sin^3 \theta + \frac{3}{40(1-e)} \sin^5 \theta + \frac{5}{112(1-e)} \sin^7 \theta + \&c. = \frac{2n\pi}{(1-e)T}$$

Reverting this series,

$$\sin \theta = \frac{2n\pi}{(1-e)T} - \frac{1}{6(1-e)} \left(\frac{2n\pi}{(1-e)T} \right)^3 + \frac{1+9e}{120(1-e)^2} \left(\frac{2n\pi}{(1-e)T} \right)^5$$

$$- \frac{1+54e+225e^2}{4040(1-e)^3} \left(\frac{2n\pi}{(1-e)T} \right)^7 + \&c.$$

$\cos \theta = \sqrt{1-\sin^2 \theta}$. Substituting in (2) we have $\cos \varphi$; and then substituting the value of $\cos \varphi$ in (1) we have r , the distance sought.

BIBLIOGRAPHICAL NOTE AND QUERY.

BY MARCUS BAKER, U. S. COAST SURVEY, WASHINGTON, D. C.

IN *The Mathematical Diary* etc., conducted by James Ryan, A. M., 12°, New York, 1832, volume II, No. 13, p. 311, we find the following:—

“QUESTION XXVI. (269.) or PRIZE QUESTION.—By *Scientific Sigma, Esq., New-York.*

“It is required to describe upon the same plane, three circles touching each other, each of which shall touch two given circles.

“For the best solution to this question, a handsomely bound complete set of the *Diary* in two volumes, is offered.”

Now this number of the *Diary*, March 1832, was, according to Dr. Hart, the last that appeared. See *ANALYST*, Vol. II, p. 134. And since it was the last no solution could have appeared in Ryan's *Diary*. Can any reader of the *ANALYST* tell whether it has ever been taken up since then and solved, and if so, when, where and by whom?

It may be added that, before it was here proposed as the Prize Question, it had been solved by Prof. Jacob Steiner of Berlin, whose solution may be found in *Crelle's Journal*, Vol. I, pp. 180—181. No proof of the solution was given by Steiner and, so far as I am aware, none has ever been given.

PROBLEM. BY W. E. HEAL, WHEELING, IND. — “I have a circular fish pond, radius, r ; a duck is swimming around the edge and my dog starts from the center in pursuit of the duck, swimming always directly towards it. Required the equation, and length, of the curve described by the dog, supposing that he swims n times as fast as the duck.

[The foregoing is substantially the same question which was published, together with the following solution by us, in the *School Friend* for Aug., 1851. Though our solution is here submitted, if any of our readers should feel sufficient interest in the question to obtain a different, or more elegant, solution, we will be pleased to insert it in a future number.—Ed.]

Solution. Let S represent the center of the pond and ACB one fourth of its circumference, and let D denote any position of the dog and C the corresponding position of the duck; then must curve $SD = n \times \text{arc } AC$.

Draw Tt tangent to the curve at D and $T't'$ tangent to the circle at C . Draw Sa and Ds indefinitely near SD and DC , respectively, and draw Db and Ce respectively perpendicular to Sa and Ds .

Equation (4) is the differential equation to the curve sought; but, as the second member is not an exact differential, it is doubtful whether the equation can be integrated in finite terms.

If any of our readers will effect an integration of (4), by series or otherwise, so as to obtain the value of x^2 in functions of y suitable for substitution in (3), and will calculate the numerical value of u for any given values of n and r we will insert the discussion, in detail, in No. 6 of the ANALYST.

THE POLYNOMIAL THEOREM.

BY MISS CHRISTINE LADD, POQUONOCK, CONN.

FOR the expansion of the n th power of m quantities the ordinary Algebras offer no better way than to apply the Binomial Theorem $m-1$ times. The process is laborious and the result is not easily brought into a symmetrical form.

Let us write down the sum of m quantities n times, and see how the actual multiplication is performed:

$$\begin{array}{c} a+b+c+d+\dots \\ a+b+c+d+\dots \\ a+b+c+d+\dots \\ \dots\dots\dots \\ \dots\dots\dots \end{array}$$

It is evident that the terms of the product will consist of every possible combination of one or more of the quantities with different exponents in such a manner that the sum of the exponents is equal to n . The only difficulty is to determine the coefficients.

The coefficient of a term expresses the number of ways in which that term can be formed. a^n , b^n &c. can be formed in only one way, therefore their coefficient is 1. The term $a^{n-1}b$ is formed by taking the b of any row and multiplying it by the a of every other row, but as any one of the n b 's can be taken there are n terms $a^{n-1}b$, or the coefficient of $a^{n-1}b$ is n ; and all terms of the same form, $b^{n-1}a$, $a^{n-1}c$, . . . have the same coefficient. To obtain the coefficient of $a^{n-2}b^2$ we observe that the number of ways in which two b 's can be selected out of n b 's is $\frac{1}{2}n(n-1)$. Take the form $a^{n-5}b^3c^2$. We can take the two c 's in $\frac{1}{2}n(n-1)$ ways. Having taken a certain combination of 2 c 's we have $n-2$ b 's out of which to select 3 b 's. This can be done in $\frac{(n-2)(n-3) \dots (n-2-3+1)}{3!}$ ways, and only one

combination of the $n-5$ a 's remains. Hence the coefficient of $a^{n-5}b^3c^2$ is

$$\frac{n(n-1)(n-2)(n-3) \dots (n-2-3+1)}{2! \quad 3!}.$$

In general, for the coefficient of

$$a^{n-\beta-\gamma-\delta} b^{\beta} c^{\gamma} d^{\delta}$$

we have the product of the combinations of n things δ at a time by the combinations of $n-\delta$ things γ at a time by the combinations of $n-\delta-\gamma$ things β at a time, which is

$$\frac{n(n-1) \dots (n-\delta+1)}{\delta!} \times \frac{(n-\delta)(n-\delta-1) \dots (n-\delta-\gamma+1)}{\gamma!} \\ \times \frac{(n-\delta-\gamma)(n-\delta-\gamma-1) \dots (n-\delta-\gamma-\beta+1)}{\beta!},$$

or
$$\frac{n(n-1) \dots (n-\beta-\gamma-\delta+1)}{\delta! \cdot \gamma! \cdot \beta!}.$$

Write $n-\beta-\gamma-\delta = a$ and multiply numerator and denominator of this by $a!$ and we have

$$\frac{n(n-1) \dots (a-1)a(a-1)(a-2) \dots 1}{\delta! \cdot \gamma! \cdot \beta! \cdot a!}$$

or
$$\frac{n!}{a! \cdot \beta! \cdot \gamma! \cdot \delta!}$$

for the coefficient of $a^a b^{\beta} c^{\gamma} d^{\delta}$, in which $a+\beta+\gamma+\delta = n$.

From the manner in which this coefficient is obtained it is evident that all the terms of the form $a^{\beta} c^{\delta} d^{\gamma} b^{\beta}$, $b^a c^{\beta} d^{\gamma} e^{\delta}$, and so on, that is to say, all the terms, formed by affecting any four quantities by these four exponents, have the same coefficient.

For example, in the expansion of $(a+b+c+d)^{10}$, the coefficient of $a^4 b^3 c^2 d$ is

$$\frac{10!}{4! \cdot 3! \cdot 2!} = 11600.$$

We may then write

$$(a+b+c+\dots)^n = \Sigma \left[\frac{n!}{a! \cdot \beta! \cdot \gamma! \dots} \Sigma (a^a b^{\beta} c^{\gamma} \dots) \right],$$

where the Σ within the brackets indicates the sum of the terms formed by all possible combinations of the m quantities with a given set of exponents, and the Σ without the brackets, the sum of all the sets of terms obtained by giving to the exponents all possible values such that

$$a+\beta+\gamma+\dots = n.$$

The number of different coefficients in the expanded form is much less than the number of terms. In the ninth power of nine quantities there are 24310 terms, but only 30 different coefficients.

The above demonstration becomes imaginary when n is fractional or negative.

ANSWER TO QUERY AT P. 64, VOL. IV, BY PROF. KERSHNER.—“The extraction of the square root of $a + b\sqrt{-1}$ is an operation to which Euclid’s geometry is competent; it requires only the bisection of an angle and the determination of a mean proportional to obtain

$$\left\{ \sqrt[4]{(a^2+b^2)}, \frac{1}{2}\tan^{-1}\frac{b}{a} \right\} \text{ from } \left\{ \sqrt{(a^2+b^2)}, \tan^{-1}\frac{b}{a} \right\}.$$

Hence it follows that wherever n is a prime number, and $n-1$ is a power of 2, the formation of the n th roots of unity is a geometrical operation, in the ancient sense. This is the discovery of Gauss, and is the most remarkable addition to the power of construction which the ancient geometry has received since the time of Euclid. Euclid mastered the cases $n=3$, $n=5$; the next one is $n=17$, and the next $n=257$.”—See De Morgan’s *Trigonometry and Double Algebra*.

GEOMETRICAL SOLUTION OF PROB. 125, BY PROF. W. P. CASEY.— AO , BO and DO are given to find the side of the square.

Join AC , BD and CO and draw Om , On , perpendiculars, and also Cs perpendicular to AO produced.

It is well known that $AO^2 + CO^2 = DO^2 + BO^2$, i. e., the sum of the squares of the lines drawn from any point to the opposite angles of a square or rectangle = the sum of the squares of the two lines drawn from the same point to the other two opposite angles, and therefore $AO^2 + CO^2$ is given, and AO is given, $\therefore CO$ is a given line.



Again, $DO^2 - BO^2 = Dn^2 - Bn^2 = 2BD \times pn = 2AC \times Om$. But $2AC \times Om = 2AO \times Cs$, hence $DO^2 - BO^2 = 2AO \times Cs$; $\therefore 2AO \times Cs$ is given and AO is given, $\therefore Cs$ is given; and CO is given, by the above, $\therefore CO^2 - Cs^2 = Os^2$ is given, and $\therefore Os$ is a given line and AO is given, hence As is given and so is Cs , hence AC is a given line, and $\therefore AC^2 = 2AB^2$ is given and hence AB is a given line.

[The above solution is here given because it is purely geometrical, whereas the solution published at p. 188, Vol. III, contains trigonometric funct's.]

ON THE ANGLE BETWEEN TWO LINES GIVEN BY THEIR EQUATIONS.

BY W. E. HEAL, WHEELING, INDIANA.

WORKS on analytic geometry give the following expressions for the angle, φ , between two right lines whose equations are

$$A x + B y + C = 0,$$

$$A_1 x + B_1 y + C_1 = 0.$$

$$\sin \varphi = \frac{(A_1 B - A B_1) \sin \omega}{\sqrt{(A^2 + B^2 - 2AB \cos \omega)} \sqrt{(A_1^2 + B_1^2 - 2A_1 B_1 \cos \omega)}},$$

$$\cos \varphi = \frac{A A_1 + B B_1 - (A B_1 + A_1 B) \cos \omega}{\sqrt{(A^2 + B^2 - 2AB \cos \omega)} \sqrt{(A_1^2 + B_1^2 - 2A_1 B_1 \cos \omega)}};$$

$$\therefore \tan \varphi = \frac{(A_1 B - A B_1) \sin \omega}{A A_1 + B B_1 - (A B_1 + A_1 B) \cos \omega}.$$

Therefore, say these authors, the lines are parallel if $(A_1 B - A B_1) \sin \omega = 0$, since the tangent then becomes zero. Likewise the lines are perpendicular if $A A_1 + B B_1 - (A B_1 + A_1 B) \cos \omega = 0$, because the tangent then becomes infinite.

This reasoning is not conclusive, since it is not proved that these expressions may not both be zero for the same values of A, A_1, B, B_1 , and thus the tangent be indeterminate.

I think a better proof of the above proposition might be given as follows.

The equation

$$[(A_1 B - A B_1) \sin \omega]^2 + [A A_1 + B B_1 - (A B_1 + A_1 B) \cos \omega]^2 = (A^2 + B^2 - 2AB \cos \omega)(A_1^2 + B_1^2 - 2A_1 B_1 \cos \omega) \quad (1)$$

is identical.

If the lines are parallel, $\cos \varphi = \cos 0^\circ = 1$. $\therefore$

$$A A_1 + B B_1 - (A B_1 + A_1 B) \cos \omega = \sqrt{(A^2 + B^2 - 2AB \cos \omega)} \times \sqrt{(A_1^2 + B_1^2 - 2A_1 B_1 \cos \omega)}. \quad (2)$$

Squaring (2), subtracting from (1), extracting the square root and dividing by $\sin \omega$, we obtain for the condition of parallelism

$$A_1 B - A B_1 = 0.$$

If the lines are perpendicular, $\sin \varphi = \sin 90^\circ = 1$; $\therefore$

$$(A_1 B - A B_1) \sin \omega = \sqrt{(A^2 + B^2 - 2AB \cos \omega)} \sqrt{(A_1^2 + B_1^2 - 2A_1 B_1 \cos \omega)}. \quad (3)$$

Squaring (3), subtracting from (1) and extracting the square root we get for the condition of perpendicularity

$$A A_1 + B B_1 - (A B_1 + A_1 B) \cos \omega = 0.$$

SOLUTION OF PROBLEM 206.

BY R. J. ADCOCK, MONMOUTH, ILLINOIS.

EDITOR ANALYST:

Having given at pages 183-4, Vol. IV, ANALYST, a correct demonstration of the principle of Least Squares, and at pages 21-22, Vol. V, the derivation of formulæ for mean error, probable error, &c., and on pages 53-54, the solution of an example according to this corrected method, and not having made a single disciple, as appears from the published solution at p. 122, of my second ex., problem 206: I therefore propose to send you a correct solution of that example, 206, for publication in No. 5, if you will insert it, and will state my objection to Prof. S's solution.

R. J. ADCOCK.

[As Mr. Adcock has nowhere, so far as we know, explicitly pointed out the *error* which his treatment of the subject is intended to correct, we have given, above, his communication in full, and subjoin the solution and objections referred to.—Ed.]

Solution.—Let $x_1, y_1, z_1, x_2, y_2, z_2, \dots, x_n, y_n, z_n$, be the coordinates of the n points, and

$$z = cx + dy + g, \quad (1)$$

the equation of the required plane, in which the quantities c, d and g are to be determined when the plane is in its most probable position.

Now it is demonstrated at page 183-4 ANALYST, Vol. IV, that any point, line or surface, to be determined from the measured coordinates of n points, has its most probable position when the sum of the squares of the normals upon it from the n points is a minimum.

By Analytical Geometry, the normal from the point x_1, y_1, z_1 , to the plane whose equation is (1) is

$$N_1 = \frac{cx_1 + dy_1 - z_1 + g}{(1 + c^2 + d^2)^{1/2}}. \quad (2)$$

Hence, writing $S(N_1^2) = N_1^2 + N_2^2 + \dots + N_n^2$, &c., and $S(x_1 y_1) = x_1 y_1 + x_2 y_2 + \dots + x_n y_n$, &c.,

$$S(N_1^2) = \frac{c^2 S(x_1^2) + d^2 S(y_1^2) + S(z_1^2) + ng^2 + 2cdS(x_1 y_1)}{1 + c^2 + d^2} + \frac{2cgS(x_1) - 2dS(y_1 z_1) + 2dgS(y_1) - 2cs(x_1 z_1) - 2gS(z_1)}{1 + c^2 + d^2}. \quad (3)$$

Then, from (3), c, d and g are to have such values as make $S(N_1^2)$ a minimum. Therefore

$$D_c S(N_1^2) = 0 \dots (4), \quad D_d S(N_1^2) = 0 \dots (5), \quad D_g S(N_1^2) = 0, \quad (6)$$

are the equations which determine c , d and g ; the solution of which can be performed by equations of the second degree, by first transferring the origin of coordinates to the centre of gravity of the n points.

My objection to Prof. Scheffer's solution is, that it does not recognize the proposition in relation to normals above referred to, nor the necessary and fundamental definition of the error of a point with respect to a line or surface, which definition is, that the error is the normal from the point to the line or surface.

SOLUTIONS OF PROBLEMS IN NUMBER FOUR.

SOLUTIONS of problems in No. 4 have been received as follows:

From Prof. W. W. Beman, 215; Marcus Baker, 213, 215, 217; George C. Comstock, 215; Geo. M. Day, 212, 213, 214, 215, 216, 217, 219; Prof. A. B. Evans, 213; Geo. Eastwood, 218; Henry Gunder, 212, 213, 214, 217; W. E. Heal, 212, 213, 214; Wm. Hoover, 212, 213, 214; Henry Heaton, 212, 213, 214, 215, 216, 217, 219, 220; George H. Harvill, 213; Prof. W. W. Johnson, 217, 220; Prof. J. H. Kershner, 212, 213, 214, 215, 216, 217, 219; Prof. D. J. Mc Adam, 212, 213, 214, 217; L. W. Meech, 218; Artemas Martin, 220; Prof. J. Scheffer, 212, 213, 214, 215, 216, 217; E. B. Seitz, 212, 213, 214, 215, 216, 217, 219, 220.

211. "Prove that every number is either a triangular number or is the sum of two, or of three triangular numbers."

[No demonstration of this proposition has been received. Prof. Scheffer writes: "This theorem is a special case of a more general one. As to this case and the following one, viz.: *Every number is either a square or the sum of two, three, or four squares*, the demonstrations have been discovered, but as to the pentagonal and higher numbers, the attempts of the greatest mathematicians at a demonstration have thus far proved futile."

The writer of the article, *number*, in *Johnson's New Encyclopædia*, says, "It is a general principle, though not capable of rigorous demonstration, that any whole number is equal to the sum of 1, 2 or 3 triangular numbers, or to the sum of 1, 2, 3 or 4 square numbers, or to the sum of 1, 2, 3, 4, or 5 pentagonal numbers, etc."]

212. "One half of a circular tract of land is cut off by an arc of a circle whose center is in the circumference of the circular tract. Find the radius with which the arc is described."

SOLUTION BY PROF. D. J. MC ADAM, WASHINGTON, PA.

Let $AB = x$, $AD = r$ and angle $CAB = \varphi$. The part cut off = sector $ACF + 2$ segment ANC , = sector $ACF + 2(\text{sector } DAC - \text{triangle } DAC)$; or

$$\frac{1}{2}\pi r^2 = x^2 \varphi + 2\left[\frac{1}{2}(\pi - 2\varphi)r^2 - \frac{1}{2}rx \sin \varphi\right]. \quad (1)$$

But $x = 2r \cos \varphi$. Substituting this value of x in (1) we have

$$\frac{1}{2}\pi r^2 = 4r^2 \varphi \cos^2 \varphi + \pi r^2 - 2r^2 \varphi - 2r^2 \sin \varphi \cos \varphi,$$

whence

$$\varphi = \frac{r^2 \sin 2\varphi - \frac{1}{2}\pi}{4r^2 \cos^2 \varphi - 2r^2} = \frac{\sin 2\varphi - 1.5708}{2(1 - 2 \sin^2 \varphi)}. \quad (2)$$

From equation (2), by the use of a table of natural sines, the value of φ is found to be $54^\circ 35' 39''$; $\therefore x = 2r \cos \varphi = 1.15872r$.



213. "Upon the three sides of any triangle construct equilateral triangles and join their centers by right lines. Prove that the triangle so formed is equilateral."

SOLUTION BY MARCUS BAKER, U. S. COAST SURVEY.

If circles be circumscribed about the three equilateral triangles these three circles will meet in a common point P , such that, A, B and C being the vertices of the triangle, $APB = BPC = CPA = 120^\circ$. Moreover, the lines joining the centers of the circles, two and two, are perpendicular to the common chord of two intersecting circles, viz., PA, PB and PC . Hence these lines by their intersection form an angle of 60° , and the resulting triangle is therefore equiangular and hence equilateral.

Note:—This is apart of Ex. 47 of Chauvenet's Geometry.

[The following demonstration of this proposition is the earliest we have seen, and was made by us and published in the *Ohio Journal of Education* for August, 1857.—Ed.]

Demonstration.—Let ABC represent any triangle, and ADB, BEC and AFC equilateral triangles described on its three sides; and let G, H and I represent, respectively, the centers of the three equilateral triangles.

Join $AG, AI, BH, BG, GH, HI, IG$, and CD . Then is $\angle GAB = \angle GBA = HBC = \angle IAC = 30^\circ$; and therefore $\angle CBD = \angle HBG$ (each being equal to $\angle HBD - 30^\circ$).



In like manner we may prove that angle $CAD = IAG$. Then, because $BD : BC :: BG : BH$, (Eucl. 6, IV) $\angle BGH = \angle BDC$. And in like manner we prove that $\angle AGI = \angle ADC$. Hence $AGI + BGH = 60^\circ$. But $AGB = 120^\circ$, therefore $HGI = 120^\circ - 60^\circ = 60^\circ$. In precisely the same way we may prove that GHI and HIG each equals 60° ; therefore the triangle GHI is equiangular, and consequently equilateral.

214. "Prove that

$$\frac{1}{\tan A + 1} + \frac{1}{\tan A + \&c.} = \frac{1}{2} [\sqrt{(\sec^2 A + 3)} - \tan A]."$$

SOLUTION BY HENRY GUNDER, NORTH MANCHESTER, IND.

Put $x = \frac{1}{\tan A + 1} + \frac{1}{\tan A + \&c.}$ Then $x = \frac{1}{\tan A + x}$, or $x^2 + x \tan A = 1$.

Therefore $x = \frac{1}{2} [-\tan A + \sqrt{(\tan^2 A + 4)}]$. Now $\tan^2 A + 4$
 $= \frac{\sin^2 A}{\cos^2 A} + 4 = \frac{\sin^2 A + \cos^2 A + 3 \cos^2 A}{\cos^2 A} = \sec^2 A + 3$.
 $\therefore x = \frac{1}{2} [\sqrt{(\sec^2 A + 3)} - \tan A].$

215. "A harbor A is so situated with reference to two headlands B and C , that the angle BAC is a right angle. A ship sails in a course making an angle of 55° with AB , 45 ms. to D , when $DB = DC$; she then sails forward on the same course 15 ms. to E , when BEC is a straight line.

Required AB , AC , DB , EB and EC ."

SOLUTION BY PROF. BEMAN, ANN ARBOR, MICH.

Let $AC = x$; $AB = y$; $CD = DB = m$; $AD = a$; $DE = b$; $DAB = \alpha$.
 From E draw EF perpendicular to AC . By similar triangles,

$$CF : FE :: CA : AB, \text{ or } x - (a+b) \sin \alpha : (a+b) \cos \alpha :: x : y,$$

$$y = \frac{(a+b) \cos \alpha}{x - (a+b) \sin \alpha} \quad (1)$$

$$m^2 = x^2 + a^2 - 2ax \sin \alpha \quad (2)$$

$$m^2 = y^2 + a^2 - 2ay \cos \alpha \quad (3)$$

Subtracting (3) from (2), and substituting values of y as given in (1) we easily obtain

$$x^2 - 2(2a+b)\sin ax^2 + (a+b)[3a - (2a+b)\cos 2a]x - 2a(a+b)^2\sin a = 0.$$

Where $a = 45$, $b = 15$, and $a = 55^\circ$, this becomes

$$x^2 - 210 \sin 55^\circ x^2 + 900(9 + 7 \sin 20^\circ)x - 324000 \sin 55^\circ = 0.$$

Having found x , y , and m , we can get CE and EB from the relations

$$CE^2 = x^2 + (a+b)^2 - 2(a+b)\sin ax,$$

$$BE^2 = y^2 + (a+b)^2 - 2(a+b)\cos ay.$$

SOLUTION BY HENRY HEATON, SABULA, IOWA.

The problem may be constructed as accurately as may be desired by the following method.

With O as center, and any radius as OH describe a circle $HAGP$, draw diameters GH and PR at right angles to each other; lay off the arc $HF = 110^\circ$, and on OR lay off $OS_1 = \frac{1}{4}OR$; draw FS_1 and prolong it until it cuts the arc HR in A_1 ; draw A_1T_1 perpendicular to OR , lay off $OS_2 = \frac{1}{4}OT_1$, draw FS_2 and prolong it until it cuts the arc HR in A_2 . Continue this until A_{n-1} is as near A_n as may be desired. Then A_n , T_n and S_n are A , T and S of the figure. On AF lay off $AD = 45m.$ and $DE = 15m.$; through E draw BC , parallel to GH , meeting AH and AG prolonged in B and C ; then A , B , C , D and E are A , B , C , D and E of the problem.

The demonstration is obvious.

To make the calculation indicated by the construction, we have $OF = r$, $OS_1 = \frac{1}{4}r$, and angle $FOS_1 = 160^\circ$. Angle $POA = 20^\circ =$ the sum and $ROA_1 =$ the difference between the other two angles.

Hence $\tan \frac{1}{2}ROA_1 = \frac{3}{5}\tan 10^\circ$; $\therefore ROA_1 = 12^\circ 13' 12''$. $OS_2 = \frac{1}{4}OT_1$ $ABC = \frac{1}{4}\cos 12^\circ 13' 12'' = .241835$.

Proceeding as before we get $\angle ROA_2 = 12^\circ 13' 26''$, $ROA_3 = 12^\circ 13' 26''$ &c. Hence angle $ROA = 12^\circ 13' 26''$. $\therefore AOH = 77^\circ 46' 34''$; $AGH = 38^\circ 53' 17''$; $\therefore ABC = 51^\circ 06' 43''$.

From the triangles AEB , AEC and ADB we find

$$AB = 74.0559 \text{ miles,}$$

$$AC = 91.8179 \text{ "}$$

$$EB = 63.1433 \text{ "}$$

$$EC = 54.8177 \text{ " and } BD = 60.7155 \text{ miles.}$$



216. "Through three given points to describe the minimum ellipse."

SOLUTION BY E. B. SEITZ, GREENVILLE, OHIO.

Let A, B, C be the three given points, and let $AB = a, AC = b$, angle $BAC = \beta$, and area of triangle $ABC = \Delta$.

Then the equation to the ellipse through A, B, C , referred to AB and AC , is of the form

$$mx^2 + ny^2 + xy - amx - bny = 0. \quad (1)$$

From (1) we readily find the area of the ellipse to be

$$u = \frac{2\pi mn(a^2m + b^2n - ab)\sin\beta}{\sqrt{(4mn-1)^3}}. \quad (2)$$

Hence we must have

$$\frac{m^2n^2(a^2m + b^2n - ab)^2}{(4mn-1)^3} = \text{a minimum}. \quad (3)$$

Differentiating (3) with respect to m , we have

$$(2mn+1)(a^2m + b^2n - ab) = a^2m(4mn-1). \quad (4)$$

Differentiating (3) with respect to n , we have

$$(2mn+1)(a^2m + b^2n - ab) = b^2n(4mn-1). \quad (5)$$

From (4) and (5) we find $m = \frac{b}{a}$, and $n = \frac{a}{b}$. Substituting the values of m and n in (1), we have

$$b^2x^2 + a^2y^2 + abxy - ab^2x - a^2by = 0,$$

$$\text{or } b^2(x - \frac{1}{3}a)^2 + a^2(y - \frac{1}{3}b)^2 + ab(x - \frac{1}{3}a)(y - \frac{1}{3}b) = \frac{1}{3}a^2b^2;$$

hence the centroid of the triangle ABC is the center of the minimum ellipse through A, B, C .

Substituting the values of m and n in (2), we have

$$u = \frac{2}{3}\pi\sqrt{3}.ab\sin\beta = \frac{4\pi}{3\sqrt{3}}\Delta; \therefore u : \Delta :: 4\pi : 3\sqrt{3}.$$

The maximum inscribed ellipse and the minimum circumscribed ellipse of the triangle are coaxial and similar, the axes of the latter being twice those of the former.

217. "The hypotenuse of a right triangle is fixed, and squares are described upon the other two sides: it is required to find the equation to the locus of the intersection of two straight lines, drawn from either extremity of the hypotenuse to the most distant corner of the square upon the opposite side. Find also the area of the curve."

SOLUTION BY PROF. W. W. JOHNSON, ST. JOHN'S COL., ANNAPOLIS, MD.

Taking the hypotenuse ($= 2a$) as axis of x and its middle point as origin, let θ denote the $\angle COA$ in the fig.; the triangles HCA and DAF are equal, and so are the triangles EGB and BHC ; hence the coordinates



of D are $(a + a \sin \theta, a - a \cos \theta)$ and those of E are $(-a - a \sin \theta, a + a \cos \theta)$.

Therefore the equation of BD is

$$y = + \frac{1 - \cos \theta}{2 + \sin \theta} (x + a), \quad (1)$$

and that of AE is

$$y = - \frac{1 + \cos \theta}{2 + \sin \theta} (x - a). \quad (2)$$

These equations give for the point of intersection P ,

$$x = a \cos \theta \text{ and } y = \frac{a \sin^2 \theta}{2 + \sin \theta}. \quad (3)$$

(The value of x showing that the point P is on CH .) Eliminating θ ,

$$y = \frac{a^2 - x^2}{2a + \sqrt{a^2 - x^2}},$$

$$\text{or} \quad x^4 + x^2 y^2 + 4ax^2 y - 2a^2 x^2 + 3a^2 y^2 - 4a^3 y + a^4 = 0. \quad (4)$$

The curve is readily traced by means of eq'ns (3). It consists of two branches meeting in cusps at A and B , the tangents at which make angles of 45° with the axis; the lower* branch corresponding to values of θ between 0 and 180° , as in the figure, has a vertex on the axis of y distant $\frac{1}{3}a$ from the origin, the upper branch, corresponding to values of θ between 180° and 360° (the squares being described on the inner sides of the lines AC and BC) has a vertex on the axis of y distant a from the origin.

For the area we have the integral

$$\int y dx = -a^2 \int \frac{\sin^3 \theta d\theta}{2 + \sin \theta}.$$

Since dx is negative from $\theta = 0$ to $\theta = 180^\circ$, the integral between these limits will give, negatively, the area between the lower branch and the axis of x ; while between the limits 180° and 360° it will give, positively, the area between the upper branch and the axis of x ; hence the req'd area is

*The vertex of the upper branch, as represented by the figure, is distant $\frac{1}{3}a$ from the origin, while that of the lower branch is distant a .—Ed.

$$-a^2 \int_0^{2\pi} \sin^2 \theta d\theta + 2a^2 \int_0^{2\pi} \sin \theta d\theta - 4a^2 \int_0^{2\pi} d\theta + 8a^2 \int_0^{2\pi} \frac{d\theta}{2 + \sin \theta}$$

The value of $\int \frac{d\theta}{2 + \sin \theta}$ is $\frac{2}{\sqrt{3}} \tan^{-1}[\sqrt{\frac{1}{3}} \cdot \tan \frac{1}{2} \theta]$; as the arc $\frac{1}{2} \theta$ passes from 0 to π it is evident that this inverse tangent also passes from 0 to π ; hence the area is $(\frac{1}{3}\sqrt{3}-9)\pi a^2$, approximately $.24\pi a^2$.

NOTE, BY THE EDITOR.—Equation (4), in the above solution of problem 217, obviously represents two real and unequal values of y for each value of x , and hence the double sign should be written before the radical; $\therefore$ the area of the upper and lower branches respectively will be represented by

$$\int_{-a}^a \frac{(a^2-x^2)dx}{2a+\sqrt{(a^2-x^2)}}, \text{ and } \int_{-a}^a \frac{(a^2-x^2)dx}{2a-\sqrt{(a^2-x^2)}}$$

and the entire area of the closed curve is

$$\begin{aligned} \int_{-a}^a \left(\frac{a^2-x^2}{2a+\sqrt{(a^2-x^2)}} + \frac{a^2-x^2}{2a-\sqrt{(a^2-x^2)}} \right) dx &= \int_{-a}^a \frac{4a(a^2-x^2)dx}{3a^2+x^2} \\ &= \frac{1}{3}\sqrt{3}\pi a^2 - 8a^2, \text{ approximately } = 1.67a^2. \end{aligned}$$

The area of the branch which lies above the axis of x is $=\pi a^2(\frac{2}{3}-\frac{1}{9}\sqrt{3})-4a^2$; therefore the area of the bell-shaped branch which lies below the axis of x is $=\pi a^2(\frac{2}{3}\sqrt{3}-\frac{2}{3})-4a^2$; and the area of the hat-shaped curve which lies between the lower curve and the upper curve *inverted*, and represents the *difference* between the two branches, is $= (\frac{1}{3}\sqrt{3}-9)\pi a^2$.

218. "Required the separate rates of dividend of two insolvent estates connected actually as follows:

John Doe's Estate. Direct liabilities \$33,425.61, also endorsements for Richard Roe \$34,949.16, less primary dividend to be paid from Roe's estate. Net proceeds to be divided \$12,395.76.

Richard Roe's Estate. Direct liabilities \$46,212.00, also endorsements for John Doe \$9,500.00, less primary dividend to be paid from Doe's estate. Net proceeds to be divided \$9,493.25, augmented by dividend on \$4,796.65 account due from John Doe's estate."

SOLUTION BY GEORGE EASTWOOD, SAXONVILLE, MASS.

Let the rate per cent of dividend on Doe's estate be designated by x , and that on Roe's estate by y .

Put Doe's direct liabilities <i>plus</i> his endorsements	= \$68374.77 = λ ,
Roe's " " " " "	= 55712.00 = λ' ,
Doe's net assets	= 12395.76 = α ,
Roe's " "	= 9493.25 = α' ,
Roe's account due from Doe's estate	= 4796.65 = β .

By the question, $\alpha' + \frac{1}{100}\beta x$ = amount to be divided among Roe's cred's.
 $\lambda' - \frac{1}{100}\alpha x$ = Roe's gross liabilities *less* primary dividend
 from Doe's estate, and

$(\lambda' - \frac{1}{100}\alpha x)\frac{1}{100}y$ = amount which Roe's estate will produce for his creditors, which from the nature of the question must be the same as $\alpha' + \frac{1}{100}\beta x$.

Equating these two expressions, and reducing,

$$y = \frac{100(100\alpha' + \beta x)}{100\lambda' - \alpha x}.$$

Again $(\alpha' + \frac{1}{100}\beta x)\frac{1}{100}y$ = $\frac{(100\alpha' + \beta x)^2}{100(100\lambda' - \alpha x)}$ = dividend on Roe's estate to be paid to Doe's estate, and

$$\lambda - \frac{(100\alpha' + \beta x)^2}{100(100\lambda' - \alpha x)}$$

= Doe's gross liabilities *less* primary dividend from Roe's state, and

$$\left(\lambda - \frac{100\alpha' + \beta x}{100(100\lambda' - \alpha x)}\right) \cdot \frac{x}{100}$$

= dividend on Doe's estate, which, from the nature of the question, must give his net assets = α .

Equating and reducing, there results,

$$x^3 + \frac{200\alpha'\beta + 100\alpha\lambda}{\beta^2}x^2 - \frac{(100)^2\lambda\lambda' + (100)^2\alpha - (100)^2\alpha'^2}{\beta^2}x + \frac{(100)^3\alpha\lambda'}{\beta^2} = 0,$$

or, taking the nearest integral coefficients,

$$x^3 + 4084x^2 - 1683265x + 30015503 = 0.$$

This equation has three real roots, two positive and one negative; but only one positive root, viz., $x = 18.68$ is applicable to the present question. This value of x gives the value of $y = 19.46$. Hence the separate rates of dividend are, approximately, 18.68 and 19.46, respectively.

[Mr. Meech finds $x = 20.1293$ and $y = 19.4399$, but has not given his analysis.]

219. "What is the locus generated by the vertex of a right angled sides given, sliding between the branches of a given paral

SOLUTION BY E. B. SEITZ.

Let BPC be the right angle, A the vertex of the parabola, AM the axis. Draw PM , BD , CE perpendicular to AM , and PF perpendicular to CE produced.

Let $BP = b$, $CP = c$, $AM = x$, $PM = y$, $\angle BPF = \theta$, and $4a$ = the parameter of the parabola.

Then from the figure we have $BD = y + b \sin \theta$, $AD = x - b \cos \theta$, $CE = y - c \cos \theta$, $AE = x - c \sin \theta$. Hence from the parabola we have

$$(y + b \sin \theta)^2 = 4a(x - b \cos \theta)^2, \quad (1)$$

$$(y - c \cos \theta)^2 = 4a(x - c \sin \theta)^2. \quad (2)$$

By eliminating θ between (1) and (2), the equation to the required locus can be found. The equation would apparently be very complicated.

[Mr. Heaton, by eliminating θ from a similar pair of equations, gets
 $(ay - bp)^2 \{ [ab(by + ap)(a^2bp - 2bpx - 2abxy - bpy^2 - ay^3) + 2a^2b^2(p^2 + y^2)^2] - 2ab(p^2 + y^2) \sqrt{[ab(by + ap)(a^2bp - 2bpx - 2abxy - bpy^2 - ay^3) + a^2b^2(p^2 + y^2)^2]} \}$
 $+ (by + ap)^2 \{ [ab(ay - bp)(2ap^2x + 2bxy - ab^2p - apy^2 - by^3) + 2a^2b^2(p^2 + y^2)^2] - 2ab(p^2 + y^2) \sqrt{[ab(ay - bp)(2ap^2x + 2bxy - ab^2p - apy^2 - by^3) + a^2b^2(p^2 + y^2)^2]} \}$
 $- a^2b^2(by + ap)^2(ay - bp)^2 = 0.]$

220. "Through a given point in the surface of a circle, a chord is drawn at random. If another chord be drawn at right angles to the first, what is the probability that they will intersect?"

SOLUTION BY PROF. JOHNSON.

Let a be the radius of the given circle, and c the distance of the given point from its centre, and denote by θ the angle between any chord and the diameter passing through the given point; then the length of the chord is

$$2 \sqrt{a^2 - c^2 \sin^2 \theta}.$$

Assuming that equally distant parallel chords are equally probable, the chance that a line, perpendicular to this chord and known to cut the circle, will cut the chord within the circle, is evidently the ratio of the length of the chord to the diameter. That is, denoting the ratio $c \div a$ by e , the chance is

$$\sqrt{1 - e^2 \sin^2 \theta}.$$

Hence, assuming all directions of the chord to be equally probable, the required probability is



$$\frac{1}{2\pi} \int_0^{2\pi} \sqrt{1 - e^2 \sin^2 \theta},$$

which is the ratio of the perimeter of an ellipse, whose eccentricity is e , to the circle on its major axis; in other words, the ratio of the perimeter of an inscribed ellipse, with the given point for a focus, to that of the given circle.

[If all *possible* chords are drawn through a point (not the center) within a circle they will not be equally dense around the point; but an indefinite number of chords, or lines, drawn at random *through* a point will be equally dense around the point, hence all values of θ , in the above solution, are, *in fact*, equally probable.

Again, perpendiculars to a chord, equally distant from each other, will not represent all *possible* chords at right angles to that chord; but if in drawing an indefinite number of chords at right angles to the chord first drawn, points be selected on the first chord (which would be likely to be done in practice) through which to draw the second random chords, these chords will be, *in fact*, equally distant from each other.

Mr. Heaton sent two solutions of this problem, corresponding, respectively to the two hypotheses here presented.—Ed.]

PROBLEMS.

221. BY PROF. ORSON PRATT, SEN., SALT LAKE CITY, UTAH.—Find the four roots of the equation,

$$x^4 + px^3 + \frac{7p^2}{4^2}x^2 + \frac{6p^3}{4^3}x + q = 0.$$

222. BY GEORGE NEWBAUER, ANSONIA, OHIO.—It is required to find four positive integral numbers, the sum of the cubes of any three of which shall be a rational cube number.

223. BY WILLIAM HOOVER, BELLEFONTAINE, OHIO.—Compare the masses of two spheres of equal radii, one of uniform density, the other increasing in density from a single point in the surface as the square of the distance measured on the diameter through that point increases.

224. BY GEO. H. HARVILL, BONNER, LA. — Two posts, each 12 feet high, stand perpendicularly, fifty feet apart, on a horizontal plane, and a rope, one end of which is fastened at the top of each post, is suspended between them so that it just touches the plane at its middle point. Required the length of the rope.

225. BY PROF. KERSHNER.—What is the locus of the angles of the inscribed squares of the respective segments of a circle when the chords or bases are parallel?

226. BY PROF. JOHNSON.—Find the algebraic equation whose roots are the real quantities found by giving integral values to k in

$$x = \cos \frac{2k\pi}{n},$$

n being a given integer. Also the equation whose solution is

$$x = \sin \frac{2k\pi}{n}.$$

227. BY A. S. HATHAWAY, CORNELL UNIV., ITHACA, N. Y.—If the equation $x^n + ax^{n-1} + \dots = 0$, whose roots are α, β , etc., be transformed into another, one of whose roots is 0, while the differences are unchanged, and then the transformed equation be divided by x ; and if this process be repeated $n-1$ times, prove that the product of the differences of the roots of the original equation is equal to the product of the absolute terms of the equations thus obtained.

228. BY A. MARTIN, M. A.—Integrate $dI = \log(a + \sqrt{x^2 + b^2})dx$.

229. BY CHAS. H. KUMMELL, DETROIT, MICH.—Evaluate

$$\int_{-x}^x \frac{\cos mx}{a+x} dx \text{ and } \int_{-x}^x \frac{\sin mx}{a+x} dx.$$

230. BY MARCUS BAKER, U. S. COAST SURVEY.— $ABCD$ is any tetrahedron. Through A pass a plane parallel to BCD , through B a plane parallel to CDA , through C a plane parallel to DAB and through D a plane parallel to ABC . A new tetrahedron is thus formed the volume of which call V' . Also let V equal the volume of the original tetrahedron.

Prove that $V' = 27V$.

ERRATA.

On page 127, line 8 from bottom, for "to D ", read, 45 miles to D .

" " " " 4 " " for "maximum", read, minimum.

" " 130, line 5, for " 3×2 ", read, 3×-2 .

" " 135, first line of Table, for the imperfect figure in 2nd column, read, 9, and for the imperfect figure in 6th column, read, 7.

" " 140, line 10, for " $+(u_2 + u_{-2})$ ", read, $-(u_2 + u_{-2})$.

" " 146, " 12, insert, fraction, at the beginning of the line.

" " " 13, for " $(a-1)a$ ", read, $(a+1)a$.

" " " 17, for " β ", as index of b , read, a .

THE ANALYST.

VOL. V.

NOVEMBER, 1878.

No. 6.

SOLUTION OF THE EQUATION OF THE FIFTH DEGREE.

TRANSLATED FROM THE THEORY OF ELLIPTIC FUNCTIONS OF
BRIOT AND BOUQUET, SECOND EDITION.

BY ALEXANDER EVANS, ESQ., ELKTON, MARYLAND.

421. It was announced by Galois that, so far as $n = 11$, the degree of the modular equation may be lowered by unity. M M. Hermite and Betti have given two demonstrations of this theorem based upon different principles. Let us consider a symmetric or alternate function (v_a, v_b) of two roots of the modular equation, then, the same function of two other roots, and so on as far as the last two roots, and let us designate by U a symmetrical function of these $\frac{1}{2}(n+1)$ quantities, we easily demonstrate, by the aid of the laws of permutation before established, that when n does not surpass 11, among the different manners of associating the roots, two by two, there obtain those for which the function U only acquires n values for each value of u and by consequence satisfies an equation of the degree n .

About each one of the critical points a root v_a remains holomorphic and the associated root v_b acquires the n other values; the quantity (v_a, v_b) and in consequence the function U takes n values, forming a circular system.

If this function only acquires n values in the whole extent of the plan, it is impossible that two of these values can have a common element (v_a, v_b) ; for by a suitable way, v_a becomes equal to the root which remains holomorphic about one of the critical points; if two values of U had a common element (v_a, v_b) without being identical, they would engender two circular systems of n values each.

We may associate two roots v_a and v_b taken at pleasure; for, about the critical point where v_a remains holomorphic, v_b acquires the n other values.

In a single value of the function U two elements cannot present the same difference of indices, because if we should describe the contour or lacing ($\bigcirc$) [*le lacet*] a certain number of times one of the elements would become equal to the other. Since we have perceived that the contour (α_0) causes the function U to acquire the same values as the loop ($\bigcirc$), we are certain that this function has only n values in the whole plan, the other lacings releading to the loops (α_0) and ($\bigcirc$).

422. These considerations allow us to find easily the favorable ways.

We should associate the roots by way of difference and form the product of the $\frac{1}{2}(n+1)$ quantites.

For $n = 5$ if we take as first factor $V-v_0$, the combination

$$(1) \quad U = (V-v_0)(v_1-v_4)(v_2-v_3)$$

is the only one in which the difference of the indices in the last factors is the same.

By the contour ($\bigcirc$) this function acquires the five values

$$(2) \quad \begin{cases} U_0 = (V-v_0)(v_1-v_4)(v_2-v_3), \\ U_1 = (V-v_1)(v_2-v_0)(v_3-v_4), \\ U_2 = (V-v_2)(v_3-v_1)(v_4-v_0), \\ U_3 = (V-v_3)(v_4-v_2)(v_0-v_1), \\ U_4 = (V-v_4)(v_0-v_3)(v_1-v_2). \end{cases}$$

After the law of permutation written in number 408 the contour (α_0) reproduces the same values in another order; from that we conclude that the function U has only the five preceding values, and by consequence that it satisfies an algebraic equation between u and U of the fifth degree in U .

For $n = 7$ there are two favorable combinations,

$$(3) \quad U = (V-v_0)(v_2-v_3)(v_4-v_6)(v_1-v_5),$$

$$(4) \quad U = (V-v_0)(v_2-v_6)(v_5-v_4)(v_1-v_3).$$

We obtain them in the following manner; upon the contour (α_0) the law of permutation is ($V, v_5, v_6, v_4, v_3, v_1, v_2$).

Let us take as first factor $V-v_0$, and as the second v_2-v_3 , for example v_2-v_3 ; when we describe the noose (α_0) the product $(V-v_0)(v_2-v_3)$ becomes $(v_5-v_0)(V-v_1)$: the first product being completed, before reproducing the second, the contour ($\bigcirc$) once run over, will contain the factor v_4-v_6 , which gives origin to v_5-v_0 , and by consequence the last factor will be v_1-v_5 . By the noose ($\bigcirc$) the function (3) acquires seven values: these same values reproduce themselves upon the loop (α_0); whence we conclude that the function U satisfies an algebraic equation between u and U , of the seventh degree in U . The function (4) possesses the same property.

For $n = 11$, we have also two favorable combinations

$$U = (V - v_0)(v_{10} - v_5)(v_3 - v_6)(v_9 - v_7)(v_2 - v_1)(v_4 - v_8),$$

$$U = (V - v_0)(v_{10} - v_9)(v_8 - v_5)(v_7 - v_3)(v_6 - v_1)(v_2 - v_4),$$

which we obtain in the same manner.

Upon the contour (a_0) the law of permutations is $(V, v_1, v_6, v_4, v_3, v_9, v_8, v_7, v_5, v_{10})$.

We will take as first factor $V - v_0$, and as second $v_{10} - v_5$. For $n = 13$ there is no favorable combination.

§23. From the depression of the degree of the modular equation for $n = 5$, M. Hermite has deduced a method for the resolution of the equation of the fifth degree by elliptic functions (*Comptes rendus*, t. XVIII).

Let us form the equation of the fifth degree in U . The values of U become finite for all finite values of u , the coefficient of U^5 is equal to unity. For $u = \infty$ the six values of v are infinite, one of the degree 5, the others of the degree $\frac{1}{5}$ (number 408): the five values of U are infinite and of the degree $\frac{2}{5}$; the sum of the negative orders of the function U being equal to the equation is of the 27th degree in relation to u (number 135).

For values of u exceedingly small the approximate values of v are

$$V = -2^{-2}u^5, v_0 = 2^{\frac{2}{5}}u^{\frac{1}{5}}, v_1 = v_0e^{\frac{2\pi i}{5}}, v_2 = v_0e^{2\frac{2\pi i}{5}}, v_3 = v_0e^{3\frac{2\pi i}{5}}, \\ v_4 = v_0e^{4\frac{2\pi i}{5}},$$

consequently those of U are,

$$U_0 = 2^{\frac{6}{5}}5^{\frac{1}{2}}u^{\frac{3}{5}}, U_1 = U_0e^{\frac{6\pi i}{5}}, U_2 = U_0e^{2\frac{6\pi i}{5}}, U_3 = U_0e^{3\frac{6\pi i}{5}}, \\ U_4 = U_0e^{4\frac{6\pi i}{5}}.$$

We have put $\xi = \frac{v}{u^5}$; putting in the same way $\phi = \frac{U}{u^{15}} = (\xi - \xi_0)$

$-\xi_4)(\xi_2 - \xi_3)$, the coefficients of the equation in ξ being entire polynomials of degree 5 (number 401), those of the equation in ϕ will possess the same property; the values of ϕ only becoming infinite for $u = 0$; this equation will be of the form

$$+\sum_{p=1}^{p=4} u^{8\beta_p} (a_p + b_p u^8 + c_p u^{16} + \dots) \phi^{5-p} + (a_5 + b_5 u^8 + c_5 u^{16} + \dots) = 0.$$

Replacing ϕ by $U + u^{15}$ and multiplying by u^3 , we obtain the equation

$$U^5 + \sum_{p=1}^{p=4} u^{8(\beta_p-9)+15p} (a_p + b_p u^8 + \dots) U^{5-p} + u^3(a_5 + a_5 u^8 + \dots) = 0.$$

For small values of u , the values of U being very small of the degree 5, forming a circular system, we have $\beta_0 = 9$, $a_5 = -2^6 5^{\frac{1}{2}}$, $8(\beta_p - 9) = 0$, and in consequence $\beta_p = 9$ or $> 10 - 2p$; we will take $\beta_p = 10 - 2p$. The equation sought for is then of the form

$$(8) \quad U^5 + \sum_{p=1}^{p=4} u^{8-p}(a_p + b_p u^8 + c_p u^{16}) U^{5-p} + u^3(a_5 + b_5 u^8 + c_5 u^{16} + d_5 u^{24}) = 0.$$

If we consider the function $(U) = [(V) - (v_0)] [(v_1) - (v_4)] [(v_2) - (v_3)]$, relative to the variable $u_1 = 1 \div u$, we have

$$(U) = \frac{-1}{V v_0 v_1 v_2 v_3 v_4} (V - v_2)(v_3 - v_1)(v_4 - v_0) = \frac{U}{u^6},$$

since the product of the roots of the modular equation (number 411) is equal to $-u^6$.

So the equation (8) ought not to change when we replace in it u by $1 \div u$, and U by $U \div u^6$. But, by this substitution the equation becomes

$$U^5 + \sum_{p=1}^{p=4} u^{7p-8} (a_p + b_p u^{-8} + c_p u^{-16}) U^{5-p} + u^3(d_5 + c_5 u^8 + b_5 u^{16} + a_5 u^{24}) = 0,$$

and we hence conclude that it contains no term in U^4 , and that we have $b_2 = c_2 = 0$, $c_3 = 0$, $b_3 = a_3$, $c_4 = a_4$, $d_5 = a_5$, $c_5 = b_5$. The equation reduces to

$$(9) \quad \begin{cases} U^5 + a_2 u^8 U^3 + a_3 u^8 (1 + u^8) U^2 + u^4 (a_4 + b_4 u^8 + a_4 u^{16}) U \\ + u^3 (a_5 + b_5 u^8 + b_5 u^{16} + a_5 u^{24}) = 0. \end{cases}$$

For $u = 1$, the root v_0 of the modular equation is equal to $+1$, and the five others to -1 (number 406); the five values of U reduce themselves to zero; the polynomes in u , coefficients of the diverse powers of U in the equation (9) ought to become null for $u = 1$; and we deduce $a_2 = 0$, $a_3 = 0$, $b_4 = -2a_4$, $b_5 = -a_5$, which reduces the equation to the simple form

$$(10) \quad U^5 + a_4 u^4 (1 - u^8)^2 U + a_5 u^3 (1 - u^8)^2 (1 + u^8) = 0.$$

We know the coefficient a_5 , it remains to determine the coefficient a_4 ; we follow the road which has been pointed out in number 411 for the calculation of the modular equation.

To a real, positive, and very small value of u , corresponds a value of ρ of the form $\rho = si$, s being positive and very great, and in consequence a value of q real, positive, and very small.

By developing $\varphi(\rho)$ in series and restricting oneself to the first two terms there results

$$\begin{aligned} u &= 2^{\frac{1}{2}} q^{\frac{1}{8}} (1 - q), \quad v_0 = 2^{\frac{1}{2}} q^{\frac{1}{40}} (1 - q^{\frac{1}{5}}), \\ v_1 &= 2^{\frac{1}{2}} q^{\frac{1}{40}} (e^{\frac{2\pi i}{5}} - q^{\frac{1}{5}} e^{-\frac{2\pi i}{5}}), \\ v_2 &= 2^{\frac{1}{2}} q^{\frac{1}{40}} (e^{\frac{4\pi i}{5}} - q^{\frac{1}{5}} e^{-\frac{4\pi i}{5}}), \\ v_3 &= 2^{\frac{1}{2}} q^{\frac{1}{40}} (e^{-\frac{4\pi i}{5}} - q^{\frac{1}{5}} e^{\frac{4\pi i}{5}}), \\ v_4 &= 2^{\frac{1}{2}} q^{\frac{1}{40}} (e^{-\frac{2\pi i}{5}} - q^{\frac{1}{5}} e^{\frac{2\pi i}{5}}), \end{aligned}$$

$$v_1 - v_4 = i2^{\frac{3}{2}} \sin \frac{2\pi}{5} q^{\frac{1}{40}} (1 + q^{\frac{1}{5}}),$$

$$v_2 - v_3 = i2^{\frac{3}{2}} \sin \frac{4\pi}{5} q^{\frac{1}{40}} (1 + q^{\frac{1}{5}}),$$

$$U_0 = 2^{\frac{3}{2}} 5^{\frac{1}{2}} q^{\frac{3}{40}} (1 + q^{\frac{1}{5}}),$$

By substituting in equation (10), dividing all the terms by $q^{\frac{1}{5}}$, then equating to zero the constant term and the coefficient of $q^{\frac{1}{5}}$, we find $a_5 = -2^6 5^{\frac{1}{2}}$, $a_4 = -2^6 5^3$; the equation sought is then

$$(11) \quad U^5 - 2^6 5^3 u^4 (1 - u^8)^2 U - 2^6 5^{\frac{5}{2}} u^2 (1 - u^8)^2 (1 + u^8) = 0.$$

424. If we put $x = hU$ this equation becomes

$$(12) \quad x^5 - 2^6 5^3 h^4 u^4 (1 - u^8)^2 x - 2^6 5^{\frac{5}{2}} h^5 u^2 (1 - u^8)^2 (1 + u^8) = 0.$$

We know that M. Jerrard has reduced the general equation of the fifth degree to the form

$$(13) \quad x^5 - Ax - B = 0.$$

But we may dispose the two parameters u , and h , which the equation (12) contains in such a manner as to identify the two preceding equations. For that, it will be sufficient to put

$$2^6 5^3 h^4 u^4 (1 - u^8)^2 = A, \quad 2^6 5^{\frac{5}{2}} h^5 u^2 (1 - u^8)^2 (1 + u^8) = B,$$

whence

$$(14) \quad h = \frac{\sqrt{5} B}{4 A} \frac{u}{1 + u^8};$$

and substituting this value of h , in the equation $2^6 5^{\frac{5}{2}} h^5 u^2 (1 - u^8)^2 (1 + u^8) = \sqrt{A}$, we arrive at the equation of the second degree

$$(15) \quad \left(u^4 - \frac{1}{u^4}\right)^2 + \frac{5^{\frac{5}{2}} B^2}{4 A^2 \sqrt{A}} \left(u^4 - \frac{1}{u^4}\right) + 4 = 0.$$

From this last equation we will deduce $u^4 - (1 \div u^4)$ and by consequence u^4 or k ; we will seek for one of the corresponding values of q or of ρ ; from it we will deduce the six values of v , and consequently by aid of the formulas (2) the five values of U ; and in multiplying them by the known quantity h , we will have at last the roots of the proposed equation (13).

425. We should see in the same manner that for n equal 7 the equation in U which is of the 52nd degree in u is of the form

$$(16) \quad \begin{cases} U^7 + a_3 u^4 (1 - u^8) U^4 + a_5 u^4 (1 - u^8)^4 U^2 + a_6 u^8 (1 - u^8)^4 U \\ + a_7 u^4 (1 - u^8)^4 (1 - u^8 + u^{16}) = 0. \end{cases}$$

The consideration of the discriminant (number 413),
the last term. We will obtain the four coefficients
equation by developing in series, as before, the function
the development as far as the fifth term, to wit;

$$u = 2^{\frac{1}{2}} q^{\frac{1}{8}} (1 - q + 2q^2 - 3q^3 + 4q^4),$$

$$v_0 = 2^{\frac{1}{2}} q^{\frac{4}{56}} (1 - q^{\frac{1}{7}} + 2q^{\frac{2}{7}} - 3q^{\frac{3}{7}} + 4q^{\frac{4}{7}}).$$

We find also for the first combination

$$U = i2^{27\frac{1}{2}} q^{\frac{1}{14}} \left(1 - \frac{1-i\sqrt{7}}{2} q^{\frac{2}{7}} + q^{\frac{4}{7}} \right),$$

whence

$$a_7 = i2^{127\frac{7}{2}}, a_5 = 0, a_3 = -i2^{47\frac{5}{2}} \frac{1-i\sqrt{7}}{2}, a_1 = 2^{87} \left[1 - \left(\frac{1-i\sqrt{7}}{1} \right) \right].$$

We deduce the second combination from the first in replacing i by $-i$.

The following is extracted from Todhunter's Theory of Equations, sections 340, 341.

"The general equation of the fifth degree can always be reduced to one of the following forms:

" $x^5 + px + q = 0$; $x^5 + px^2 + q = 0$; $x^5 + px^3 + q = 0$; $x^5 + px^4 + q = 0$.
[Due to Mr. Jerrard or to E. S. Bring. Quarterly Journal of Mathematics, Vol. VI.]

"Mr. Jerrard considered that the algebraical solution of equations of the 5th degree could be effected; his proposed method formed the subject of an enquiry by Sir W. R. Hamilton in the *Reports of the British Association*, Vol. VI. Most mathematicians admit that ABEL has demonstrated the impossibility of the algebraical solution of equations of a higher degree than the fourth. An abstract of Sir W. R. Hamilton's exposition of *Abel's* argument will be found in the *Quarterly Journal of Mathematics*, Vol. V. A simpler demonstration due to *Wantzel* will be found in *Serret's Cours d'Algebre Supérieure*." [See ANALYST, p. 65-70, Vol. IV.] "An *Essay* on the resolution of algebraical equations by the late *Judge Hargreave* has been recently published: the results arrived at are to some extent at variance with those of *Abel* and *Sir W. R. Hamilton*."

TO DRAW A CIRCLE TANGENT TO THREE GIVEN CIRCLES.

BY ISAAC H. TURRELL, CINCINNATI, OHIO.

THIS problem is a celebrated one in the history of pure geometry. When or by whom it was first proposed I am unable to ascertain; but the best mathematicians of modern times have not thought it unworthy

their special attention. In Salmon's Conics (5th edition) not less than four different solutions are given. In the 8th volume of Smithsonian Contributions to Knowledge, Major Alvord, by a different and very elegant method, solves, not only this, but the more general problem—"To draw a sphere tangent to four given spheres."

The following solution so far as the writer is aware is original. It depends upon the following elementary propositions.

I. A circle orthogonal to two tangent circles passes through their point of contact.

II. All the circles touching two given circles externally, are cut orthogonally by an auxiliary circle whose center is the external center of similitude of the two circles.

III. An axis of similitude of three given circles, and the radical axis of a pair of circles touching them are coincident.

Let us first take the case in which the circles O , O_1 , O_2 , which are drawn auxiliary (Pr. II) to the pairs AB , AC and BC , respectively, meet in two points P , P_1 . Since these auxiliaries have their centers on the radical axis (Prop. III) of the req'd pair S , S_1 , which they cut orthogonally, they must pass through their limiting



points; hence P , P_1 are the limiting points, or infinitesimal circles, of the system determined by S , S_1 .

To find the points of tangency on A , draw the radical axis of P and A meeting OO_1 in M . The circle M , passing through P , evidently cuts the three circles S , S_1 and A orthogonally; hence (Prop. I) it passes through their points of contact N , N_1 . In a similar manner the points of tangency of S , S_1 with B and C can be obtained.

It may happen that the auxiliaries O , O_1 , O_2 , will not intersect; but since they belong to a system complementary to that of S , S_1 , the latter will pass through their limiting points, which will lie on the line OO_1 .

In this case, which a slight change in the diagram will illustrate, find the limiting points, P , P_1 of the system O , O_1 , O_2 , through which draw

any circle V . This circle will belong to the system S, S_1 . From M , the intersection of the radical axis of V and A with the line OO_1 , draw a circle cutting V orthogonally; it will evidently cut S, S_1, A orthogonally, and pass through their points of contact N, N_1 , as in the former case.

Repeating this process with the other three axes of similitude we can determine the other six touching circles.

This solution can be readily extended to include Gen'l Alvord's problem on the tangencies of Spheres.

SUMMATION OF TWO SERIES.

BY PROF. D. TROWBRIDGE, WATERBURGH, NEW YORK.

REQUIRED the sums of the following series:

$$S_x^{(n)} = (a+r)^n + (a+2r)^n + (a+3r)^n + \dots + (a+rx)^n, \quad (1)$$

$$S_x^{(n)} = (a+r)^n + (a+3r)^n + (a+5r)^n + \dots + (a+2rx-r)^n. \quad (2)$$

In these series x is the number of terms, and hence, when $x=0$, the sums will equal nothing. It is plain that the sum of either series will be a function of x ; that is it will depend on x in such a way that when $x=0$, the sum will be 0; and we easily see that the sum will involve the powers of x from x^{n+1} down to x ; we say the $(n+1)$ th power, because the expansion of the last term will give the n th power, and since there are x terms, all similar, there will be the $(n+1)$ th power of x , at least. We may therefore assume

$$(a+r)^n + (a+2r)^n + \dots + (a+rx)^n = A_1 x^{n+1} + A_2 x^n + \dots + A_{n+1} x, \quad (3)$$

$$(a+r)^n + (a+3r)^n + \dots + (a+2rx-r)^n = B_1 x^{n+1} + B_2 x^n + \dots + B_{n+1} x. \quad (4)$$

In these series let $x+1$ be written for x —for they are true for at least all positive integral powers of x —and we shall have

$$(a+r)^n + (a+2r)^n + \dots + (a+rx)^n + (a+rx+r)^n = A_1 (x+1)^{n+1} + A_2 (x+1)^n + \dots + A_{n+1} (x+1), \quad (5)$$

$$(a+r)^n + (a+3r)^n + \dots + (a+2rx-r)^n + (a+2rx+r)^n = B_1 (x+1)^{n+1} + B_2 (x+1)^n + \dots + B_{n+1} (x+1). \quad (6)$$

Now substitute (3) from (5) and (4) from (6) and we have

$$(a+rx+r)^n = A_1 [(x+1)^{n+1} - x^{n+1}] + A_2 [(x+1)^n - x^n] + \dots + A_{n+1}, \quad (7)$$

$$(a+2rx+r)^n = B_1 [(x+1)^{n+1} - x^{n+1}] + B_2 [(x+1)^n - x^n] + \dots + B_{n+1}. \quad (8)$$

Now make $a+r = b$, and $2r = r'$ in (8) and the two series will have the same form. We can, therefore, easily find the B 's from the A 's.

If we expand the several terms in (7) and equate the coefficients of the like powers of x we can determine the values of $A_1, A_2, \&c.$; and we shall further see that had we assumed a power of x in the sum, greater than $n+1$ its coefficient would have been 0.

We have

$$\begin{aligned}(a+rx+r)^n &= (rx+b)^n = r^n x^n + nbr^{n-1}x^{n-1} + \frac{n(n-1)}{1.2}b^2r^{n-2}x^{n-2} + \dots \\ &= A_1 \left[(n+1)x^n + \frac{(n+1)n}{1.2}x^{n-1} + \frac{(n+1)n(n-1)}{1.2.3}x^{n-2} + \dots \right] \\ &+ A_2 \left[nx^{n-1} + \frac{n(n-1)}{1.2}x^{n-2} + \frac{n(n-1)(n-2)}{1.2.3}x^{n-3} + \dots \right] \\ &+ A_3 \left[(n-1)x^{n-2} + \frac{(n-1)(n-2)}{1.2}x^{n-3} + \dots \right] \\ &+ A_4 \left[(n-2)x^{n-3} + \dots \right].\end{aligned}\tag{9}$$

From this equation we have

$$\left. \begin{aligned}r^n &= (n+1)A_1, \quad nbr^{n-1} = \frac{(n+1)n}{1.2}A_1 + nA_2, \\ \frac{n(n-1)}{1.2}b^2r^{n-2} &= \frac{(n+1)n(n-1)}{1.2.3}A_1 + \frac{n(n-1)}{1.2}A_2 + (n-1)A_3, \\ \frac{n(n-1)(n-2)}{1.2.3}b^3r^{n-3} &= \frac{(n+1)n(n-1)(n-2)}{1.2.3.4}A_1 \\ &+ \frac{n(n-1)(n-2)}{1.2.3}A_2 + \frac{(n-1)(n-2)}{1.2}A_3 + (n-2)A_4, \&c.\end{aligned} \right\} \tag{10}$$

$$\begin{aligned}\frac{n(n-1)(n-2)\dots(n-p)}{1.2.3\dots(p+1)}b^{p+1}r^{n-p-1} &= \frac{(n+1)n(n-1)\dots(n-p)}{1.2.3\dots(p+2)}A_1 \\ &+ \frac{n(n-1)\dots(n-p)}{1.2.3\dots(p+1)}A_2 + \dots + (n-p)A_{p+2}.\end{aligned}\tag{11}$$

From these equations we find

$$\left. \begin{aligned}A_1 &= \frac{r^n}{n+1}, \quad A_2 = \frac{r^{n-1}}{2} [2b-r], \quad A_3 = \frac{nr^{n-2}}{12} [6b^2-6br+r^2], \\ A_4 &= \frac{n(n-1)br^{n-3}}{12} [2b^2-3br+r^2], \quad \&c.\end{aligned} \right\} \tag{12}$$

If we now substitute $a+r$ for b we shall have for the sum of series (1)

$$\begin{aligned}S_x^{(n)} &= \frac{r^n}{n+1}x^{n+1} + \frac{r^{n-1}}{2}(2a+r)x^n + \frac{nr^{n-2}}{12}(6a^2+6ar+r^2)x^{n-1} \\ &+ \frac{n(n-1)a^2r^{n-3}}{6}(a+2r)x^{n-2} + \dots\end{aligned}\tag{13}$$

If we now substitute $a+r$ for b and $2r$ for r in (12) we shall find for the sum of series (2)

$$S_2^{(n)} = \frac{(2r)^n}{n+1} x^{n+1} + (2r)^{n-1} a x^n + \frac{n(2r)^{n-2}}{6} (3a^2 - r^2) x^{n-1} \\ + \frac{n(n-1)(2r)^{n-3}}{6} a (a^2 - r^2) x^{n-2} + \dots \quad (14)$$

If we make $a = 0$ and $r = 1$, we shall have

$$1^n + 2^n + 3^n + \dots + x^n = \frac{1}{n+1} x^{n+1} + \frac{1}{2} x^n + \frac{n}{12} x^{n-1} + \dots \quad (15)$$

$$1^n + 3^n + 5^n + \dots + (2x-1)^n = \frac{2^n}{n+1} x^{n+1} - \frac{n \cdot 2^{n-3}}{3} x^{n-1} + \dots \quad (16)$$

If we make $a = 0$ and $r = 2$ in series (1) we shall have

$$2^n + 4^n + 6^n + \dots + (2x)^n = \frac{2^n}{n+1} x^{n+1} + 2^{n-1} x^n + \frac{n \cdot 2^{n-2}}{3} x^{n-1} + \dots$$

REVISED SOLUTION OF PROBLEM 218.

EDITOR ANALYST:

MR. MEECH, the ingenious proposer of problem 218, having furnished me with the data from which that question was constructed, and requested me to make a general solution, under fuller conditions, for publication in the ANALYST, I hereby cheerfully comply with his request.

GEORGE EASTWOOD.

The problem, under its new aspect, may be stated as follows:

Required the separate rates of dividend of two insolvent estates connected as follows:

JOHN DOE'S ESTATE. Direct liabilities = λ ; his endorsements for Richard Roe = λ_1 less a first dividend on the same to be paid out of Roe's estate. His net assets = a to be increased by dividend on account, = β' , due from Roe's Estate.

RICHARD ROE'S ESTATE. Direct liabilities = λ' ; his endorsements for John Doe = λ_2 , less a first dividend on same to be paid out of Doe's estate. His net assets = a' to be increased by dividend on account, = β , due from Doe's estate.

We have Doe's direct liabilities = λ ; his endorsements = λ_1 ;
 Roe's direct liabilities = λ' ; his endorsements = λ_2 ;
 Doe's net assets = a ;
 Roe's " " = a' ;
 Doe's account due from Roe's estate = β' ;
 Roe's " " " Doe's " = β .

Therefore, x being the rate per cent of dividend on Doe's estate and y the rate on Roe's, we have

First dividend on Roe's endorsement for Doe = $\lambda_2 \cdot \frac{1}{100}x$;

First dividend on Doe's endorsement for Roe = $\lambda_1 \cdot \frac{1}{100}y$.

$$\lambda_2 - \lambda_2 \frac{x}{100} = \lambda_2 \left(\frac{100-x}{100} \right) = \text{balance of Roe's endorsements,}$$

$$\lambda_1 - \lambda_1 \frac{y}{100} = \lambda_1 \left(\frac{100-y}{100} \right) = \text{balance of Doe's endorsements.}$$

$\alpha' + \frac{1}{100}\beta x$ = amount to be dividend among Roe's creditors ;

$\alpha + \frac{1}{100}\beta' y$ = " " " " " " Doe's " ; also

$$\left[\lambda' + \lambda_2 \left(\frac{100-x}{100} \right) \right] \frac{y}{100} = \text{final dividend on Roe's state,}$$

$$\left[\lambda + \lambda_1 \left(\frac{100-y}{100} \right) \right] \frac{x}{100} = \text{" " " " Doe's estate.}$$

Hence, from the nature of the question,

$$\left[\lambda' + \lambda_2 \left(\frac{100-x}{100} \right) \right] \frac{y}{100} = \alpha' + \frac{\beta x}{100}, \quad (1)$$

$$\left[\lambda + \lambda_1 \left(\frac{100-y}{100} \right) \right] \frac{x}{100} = \alpha + \frac{\beta' y}{100}. \quad (2)$$

From (1) and (2) we deduce, respectively,

$$y = \frac{100(100\alpha' + \beta x)}{100\lambda' + \lambda_2(100-x)} \text{ and } y = \frac{100[(\lambda + \lambda_1)x - 100\alpha]}{\lambda_1 x + 100\beta'}.$$

Equating these values of y , reducing and arranging like powers of x , we find

$$x^2 - \frac{100[(\lambda + \lambda_1)(\lambda' + \lambda_2) + \alpha\lambda_2 - \alpha'\lambda_1 - \beta\beta']}{\beta\lambda_1 + \lambda_2(\lambda + \lambda_1)}x = -\frac{(100)^2[\alpha(\lambda' + \lambda_2) + \alpha'\beta']}{\beta\lambda_1 + \lambda_2(\lambda + \lambda_1)}. \quad (3)$$

Put the coefficient of x in (3) = $100A$ and the right-hand member of the equation = $(100)^2B$, then equation (3) becomes

$$x^2 - 100Ax = -(100)^2B,$$

whence

$$x = 50A \mp 50\sqrt{A^2 - 4B},$$

the upper sign of which is to be applied, and thence the value of y is easily determined.

If, as in the former problem, $\beta' = 0$, then the final equation in x becomes

$$x^2 - \frac{100[(\lambda + \lambda_1)(\lambda' + \lambda_2) + \alpha\lambda_2 - \alpha'\lambda_1]}{\beta\lambda_1 + \lambda_2(\lambda + \lambda_1)}x = -\frac{(100)^2\alpha(\lambda' + \lambda_2)}{\beta\lambda_1 + \lambda_2(\lambda + \lambda_1)}. \quad (4)$$

Equation (4) may be written

$$x^2 - 100A'x = -(100)^2B'. \quad \therefore x = 50A' - 50\sqrt{A'^2 - 4B'}$$

Substituting the numerical values given in prob. 218, I find
and $y = 19.4399$.

VERIFICATION. —In accordance with the conditions, the rate of dividend of Doe's estate is found to be 20.1293 per cent of the liabilities. And the rate of dividend of Roe's estate is found to be 19.4399 per cent of the liabilities. These rates can be verified as follows :

John Doe's estate—direct liabilities,	\$33,425.61
Endorsements for Richard Roe \$34,949.16 }	
Less dividend from " " 6,794.08 }	28,155.08
Total liabilities,	\$61,580.69
Total dividends \$12,395.76, equal to 20.1293 per cent of liabilities.	
Richard Roe's estate—direct liabilities,	\$46,212.00
Endorsements for John Doe \$9,500.00 }	
Less dividend from " " 1,912.28 }	7,587.72
Total liabilities,	\$53,799.72
Total dividends, \$10,458.78, equal to 19.4399 per cent of total liabilities.	

APPROXIMATE MULTISECTION OF AN ANGLE AND HINTS FOR REDUCING THE UNAVOIDABLE ERROR TO THE SMALLEST AMOUNT.

BY CHAS. H. KUMMELL, DETROIT, MICHIGAN.

THE method of Query, page 96, is applicable also for multisection of angles or for dividing angles in a given ratio, approximately.

For example, if BCD shall be trisected, draw ACA' perpendicular to BC and describe, with any radius CA , the semicircle $ADBA'$; make $AE = A'E = AA'$, join ED intersecting AA' at F ; trisect CF at f and f' and draw EfD' and $Ef'D''$, then the arc BD will be approximately trisected in D' and D'' .



The answer to the query by Mr. E. B. Seitz at page 125, 126 is quite sufficient to prove this construction to be approximately true; yet for my purpose I shall present a different treatment.

The lines Ef and Ef' will intersect the circle ABA' at points D' and D'' which are more or less distant from the true points required; they may also be on the right or on the left of the true points. Let $CA = CB = 1$; $BCD'' = \varphi$; $BED'' = \psi$; $Cf' = x$. Let φ_0 be the true angle, then

$$A = \varphi_0 - \varphi \quad (1)$$

is the correction to the constructed angle to obtain the true angle.

By construction we have

$$x = \frac{2\varphi_0}{\pi} \quad (2)$$

In the triangle ECD'' we have $CD'' = 1$; $CE = \sqrt{3}$; $ECD'' = \pi - \varphi$.
Therefore

$$\tan \psi = \frac{\sin \varphi}{\sqrt{3} + \cos \varphi} \quad (3)$$

Now
$$x = \sqrt{3} \cdot \tan \psi = \frac{\sqrt{3} \cdot \sin \varphi}{\sqrt{3} + \cos \varphi}; \quad (4)$$

by (2)
$$\varphi_0 = \frac{\pi}{2} \cdot \frac{\sqrt{3} \cdot \sin \varphi}{\sqrt{3} + \cos \varphi}, \quad (5)$$

and by (1)
$$A = \frac{\pi}{2} \cdot \frac{\sqrt{3} \cdot \sin \varphi}{\sqrt{3} + \cos \varphi} - \varphi. \quad (6)$$

This equation gives $A = 0$ if $\varphi = 0$, $\varphi = \frac{1}{3}\pi$, or $\varphi = \frac{1}{2}\pi$.

To find the maximum and minimum values of A we place

$$\frac{dA}{d\varphi} = \frac{1}{2}\pi\sqrt{3} \frac{1 + \sqrt{3} \cdot \cos \varphi}{(\sqrt{3} + \cos \varphi)^2} - 1 = 0,$$

or
$$\cos^2 \varphi - 2\left(\frac{3}{4}\pi - \sqrt{3}\right)\cos \varphi = \frac{1}{2}\pi\sqrt{3} - 3,$$

whence
$$\cos \varphi = \frac{3}{4}\pi - \sqrt{3} \pm \sqrt{\left(\frac{9}{16}\pi^2 - \pi\sqrt{3}\right)}.$$

From this we have, for the upper sign,

$$\varphi_+ = 17^\circ 01' 21''; \therefore A_+ = -0.000817 = -02' 48'' \text{ (minimum),}$$

and for the lower sign,

$$\varphi_- = 73^\circ 01' 01''; \therefore A_- = +0.011327 = +38' 56'' \text{ (maximum).}$$

We learn from this that the constructed angle is slightly too great in the interval $\varphi = 0$ to $\varphi = \frac{1}{3}\pi$, the greatest error $2' 48''$ occurring at about 17° or near the middle of the interval. Also the constructed angle is by a greater amount too small in the interval $\varphi = \frac{1}{3}\pi$ to $\varphi = \frac{1}{2}\pi$, the greatest error $38' 56''$ occurring at about 73° or at nearly $\frac{1}{2}$ from the end of the interval.

Having multisectioned an angle by this method, the parts will not be strictly equal, but we can decide now which is freest of error. The first choice should be that part extending nearly at equal distances from the minimum point; the second best would be a part $\frac{1}{2}$ of which extends beyond the maximum point, and the third best, any part about the minimum interval.

In any case the error of draughting may or may not combine favorably with these theoretical errors, but this we cannot control and we have to leave these accidental errors out of consideration.

It should be observed that if φ is negative the errors are numerically equal but of opposite signs to those in the positive quadrant.

NOTE.—Equation (2) is strictly true, however, only if $\varphi = \frac{1}{3}\pi$ or $= \frac{2}{3}\pi$, and for the approximate construction of a polygon. For any angle φ to be multisectioned we have, if φ_1 is one part reckoned from B , strictly

$$\frac{x}{x_1} = \frac{\varphi}{\varphi_1} \text{ where } x = \frac{r\sqrt{3} \cdot \sin \varphi}{\sqrt{3} + \cos \varphi}, \text{ and } x_1 = \frac{r\sqrt{3} \cdot \sin (\varphi_1 - \Delta)}{\sqrt{3} + \cos (\varphi_1 - \Delta)}$$

and Δ is the correction to be applied to the angle $\varphi_1 - \Delta$, which the construction gives. We have then the equation:

$$0 = \varphi(\sqrt{3} + \cos \varphi) \sin (\varphi_1 - \Delta) - \varphi_1 \sin \varphi [\sqrt{3} + \cos (\varphi_1 - \Delta)],$$

and the maximum or minimum condition is

$$0 = \varphi(\sqrt{3} + \cos \varphi) \cos (\varphi_1 - \Delta) - \sin \varphi [\sqrt{3} + \cos (\varphi_1 - \Delta)] + \varphi_1 \sin \varphi \times \sin (\varphi_1 - \Delta).$$

CORRESPONDENCE.

EDITOR ANALYST:

Since reading your note on page 150 of the ANALYST, I have made some vain essays at a demonstration of the proposition stated in Problem 211. Though I failed in my main object, some of the results I attained in regard to triangular numbers seem to me of some interest, and may appear to you of sufficient interest to warrant their insertion in your Journal.

JOHN MACNIE.

Let $N = \frac{1}{2}(n^2 + n) = t_n$ be the n th triangular number, then

$$8N + 1 = (2n + 1)^2.$$

Hence, in order that an integer N , be a triangular number, it is necessary and sufficient that $8N + 1$ be a square number. Also, any odd square number is of the form $8t_n + 1$.

Ex. 1. Let $N = 55$; $8N + 1 = 441 = 21^2$. $\therefore 55 = t_{10}$.

In the same way we find that for N to be the sum of two triangular numbers, t_n and t_p , it is necessary and sufficient that $8N + 2 = (2n + 1)^2 + (2p + 1)^2$, i. e., be the sum of two odd squares.

Ex. 2. Let $N = 93$; $8N + 2 = 746 = 25^2 + 11^2$. $\therefore 93 = t_{12} + t_5$.

In order that N may be the sum of three triangular numbers, t_n, t_p, t_r , it is necessary and sufficient that $8N+3 = (2n+1)^2 + (2p+1)^2 + (2r+1)^2$, i. e., be the sum of three odd squares.

Ex. 3. Let $N = 1878$; $8N+3 = 15027 = 121^2 + 19^2 + 5^2$. $\therefore 1878 = t_{60} + t_9 + t_2$.

Since every odd square number, as we have seen above, is of the form $8t_n+1$, the sum of any three odd squares must be of the form $8N+3$. But of the converse proposition, that $8N+3$ is always the sum of three odd squares, I have been unable to obtain a demonstration, though the fact can always be verified upon trial. The greatest square $(2n+1)^2$ is usually the greatest odd square in $8N+3$, or the next odd square below; the second square in the sum is, in like manner, usually the greatest odd square in $8N+3 - (2n+1)^2$.

Perhaps some reader of the ANALYST can find a demonstration of one or both of the following propositions, in which N, x, y, z , may be integers or zero.

$$8N+3 = (2x+1)^2 + (2y+1)^2 + (2z+1)^2. \quad (1)$$

$$4N+1 = (2x+1)^2 + 4y^2 + 4z^2. \quad (2)$$

NOTE ON THE SOLUTION OF PROB. 217, BY THE EDITOR.—When the note at the foot of page 155 was written, we supposed that Prof. Johnson's solution necessarily gave one branch of the curve on each side of the axis of x , as it was supposed that the triangles were only drawn on *one side* of the hypotenuse; and the integration we gave at page 156 was made with that understanding.

By drawing the triangles on *both sides* of the hypotenuse, and describing squares on the *outside* of the legs *above* the hypotenuse, and on the *inside* of the legs *below* the hypotenuse, both branches of the locus will be *above* the axis of x , and consequently they will contain the area $(\frac{1}{3}\sqrt{3} - 9)\pi a^2$, as found by Prof. Johnson. On the other hand, if we consider only the triangles that can be drawn on one side of the hypotenuse, and describe the squares on both sides of the legs, we obtain two branches respectively equivalent to those found by Prof. Johnson, but one lying above and the other below the axis of x , and consequently enclosing an area $= \frac{1}{3}\sqrt{3} \cdot \pi a^2 - 4a^2$, as given by us at page 156.

Both branches as thus found are represented by equation (4), page 155, but one, the outer branch, is excluded by the wording of the question, and hence Messrs. Seitz, Heaton and Baker, by describing the squares only on the outside of the legs, find two equal branches, one above and the other below the axis of x , which enclose an area $= (9 - \frac{3}{2}\sqrt{3})\pi a^2 - 8a^2$.

Equation (4) may be written

$$y = \pm \frac{a^2 - x^2}{2a \pm \sqrt{(a^2 - x^2)'}}$$

and hence represents four branches, as in the marginal diagram. $a d' b d a$ is the area as found by Prof. Johnson; $a e' b d a$ is the area as found by us, and $a e' b d' a$, is the area as found by Messrs. Seitz, Heaton and Baker.



Prof. Johnson writes as follows:—

“The expression given on page 156, line 2, is the value of

$$\int \frac{d\theta}{2 + \cos \theta} \text{ and not of } \int \frac{d\theta}{2 + \sin \theta},$$

but the result is not affected, since, if we put $\theta = \theta' - \frac{1}{2}\pi$, we have

$$\int_0^{2\pi} \frac{d\theta}{2 + \cos \theta} = \int_{\frac{3}{2}\pi}^{\frac{5}{2}\pi} \frac{d\theta'}{2 + \sin \theta'} = \int_0^{2\pi} \frac{d\theta}{2 + \sin \theta}.”$$

NOTE ON THE POLYNOMIAL THEOREM.

BY PROF. W. W. JOHNSON.

THE Binomial Theorem may be written in the form

$$\frac{(a+b)^n}{n!} = \frac{a^n}{n!} + \frac{a^{n-1}}{n-1!} \cdot \frac{b}{1} + \frac{a^{n-2}}{n-2!} \cdot \frac{b^2}{2!} + \dots$$

and if we put ${}_r a = \frac{a^r}{r!}$ (we might call ${}_r a$ the r th pyramid of a , since ${}_2 a$ is the area of a triangle whose base and altitude are a , and ${}_3 a$, the volume of a pyramid whose base is ${}_2 a$ and altitude a), this becomes

$${}_n(a+b) = \sum {}_r a {}_s b,$$

where $r + s = n$ and r admits of all values from 0 to n inclusive. It follows at once that

$${}_n(a+b+c) = \sum {}_r(a+b) {}_s c = \sum {}_r a {}_s b {}_t c,$$

where $r + s + t = n$; and in general

$${}_n(a+b+c+\dots) = \sum {}_r a {}_s b {}_t c \dots$$

where $r + s + t + \dots = n$. This last equation is a form of the multinomial theorem.

TO FIND THE DIFFERENTIAL OF A VARIABLE QUANTITY
WITHOUT THE USE OF INFINITESIMALS OR LIMITS.

BY PROF. JOSEPH FICKLIN, COLUMBIA, MISSOURI.

I. If any function increases at a uniformly accelerated rate, the increment of that function in one unit of time is equal to the rate at which that function increases at the middle of that unit of time.

II. Let $u = x^2$, and suppose dx to be the rate at which x varies; then

$$u' - u = 2xdx + dx^2 = \text{increment of } u \text{ in first unit of time;}$$

$$u'' - u' = 2xdx + 3dx^2 = \text{ " " " sec'd " " " ;}$$

$$u''' - u'' = 2xdx + 5dx^2 = \text{ " " " third " " " ;}$$

$$u'''' - u''' = 2xdx + 7dx^2 = \text{ " " " fo'rth " " " ;}$$

.

Hence u increases faster each instant; and, since there is a constant difference ($2dx^2$) in the increments of u , the acceleration in the rate of increase of u is uniform.

Now, since the difference between u' and u is $2xdx + dx^2$, and, since u is increasing at a uniformly accelerated rate,

$$2xdx + dx^2$$

is not only the mean rate at which u increases during the first unit of time, but u increases at that rate at the middle of that unit of time (I).

But the rate at which u increases is increased by $2dx^2$ in one unit of time, for u is increasing at the rate of $2xdx + dx^2$ at the middle of the first unit, and at the rate of $2xdx + 3dx^2$ at the middle of the second unit; hence, in half a unit of time, the rate is increased by dx^2 . Therefore, if du represents the rate at which u increases at the instant $u = x^2$, then

$$du + dx^2$$

will be the rate at which u increases at the middle of the first unit of time.

$$\therefore du + dx^2 = 2xdx + dx^2;$$

whence,

$$du = 2xdx.$$

The same result may be obtained by equating the two expressions for the rate at which u increases at the middle of any other unit of time. Thus,

$$2xdx + 5dx^2$$

is the rate at which u increases at the middle of the third unit of time, and, since the rate at which u increases is increased by $2dx^2$ in one unit,

$$du + 2\frac{1}{2} \times 2dx^2$$

is also an expression for that rate;

$$\therefore du + 2\frac{1}{2} \times 2dx^2 = 2xdx + 5dx^2;$$

whence,
$$du = 2xdx.$$

III. Let $u = xy$; then

$$u' - u = xdy + ydx + dxdy;$$

$$u'' - u' = xdy + ydx + 3dxdy;$$

$$u''' - u'' = xdy + ydx + 5dxdy;$$

$$u'''' - u''' = xdy + ydx + 7dxdy;$$

In this case the constant difference in the increments of u is $2dxdy$; and it may be shown, by a process of reasoning entirely similar to that used in II, that

$$xdy + ydx + dxdy$$

is the rate at which u increases at the middle of the first unit of time, and that

$$du + dxdy$$

is also an expression for that rate;

$$\therefore du + dxdy = xdy + ydx + dxdy;$$

whence,
$$du = xdy + ydx.$$

If we put $y = x^2$, we have

$$du = d(xy) = d(x^3) = xd(x^2) + x^2dx.$$

But it has been shown that $d(x^2) = 2xdx$;

$$\therefore du = d(x^3) = x \times 2xdx + x^2dx = 2x^2dx + x^2dx = 3x^2dx.$$

In this way it may be shown that, if n be any positive integer,

$$du = d(x^n) = nx^{n-1}dx.$$

It may now be shown, in the usual way, that this equation is true, whether n be positive or negative, integral or fractional.

THE METEOR OF AUGUST 11, 1878.

BY PROFESSOR DANIEL KIRKWOOD.

1. *Observations at Bloomington, Ind.*—A few minutes after 10 o'clock* on Sunday evening, Aug. 11, 1878, Rev. John A. Bower of Bloomington, Ind., saw a brilliant meteor near the eastern horizon. Mr. B. had just taken

*The precise time was not noted.

a position facing an open eastern window. The meteor became visible very nearly east of Bloomington—perhaps a few degrees south of east—and about ten degrees above the horizon. Its motion was from south to north. The length of its apparent track was 20° or 25° . The first half of its course was but slightly inclined to the earth's surface; the inclination, however, became sensibly greater towards the point of disappearance, which was N. about 70° E., and very near the horizon. The apparent diameter of the meteor was at least one-third that of the moon. The motion was extremely rapid; the time of flight not exceeding two seconds. No detonation was heard, nor did the meteor separate into fragments at the time of disappearance.

The observations of Mr. Bower were given me verbally. To verify their accuracy I placed myself in the position which he occupied, and had him point out the meteor's course as he had seen it. The foregoing statement, I am satisfied, must be very nearly correct, except as to the time of flight, which is admitted by the observer to be very uncertain.

2. *Titusville, Crawford County, Pennsylvania.*—The following telegram to the associated press appeared in the papers of Tuesday morning, August 13:

"TITUSVILLE, August 12.—A beautiful meteoric display was witnessed from here last evening. The meteor made its appearance in the west at 10:30, moving in a northerly direction. It was of a greenish color and shone with great brilliancy, lighting up the entire surrounding country with a light that for the time prevailed over that of the full moon. Its appearance was only momentary, when it burst and divided into three fragments, two of which assumed a reddish color. Calculating from the time the explosion was seen until it was heard, the meteor was about 25 miles distant."

3. *Oil City, Venango County, Pennsylvania.*—The papers of the same date contained also the following:

"Oil City, August 12.—A meteor of unusual brilliancy passed here last evening a few minutes after 10 o'clock. It was nearly twice the size of a cannon ball. Its course was north."

All accounts agree that the meteor's course was nearly north. It was seen somewhat west of Titusville; and as the final explosion occurred about 25 miles from that city we may conclude that the track terminated over Crawford County, Pennsylvania. The observations at Bloomington, Ind., indicate that the body first became visible over Western Virginia. The distance directly east from Bloomington to the meridian which bounds Venango county Pa., on the west is 348 miles. Hence when first seen the meteor altitude was about 77 miles. The length of the visible track was betw

170 and 180 miles—say 175. The most probable velocity therefore decidedly indicates a hyperbolic orbit.

Owing perhaps to the late hour at which the meteor appeared but few observations of the phenomenon were reported. Several letters of inquiry brought no available response, and Prof. Cleveland Abbe of the U. S. Signal Service informed me that no accounts of the meteor were received at the Washington Office.

Bloomington, Indiana, Sept., 1878.

QUINQUESECTION OF THE CIRCUMFERENCE OF A CIRCLE.

BY PROF. L. G. BARBOUR, RICHMOND, KENTUCKY.

Theorem.—Let C be the centre of a circle; AD , a diameter. Divide AC in extreme and mean ratio, putting the larger segment next the centre. Then from D as a centre, with a radius equal to DB , describe an arc cutting the circumference in F .

The arc AF will be one fifth of the circumference.

Demonstration.—Join BF , CF and DF and draw CE parallel to BF

By hypothesis

$$AB : BC :: BC : AC.$$

By composition, because $AC = CD$,

$$CD : BC :: BD : CD;$$

$$\therefore BC : CD :: CD : BD.$$

The triangles BCF and CFD , having the same altitude, are to each other as their bases; $\therefore BFC : CFD :: BC : CD$. Similarly

$$CFD : BFD :: CD : BD.$$

But the last couplets of these proportions, themselves form a proportion;

$$\therefore BFC : CFD :: CFD : BFD.$$

Also, since CE is parallel to BF , we have

$$DCE : CFD :: CFD : BFD.$$

Comparing this with the last proportion, we find $BFC = DCE$. These two triangles then are equal in area; the base DE of the one, is equal to the base CF of the other, for CED is isosceles, because similar to BFD , and hence $DE = CD$. Moreover, the angle DCE opposite DE is equal to the angle DBF . But any two triangles of equal areas, equal bases, and having the angles opposite the bases, equal, each to each, are equal in all



their parts. $\therefore$ angle BFC = angle at D . $BCF = CBF = DCE$. But BCF , being measured by the arc AF , is double the angle at D , which is measured by half the arc AF ; $\therefore \angle BCF = 2 \angle BFC$.

The angles BFC , BCF and CBF are together equal to two right angles, $\therefore BCF = \frac{2}{3}$ of two right angles, or $\frac{1}{3}$ of four right angles; $\therefore AF$ is $\frac{1}{3}$ of the circumference. Q. E. D.

SCHOLIA.—1. Taking the angle D as the unit of measurement, we have $D = BFC = FCE = CFE$, $2D = FBC = FCB = BFD = DCE = DEC$; $3D = CEF = BCE = FCD = FBA$.

2. $DE = DC = CF = BF =$ radius; $BC = CE = EF =$ greater segment of radius. Also the triangles CDE , CEF , DCF , BCF and BFD , i. e., all the triangles in the figure, are isocceles.

3. Laying off from F towards D an arc equal to AF , we get a remainder DG which is $\frac{1}{10}$ of the circumference. We know from Euclid that the chord from D to G is equal to BC ; hence BC , CE and EF are each equal to the chord of a decagon, and DE , &c., to the chord of a hexagon. Also, $BD = DF =$ chord of $\frac{3}{5}$ of two right angles, or $\frac{5}{6}$ of one right angle, $\therefore FCD = FBA$ &c. = angle between two consecutive sides of a regular pentagon.

4. Since $BC : CD :: CD : BD$, chord of a regular inscribed decagon : chord of hexagon :: chord of hexagon : chord of 108° .

Putting $CD =$ radius = 1, chord $36^\circ = \frac{1}{\text{chord } 108^\circ}$; $\therefore \sin 18^\circ = \frac{1}{\sin 54^\circ}$. Since $BD = CD + BC$, chord $108^\circ = 1 + \text{chord } 36^\circ$. $\therefore \sin 54^\circ = .5 + \sin 18^\circ$, as may be seen in a table of natural sines.

To find the length of BC , which we will call x , $x : 1 :: 1 : x + 1$; $\therefore x^2 + x = 1$, whence $x = -\frac{1}{2} \pm \frac{1}{2}\sqrt{5} = BC$; $\therefore BD = DF = +\frac{1}{2} + \frac{1}{2}\sqrt{5}$.

$AF^2 = AD^2 - DF^2 = 4 - \frac{1}{4} - \frac{1}{2}\sqrt{5} - \frac{5}{4} = \frac{5}{2} - \frac{1}{2}\sqrt{5} = BC^2 + CD^2$; $\therefore$ the side of a regular inscribed pentagon is the hypotenuse of a right angled triangle, the other sides being the side of a regular inscribed hexagon and that of a regular inscribed decagon. This has been noticed by Young.

A PROBLEM AND ITS SOLUTION.

BY DR. H. EGGERS, MILWAUKEE, WISCONSIN.

Problem. — Given in a plane three fixed right lines, L_1, L_2, L_3 , and in each of them a fixed point, respectively, A_1, A_2, A_3 : required a right line M , which shall cut off on L_1, L_2, L_3 , three equal distances counting from

A_1, A_2, A_3 , that is, the line M shall mark on L_1, L_2, L_3 , respectively, three points, B_1, B_2, B_3 , so that $A_1B_1 = A_2B_2 = A_3B_3$.

Solution. — Taking on one of the given lines, on L_1 , for instance, any number of distances, counting from A_1 , and on the line L_2 the same distances, counting from A_2 , then the corresponding terminating points on both lines will form two homographic systems of points, the projecting rays of which envelop a parabola touching L_1 and L_2 .

Combining in the same manner L_1 and L_3 , a second parabola will be generated, touching L_1 and L_3 . These two parabolas have L_1 and the line at infinity L_∞ as common tangents. The other pair of common tangents will evidently solve the problem. For in virtue of the first parabola they severally cut off equal distances on L_1 and L_2 , and in virtue of the second parabola they cut off equal distances on L_1 and L_3 .

The two required common tangents, T_1 and T_2 , can be constructed as follows:

Let X be their point of intersection and Y_∞ the point of intersection of L_1 with the line in infinity, L_∞ . These two points, X and Y_∞ , constitute one of the conics enveloped by the four common tangents, T_1, T_2, L_1, L_∞ .

If we take now an arbitrary point E and draw from E the two pairs of tangents on the two parabolas, and pass also from E two lines through the two points X and Y_∞ , these six lines form an involution, of which five lines are given; therefore the sixth line, which must pass through X , can be constructed. Taking now another arbitrary point E_1 and performing the analogous construction, we gain a second line E_1X , which must pass through X . Therefore the point of intersection X of the two tangents T_1 and T_2 can be found. All that remains is, to construct through X the two tangents to one of the parabolas. These will touch the other parabola also.

If the direction of counting the distances from A_1, A_2, A_3 is not determined beforehand, there will be four pairs of solutions possible, of which some may be imaginary. For we may on each of the given lines L_1, L_2, L_3 discern two directions of counting, $+$ and $-$, the combinations of which will furnish four different cases, viz.;

$$\begin{array}{c} L_1 \\ L_2 \\ L_3 \end{array} \begin{array}{c} + \\ + \\ + \end{array} \begin{array}{c} + \\ + \\ - \end{array} \begin{array}{c} + \\ - \\ + \end{array} \begin{array}{c} - \\ + \\ + \end{array}$$

Of the four other combinations of signs yet possible, any one will coincide with one of the four above given; for example, three minus signs give the same solution as three plus signs, because the variable points of division which mark equal distances on L_1, L_2, L_3 , pass at the same time to infinity and leap from there simultaneously on the opposite half of these lines.

Each of the four combinations of signs now gives rise to two solutions, which in regard to method do not differ.

If the distances to be cut off, instead of being equal, shall have given ratios

$$m : n : p$$

the method of solution remains the same.

The solution of this problem becomes more simple when two of the given lines are parallel.

SOLUTIONS OF PROBLEMS IN NUMBER FIVE.

SOLUTIONS of problems in No. 5 have been received as follows:

From Prof. L. G. Barbour, 224; Marcus Baker, 225, 230; Prof. P. E. Chase, 230; Newton Fitz, 221, 230; Henry Gunder, 221, 223, 224, 230; Henry Heaton, 221, 223, 224, 225, 226, 227, 228, 229, 230; W. E. Heal, 221, 227; Prof. E. W. Hyde, 225, 230; Prof. Joseph H. Kershner, 221, 222, 223, 224, 225, 226, 228, 229, 230; Chas. H. Kummell, 221, 224, 227, 228, 229, 230; Prof. D. J. Mc Adam, 223, 224, 228; Prof. Orson Pratt, 221; P. Richardson, 221, 225; Prof. J. Scheffer, 221, 224, 225, 226, 228; E. B. Seitz, 221, 223, 224, 225, 227, 228, 230.

221. "Find the four roots of the equation,

$$x^4 + px^3 + \frac{7p^2}{4^2}x^2 + \frac{6p^3}{4^3}x + q = 0."$$

SOLUTION BY PROF. ORSON PRATT, SEN., SALT LAKE CITY, UTAH.

Increase the roots by $\frac{1}{4}p$, and the second and fourth terms vanish, and the resulting equation becomes a quadratic; reducing, and diminishing the roots by $\frac{1}{4}p$, we have

$$x = -\frac{p}{4} + \frac{1}{4\sqrt{2}}\sqrt{\left[-p^2 + \sqrt{(9p^4 - 4^5q)}\right]};$$

$$x = -\frac{p}{4} - \frac{1}{4\sqrt{2}}\sqrt{\left[-p^2 + \sqrt{(9p^4 - 4^5q)}\right]};$$

$$x = -\frac{p}{4} + \frac{1}{4\sqrt{2}}\sqrt{\left[-p^2 - \sqrt{(9p^4 - 4^5q)}\right]};$$

$$x = -\frac{p}{4} - \frac{1}{4\sqrt{2}}\sqrt{\left[-p^2 - \sqrt{(9p^4 - 4^5q)}\right]}.$$

SOLUTION BY P. RICHARDSON, BROOKLYN NEW YORK.

Put $p = 4t$, then the equation becomes

$$x^4 + 4tx^3 + 7t^2x^2 + 6t^3x + q = 0.$$

Put $x = y - t$, then is

$$y^4 + t^2y^2 - 2t^4 + q = 0;$$

$$\therefore y = \pm \sqrt{[-\frac{1}{2}t^2 \pm \sqrt{(\frac{3}{4}t^4 - q)]}.$$

But $y - t = x$;

$$\therefore x = -t \pm \sqrt{[-\frac{1}{2}t^2 \pm \sqrt{(\frac{3}{4}t^4 - q)]}.$$

222. "It is required to find four positive integral numbers, the sum of the cubes of any three of which shall be a rational cube number."

SOLUTION BY PROF. JOSEPH H. KERSHNER, MERCERSBERG, PA.

Let ax^2, bx^2, cx^2, dx^2 be the numbers. If $a + b + c = x, a + c + d = x, a + b + d = x$, and $b + c + d = x$, the conditions are fulfilled. Adding these equations, $x = \frac{2}{3}(a + b + c + d)$.

Assume $x - \frac{2}{3}a = p^3, x - \frac{2}{3}b = q^3, x - \frac{2}{3}c = r^3$, and $x - \frac{2}{3}d = s^3$, then

$$x = \frac{1}{3}(4x - x) = \frac{1}{3}(p^3 + q^3 + r^3 + s^3).$$

Let $3m = p, 3m + 3 = q, 3m + 6 = r$, and $6m = s$; thus

$$a = \frac{4}{9}(-2p^3 + q^3 + r^3 + s^3),$$

$$b = \frac{4}{9}(p^3 - 2q^3 + r^3 + s^3),$$

$$c = \frac{4}{9}(p^3 + q^3 - 2r^3 + s^3),$$

$$d = \frac{4}{9}(p^3 + q^3 + r^3 - 2s^3).$$

Therefore, if $p^3 + q^3 + s^3 > 2r^3$, or $(3m)^3 + (3m + 3)^3 + (6m)^3 > (3m + 6)^3$, a, b, c, d , and x shall all be positive and integral; that is,

$$8(3m)^3 - 27(3m)^2 - 189 \times 3m - 405 > 0.$$

If $m = 3$, or more, the four numbers are known.

[The above is the only solution of problem 222 that has been received, hence we insert it, though we think it is defective.

In assuming $x = a + b + c = a + c + d$, &c., it is virtually assumed that $a = b = c = d$, consequently $p = q = r = s$; therefore the assumption, $3m = p, 3m + 3 = q$, &c., is not admissible.—Ed.]

223. "Compare the masses of two spheres of equal radii, one of uniform density, the other increasing in density from a single point in the surface as the square of the distance measured on the diameter through that point increases."

SOLUTION BY E. B. SEITZ, GREENVILLE, OHIO.

Let m be the mass of the sphere of uniform density, m' , the mass of the other sphere, r the radius of each sphere, and take a unit of volume of the sphere of uniform density for the unit of mass, and let the density of the second sphere, at the distance a from the point in the surface, be equal to the density of the first sphere.

Then $m = \frac{4}{3}\pi r^3$, and the mass of an elemental segment of the second sphere whose base is perpendicular to the diameter through the point in the surface, and is distant x from that point, is

$$\frac{\pi x^2}{a^2}(2rx - x^2)dx; \therefore m' = \int_0^{2r} \frac{\pi x^2}{a^2}(2rx - x^2)dx = \frac{8\pi r^5}{5a^2}.$$

$$\therefore m : m' :: 5a^2 : 6r^2.$$

224. "Two posts, each 12 ft. high, stand perpendicularly, fifty feet apart, on a horizontal plane, and a rope, one end of which is fastened at the top of each post, is suspended between them so that it just touches the plane at its middle point. Required the length of the rope."

SOLUTION BY HENRY GUNNER, NORTH MANCHESTER, IND.

The equations to the catenary are

$$y = -a + \frac{a}{2}\left(\varepsilon^{\frac{x}{a}} + \varepsilon^{-\frac{x}{a}}\right) \dots (1), \text{ and } z = \frac{a}{2}\left(\varepsilon^{\frac{x}{a}} - \varepsilon^{-\frac{x}{a}}\right), \quad (2)$$

in which x and y are the co-ordinates of the curve, the origin being at the lowest point, and the axis of x being tangent to the curve at the origin; and where a is the length of a portion of the rope the weight of which is equal the tension at the origin, and z , its length from the origin to the point (x, y) .

Putting $x = 25$ and $y = 12$, (1) gives $a = 27.8$ ft., nearly, and putting this value of a in (2) we get $2z = 56.9$ ft., nearly, the length required.

SOLUTION BY E. B. SEITZ.

Let $h = 12$ feet, $2b = 50$ feet, $2l =$ the length of the rope, and let $a =$ the tension at the lowest point of the rope.

Then from the properties of the catenary we have

$$l^2 = h^2 + 2ah, \quad (1)$$

$$b = a \log \left\{ [l + \sqrt{l^2 + a^2}] \div a \right\}. \quad (2)$$

From (1) and (2), by eliminating a , we find

$$b = \left(\frac{l^2 - h^2}{2h} \right) \log \left(\frac{l+h}{l-h} \right). \quad (3)$$

From (3), by the method of Double Position, we find

$$2l = 56.996 \text{ feet.}$$

225. "What is the locus of the angles of the inscribed squares of the respective segments of a circle when the chords or bases are parallel?"

BY MARCUS BAKER, U. S. COAST SURVEY, WASHINGTON, D. C.

$OB = x$, $PB = y$. Therefore for the point P we have

$$(2x+y)^2 + x^2 = R^2;$$

and for the point P' , i. e., when x becomes half the side of the square in the greater segment, we have

$$(2x-y)^2 + x^2 = R^2; \therefore 5x^2 \pm 4xy + y^2 = R^2,$$

and consequently the locus is an *ellipse*.



226. "Find the algebraic equation whose roots are the real quantities found by giving integral values to k in

$$x = \cos \frac{2k\pi}{n},$$

n being a given integer. Also the equation whose solution is

$$x = \sin \frac{2k\pi}{n}."$$

SOLUTION BY PROF. J. SCHEFFER, MERCERSBERG, PA.

By De Moivre's formula we have

$$\left(\cos \frac{2k\pi}{n} + \sin \frac{2k\pi}{n} \sqrt{-1} \right)^n = \cos 2k\pi + \sin 2k\pi \sqrt{-1} = 1.$$

Developing the first member in a series by the Binomial Theorem, and putting $\cos(2k\pi \div n) = x$, we get the equation

$$x^n - \binom{n}{2} x^{n-2} (1-x^2) + \binom{n}{4} x^{n-4} (1-x^2)^2 - \dots - 1 = 0;$$

of which in any concrete case we choose as many terms as is compatible with $n > r$ in $\binom{n}{r}$. Putting now $\sin \frac{2k\pi}{n} = x$, we get, if n is odd,

$$x^n - \binom{n}{2} x^{n-2} (1-x^2) + \binom{n}{4} x^{n-4} (x^2-1)^2 - \dots = 0;$$

and if n is even,

$$x^n - \binom{n}{2} x^{n-2} (1-x^2) + \binom{n}{4} x^{n-4} (1-x^2)^2 - \dots \pm 1 = 0;$$

where we use the upper sign if n is of the form $4m-2$, and the lower, if n is of the form $4m$.

SOLUTION BY H. HEATON, SABULA, IOWA.

Put $\frac{2k\pi}{n} = \theta$, and let $2x = 2 \cos \theta$. Then $2x = e^{\theta\sqrt{-1}} + e^{-\theta\sqrt{-1}}$, and $2\sqrt{x^2-1} = 2\sqrt{-1} \sin \theta = e^{\theta\sqrt{-1}} - e^{-\theta\sqrt{-1}}$; $\therefore e^{\theta\sqrt{-1}} = x + \sqrt{x^2-1}$ and $e^{-\theta\sqrt{-1}} = x - \sqrt{x^2-1}$. Also $2 \cos n\theta = e^{n\theta\sqrt{-1}} + e^{-n\theta\sqrt{-1}} = [x + \sqrt{x^2-1}]^n + [x - \sqrt{x^2-1}]^n$.

But $\cos n\theta = \cos 2k\pi = 1$, k having any integral value. Hence the roots of the equation, $[x + \sqrt{x^2-1}]^n + [x - \sqrt{x^2-1}]^n = 2$, are the different values of $\cos(2k\pi/n)$.

If $x = \sin \theta$, $2\sqrt{-1}x = e^{\theta\sqrt{-1}} - e^{-\theta\sqrt{-1}}$, and $2\sqrt{1-x^2} = e^{\theta\sqrt{-1}} + e^{-\theta\sqrt{-1}}$; $\therefore e^{\theta\sqrt{-1}} = \sqrt{-1}[x + \sqrt{x^2-1}]$ and $e^{-\theta\sqrt{-1}} = \sqrt{-1}[\sqrt{x^2-1} - x]$. Hence $2\sqrt{-1} \sin n\theta = e^{n\theta\sqrt{-1}} - e^{-n\theta\sqrt{-1}} = (-1)^{1/2n} [x + \sqrt{x^2-1}]^n - (-1)^{1/2n} [\sqrt{x^2-1} - x]^n = 0$.

If n be even, and the expression $[x + \sqrt{x^2-1}]^n - [\sqrt{x^2-1} - x]^n$ be developed it will be found to be of the $2n$ th degree, and its roots, when it equals nothing, are the different values of $2k\pi/2n$, k taking any integral value less than $2n$.

Therefore if n be even the required equation is

$$[\sqrt{x^2-1} + x]^{1/2n} - [\sqrt{x^2-1} - x]^{1/2n} = 0, \text{ or } [\sqrt{x^2-1} + x]^n + [\sqrt{x^2-1} - x]^n + 2 = 0. \text{ If } n \text{ is odd the eq'n is } [\sqrt{x^2-1} + x]^n - [\sqrt{x^2-1} - x]^n = 0.$$

[Since the above was put in type we have received a solution of prob. 226 from Prof. Johnson, containing an interesting discussion of the required equations, which will appear, entire, in No. 1, Vol. VI.]

227. "If the equation $x^n + ax^{n-1} + \dots = 0$, whose roots are α, β , etc., be transformed into another, one of whose roots is 0, while the differences are unchanged, and then the transformed equation be divided by x ; and if this process be repeated $n-1$ times, prove that the product of the differences of the roots of the original equation is equal to the product of the absolute terms of the equations thus obtained."

SOLUTION BY WILLIAM E. HEAL, WHEELING, INDIANA.

In the given equation, substitute $x-a$ for x ; then the roots of the equation will be $(a-a) (= 0)$, $(a-\beta)$, $(a-\gamma)$, &c. Hence the absolute term, being the product of these roots, will vanish and the equation will be divisible by x ; and the absolute term of the new equation will be $(a-\beta)(a-\gamma) \times (a-\delta)$ &c. For, from the manner of forming this term it must be equal to the sum of the terms formed by taking every combination of the quantities $(a-a)$, $(a-\beta)$, $(a-\gamma)$, $(a-\delta)$, &c., taken in sets of $n-1$; but all the terms except that written above will contain the factor $(a-a)$ and will therefore be equal to zero.

Substituting $(x-\beta)$ for x in the given equation we find, in like manner, for the absolute term of the derived equation $(\beta-a)(\beta-\gamma)(\beta-\delta)$ &c.

Hence, by repeating the substitution $n-1$ times and multiplying the absolute terms together, we obtain $(a-\beta)(\beta-a)(a-\gamma)(\gamma-a)(a-\delta)(\delta-a)$ &c., which is the product of differences of the given equation.

228. "Integrate $dI = \log(a + \sqrt{x^2 + b^2})dx$."

SOLUTION BY CHAS H. KUMMELL, U. S. LAKE SURVEY, DETROIT, MICH.

Integrating by parts we have

$$\begin{aligned} I &= x \log[a + \sqrt{x^2 + b^2}] - \int \frac{x^2 dx}{\sqrt{x^2 + b^2}[a + \sqrt{x^2 + b^2}]} \\ &= x \log[a + \sqrt{x^2 + b^2}] - I'. \end{aligned} \quad (1)$$

To rationalize I' , assume

$$\sqrt{x^2 + b^2} = y - x = \frac{y^2 + b^2}{2y}; \quad x = \frac{y^2 - b^2}{2y}; \quad dx = \frac{y^2 + b^2}{2y^2} dy, \quad (2)$$

$$\begin{aligned} \text{then } I' &= \int \frac{(y^2 - b^2)^2 dy}{2y^2(y^2 + 2ay + b^2)} = \frac{1}{2}y - \frac{1}{2} \int dy \frac{2ay^3 + 3b^2y^2 - b^4}{y^2(y^2 + 2ay + b^2)} \\ &= \frac{1}{2}y - \frac{1}{2}I''. \end{aligned} \quad (3)$$

$$\text{We have } \frac{2ay^3 + 3b^2y^2 - b^4}{y^2(y^2 + 2ay + b^2)} = -\frac{b^2}{y^2} + \frac{2a}{y} - \frac{4(a^2 - b^2)}{y^2 + 2ay + b^2},$$

$$\therefore I'' = \frac{b^2}{y} + 2a \log y - 2\sqrt{a^2 - b^2} \log \frac{y + a - \sqrt{a^2 - b^2}}{y + a + \sqrt{a^2 - b^2}};$$

$$\therefore \text{ by (3), } I' = \frac{1}{2}y - \frac{b^2}{2y} - a \log y + \sqrt{a^2 - b^2} \log \frac{y + a - \sqrt{a^2 - b^2}}{y + a + \sqrt{a^2 - b^2}},$$

$$= x - a \log [x + \sqrt{(x^2 + b^2)}] + \sqrt{(a^2 - b^2)} \\ \times \log \frac{x + \sqrt{(x^2 + b^2)} + a - \sqrt{(a^2 - b^2)}}{x + \sqrt{(x^2 + b^2)} + a + \sqrt{(a^2 - b^2)}} \text{ by (2);}$$

Therefore $I = x \log [a + \sqrt{(x^2 + b^2)}] - x + a \log [x + \sqrt{(x^2 + b^2)}]$

$$- \sqrt{(a^2 - b^2)} \log \frac{a + x + \sqrt{(x^2 + b^2)} - \sqrt{(a^2 - b^2)}}{a + x + \sqrt{(x^2 + b^2)} + \sqrt{(a^2 - b^2)}}, \quad a > b.$$

229. "Evaluate

$$\int_{-\infty}^{\infty} \frac{\cos mx}{a+x} dx \text{ and } \int_{-\infty}^{\infty} \frac{\sin mx}{a+x} dx."$$

SOLUTION BY H. HEATON.

We have $\int_0^{\infty} \frac{\sin rx}{x} dx = \frac{\pi}{2}$. (See Todhunter's Integral Calculus, p. 271.)

Putting $-x$ for x , we have

$$\int_{-\infty}^0 \frac{\sin rx}{x} dx = \frac{\pi}{2}; \therefore \int_{-\infty}^{\infty} \frac{\sin rx}{x} dx = \pi.$$

Suppose $\int_0^{\infty} \frac{\cos rx}{x} dx = p$, and put $-x = x$; then is $\int_{-\infty}^0 \frac{\cos rx}{x} dx = -p$.

$$\int_{-\infty}^{\infty} \frac{\cos rx}{x} dx = 0.$$

Put $y = a + x$ for x in the given problem; then

$$\int_{-\infty}^{\infty} \frac{\cos mx}{a+x} dx = \int_{-\infty}^{\infty} \frac{\cos ma \cos my + \sin ma \sin my}{y} dy = \pi \sin ma.$$

$$\int_{-\infty}^{\infty} \frac{\sin mx}{a+x} dx = \int_{-\infty}^{\infty} \frac{\cos ma \sin my - \sin ma \cos my}{y} dy = \pi \cos ma.$$

[Mr. Kummell has given an interesting discussion of these integrals which we are compelled to omit, at present, for want of room.]

230. " $ABCD$ is any tetrahedron. Through A pass a plane parallel to BCD , through B a plane parallel to CDA , through C a plane parallel to DAB and through D a plane parallel to ABC . A new tetrahedron is thus formed the volume of which call V' . Also let V equal the volume of the original tetrahedron. Prove that $V' = 27V$."

SOLUTION BY NEWTON FITZ, NORFOLK, VIRGINIA.

Let $ABC-D$ represent the tetrahedron V , and let a be its altitude.

A plane through A parallel to the plane DBC will cut the plane of the base in EF parallel to BC . A plane through B parallel to DAC will cut the plane of the base in EG parallel to AC ; and a plane through C parallel to DAB will cut the plane of the base in FG parallel to AB . The intersections of these three planes will be the lines EN , FN , GN , parallel respectively to DC , DB and DA .

The tetrahedron $EFG-N$ will be similar to V and to V' . Because $AC = EB = BG$ the altitude of $EFG-N = 2a$. The altitude of the frustum remaining to V' after $EFG-N$ is removed is equal to the altitude of V , $= a$. Hence the altitude of $V' = 3a$ and $V' = 27V$. Q. E. D.

[Prof. Hyde's solution of this problem and also his solution of 225 are by quaternions. It was our intention to insert one of these solutions but the space remaining will not permit; and for like reason a solution of Mr. Baker's question, on p. 143, by Mr. Eastwood, and of Mr. Heal's question, on the same page, by Mr. Adcock, are, at present, excluded.]



PROBLEMS.

231. BY PROF. ORSON PRATT, SEN. — What is the sum expressed in terms of m , of the values of all the determinants, from the second to the n th orders inclusive, which can be formed from the m -gonal series of numbers, represented by $1, a_2, a_3, a_4, \dots, a_n$, the arrangement of the constituents of the respective determinants being after the following form:

$$\begin{vmatrix} 1 & a_2 \\ a_3 & a_4 \end{vmatrix} + \begin{vmatrix} 1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{vmatrix} + \begin{vmatrix} 1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} \end{vmatrix} + \&c.?$$

232. BY PROF. J. H. KERSHNER. — From two given points on a circle to draw straight lines through a point C in the circumference so they shall form with a line MN , given in position, a triangle CMN of given area.

233. *Selected, by Prof. M. L. Comstock.*—If $ABCD$ be a spherical quadrilateral whose sides AB, DC are produced to meet at P , and AD, BC , to meet at Q , and whose diagonals AC, BD intersect at R , then

$$\sin AB \sin CD \cos P - \sin AD \sin BC \cos Q = \pm \sin AC \sin BD \cos R.$$

234. *By P. Richardson.*— $ABCD$ is a trapezium, $AB = a$, $CD = b$, $AD = c$; angle ABC is a right angle, and E is a point on AD such that angle BCE is a right angle and $CE = CD = b$.

It is required to find BC, DE and AE .

235. *By W. E. Heal.*—Find the condition that the general equation of the n th degree may have q equal roots.

236. *By Christine Ladd, Baltimore Md.*—If A_1A_2, B_1B_2, C_1C_2 are three lines which meet in a point O , then there are four different ways in which the points $A_1, B_1, C_1, A_2, B_2, C_2$, can be combined into two homologous triangles, and for each combination there is a different axis of homology. Show that these four axes form a complete quadrilateral whose diagonals intersect each other on the lines A_1A_2, B_1B_2, C_1C_2 .

237. *By Prof. D. J. McAdam.*—Prove that

$$\int_0^\pi \frac{\cos \frac{1}{2}(n-1)(\theta+\pi) \sin \frac{1}{2}[n(\theta+\pi)]}{\sin \frac{1}{2}(\theta+\pi)} d\theta = \pi.$$

238. *By Artemas Martin, M. A.*—Two points are taken at random in the surface of a circle, radius r , one of them being confined to a given radius, and a chord drawn through the points. Find (1) the chance that the length of the chord does not exceed $2a$, and (2) the chance that it does not exceed the radius of the circle.

239. *By Prof. A. B. Evans.*—A can plant thirty-six per cent of his arrows within a circular target ten inches in diameter at the distance of one hundred yards; B can plant sixty-four per cent of his arrows within a circle thirteen and one third inches in diameter at the same distance.

Prove that B's skill is greater than A's.

240. *By Prof. Johnson.*—If the angles θ, φ and ψ are connected by a certain relation any two of them may be the oblique angles of a spherical right triangle, and the third will be the complement of the perpendicular from the right-angle to the hypotenuse. Give a geometrical construction of the three triangles thus connected, and find the relations that exist between their sides.

ANNOUNCEMENT. — Five years ago we commenced the publication of the ANALYST contrary to the advice of friends, who were nearly unanimous in cautioning us that such a publication, at this place, would not be supported. But, from five years experience, we are pleased to say that the support received has exceeded our expectations; and that neither the locality of its publication nor the obscurity of its editor has prevented the ANALYST from receiving the patronage and support of many of the best mathematicians, educators, and public institutions in America.

We embrace this opportunity therefore to inform our readers that the ANALYST will be continued as long as our health permits and the interest of subscribers in its continuance is manifested by suitable contributions for publication.

It is not expected nor desired that subscribers will discommode themselves to supply us with material for publication; but we invite *all* our readers to send us any mathematical deductions or investigations that may be thought interesting or useful, so that we may have on hand at all times a *variety* of material from which to make selections for each number.

We again request present subscribers who may desire to discontinue their subscriptions at the end of Vol. V to *notify* us before the first of December.

No. 1, Vol. VI, will be mailed to subscribers by the 25th of December.

J. E. HENDRICKS.

PUBLICATIONS RECEIVED.

On the Origin of Comets. By H. A. NEWTON. [From the American Journal of Science and Arts, Vol. XVI, Sept. 1878.] 15 pp., 8vo.

Report of the Committee to Collect Information Relative to the Meteor of December 24, 1873 (read April 7, 1877) to the Philosophical Society of Washington, by CLEVELAND ABBE, on behalf of the Committee. 22 pp., 8vo, with map.

Researches in Graphical Statics. By HENRY T. EDDY, PH. D. 57 pp. 8vo, with plates. D. Van Nostrand. New York.

Observations and Orbits of the Satellites of Mars. By ASAPH HALL, Professor of Mathematics U. S. Navy. 46 pp. 4to. Washington. 1878.

ERRATA.

On page 144, eqn. (4), for $nr dx + y dy$, read, $-nr dx + y dy$.

" " 153, line 13 from bottom, after "construction", insert, in the triangle FOS_1 .

" " " " 12 " " for " POA ", read, POF ; and in line 9 from bot., *dele* ABC .

" " 158, in the second member of (1) and (2), *dele* exponent 2.

" " 161, line 17, for "by", read in; and in line 20, for, " U takes", read, U may take.

" " 163, " 4, *dele* s in "permutations"; and in line 25, insert, in, before u^8 .

" " 165, " 22, for "by consequence" read consequently.

" " 166, " 6, insert 2 for exponent after the parenthesis.

THE ANALYST.

A JOURNAL OF

PURE AND APPLIED MATHEMATICS.

EDITED AND PUBLISHED BY

J. E. HENDRICKS, A. M.

VOLUME VI.

DES MOINES, IOWA:
MILLS & Co., BOOK AND JOB PRINTERS.
1879.



ALPHABETICAL INDEX.

	PAGE		PAGE
An account of Cauchy's Calcul des		On the Arithmetico Geometric Mean,	10
Residus, - - - - 1, 41,	173	On the Removal of terms from an	
Answer to Query on page 143, Vol. V,	25	Equation of the Fifth Degree,	78
" " " " " 16, -	55	On the Trisection of an angle,	117
" " " " 1, " 64, -	85	On Unsymmetrical Adjustments, and	
Announcement of Vol. VII, -	191	their Limits,	140, 161
Adcock, R. J., - - - - 113,	151	On a Theorem of Lambert's	171
Arnold, J. M., - - - - 54,	182	Problems, . 31, 64, 95, 127, 159, 190	
Barbour, Prof. L. G., - - - - 108		Publications Rec'd, 64, 96, 128, 160, 192	
Correspondence, - - - - 54,	182	Query, ... 16, 64, 128, 191	
Curve of Pursuit Generalized, -	108	Revision of Proof of formula for Errors	
Computation of the Cube Root of 2,	117	of Observation, ... 80	
Construction of the Metian Ratio, -	189	Remarks on the Resolution, in Integers;	
Comstock, Prof. M. L., - - - - 54		of the Eq. $(x+1)(x+2) \dots (x+n)=y^2$,	149
Chase, Prof. P. E., - - - - 127,	189	Solutions of Problems in No. 1,	56
Demonstration of a Proposition, 83,	151	" " " " " 2, ... 88	
De Forest, E. L., - - - - 65, 140,	161	" " " " " 3, ... 120	
Evans, Prof. A. B., - - - - 149		" " " " " 4, ... 152	
Errata, - 31, 64, 96, 128, 160, 192		" " " " " 5, ... 184	
Further Remarks on the Solut'n of 260,	183	" " " No. 6, Vol. V, 25, 47	
Frisby, Prof. Edgar, - - - - 10		" " Problem 226, ... 17	
General Solution of Problem 143, -	81	" " " 215, ... 115	
Hall, Prof. Asaph, - - - - 33, 129,	171	" " " 85, ... 189	
Harvill, G. W., - - - - 56		" " a Problem in Mensuration,	112
Hendrickson, Prof. W. W., - 115,	180	Stellar Parallax, ... 33	
Johnson, Prof. W. W., . 105, 119,	177	Symmetrical Functions of the sines of	
Kummell, Chas. H., 1, 41, 80, 97,	173	the Angles included in the Expres-	
Kershner, Prof. J. H., - - - - 53		sion $a_0 + (2k\pi \div n)$, ... 105	
Martin, Artemas, - - - - 23,	117	Solution of the General Case of Prob-	
Note on Differentiation, - - - - 22		lem 87, Math. Visitor, No. 2, . 177	
Note, - - - - 54,	127	Scheffer, Prof. J., ... 14, 117	
Note on the Catenary, - - - - 119		The Summation of Series whose Coeffi-	
Note on the Solution of Problem 260,	151	cients form an Arithmetical Progres-	
Note on a Form of the Equation of		sion of any Order, ... 14	
the Tangent to a Conic. - - - - 180		The Tangency Problem, ... 53	
Nelson, Dr. A. B., - - - - 73			



THE ANALYST.

VOL. VI.

JANUARY, 1879.

No. 1.

AN ACCOUNT OF CAUCHY'S "CALCUL DES RESIDUS."

BY CHAS. H. KUMMELL, ASSISTANT U. S. LAKE SURVEY, DETROIT, MICH.

§ 1.

THIS very curious method for evaluating a great variety of definite integrals or for establishing relations between more complex and simple ones of the same class, is founded on the theorem that, if in a double integral the limits of both integrations are independent, the same result is obtained if the order of integration is inverted, provided the element function does not become infinite for one or more systems of simultaneous values of the variables comprised within the limits of integration.

Thus

$$\int_{x_0}^{x_n} dx \int_{y_0}^{y_n} dy f(x, y) = \int_{y_0}^{y_n} dy \int_{x_0}^{x_n} dx f(x, y), \quad (1)$$

is an identical equation if $f(x, y)$ does not become infinite for certain values x_1 and y_1 for which

$$\begin{array}{l} \text{or} \quad x_n > x_1 > x_0; y_n > y_1 > y_0, \\ \text{or} \quad x_n > x_1 > x_0; y_n > y_1 = y_0, \\ \text{or} \quad x_n > x_1 > x_0; y_n = y_1 > y_0, \\ \text{or} \quad x_n > x_1 = x_0; y_n > y_1 > y_0, \\ \text{or} \quad x_n = x_1 > x_0; y_n > y_1 > y_0. \end{array} \quad (2)$$

But suppose $f(x_1, y_1) = \infty$ then the corresponding element $dx dy f(x_1, y_1)$ is of the form $0 \times \infty$ and may therefore be zero, a finite quantity or infinity. In the first case only, the integration by the regular process is correct, but in the second and third case it is not because the element, although finite or infinite, is treated as an infinitesimal and therefore neglected in comparison with the sum of the remaining elements. To establish identity in equation (1) in that case, a correction must be added. This is called by Cauchy the

correction for discontinuity and the process to determine it the "Calcul des Residus."

To illustrate, let us take the double integral

$$u = \int_{-b}^a dx \int_{-d}^c dy \frac{y^2 - x^2}{(x^2 + y^2)^2} = -\tan^{-1} \frac{a}{c} - \tan^{-1} \frac{b}{c} - \tan^{-1} \frac{a}{d} - \tan^{-1} \frac{b}{d},$$

$$u' = \int_{-d}^c dy \int_{-b}^a dx \frac{y^2 - x^2}{(x^2 + y^2)^2} = \tan^{-1} \frac{c}{a} + \tan^{-1} \frac{c}{b} + \tan^{-1} \frac{d}{a} + \tan^{-1} \frac{d}{b};$$

therefore $u' - u = 4 \times \frac{1}{2} \pi = 2\pi$. The cause of this is the discontinuous element for $x = y = 0$, and we have 2π for the correction for discontinuity, which is here determined by common integration.

However, in the application of these principles to the evaluation of definite integrals these integrations cannot in general be performed and it will be shown that the corrections for discontinuity may, nevertheless, be determined in many cases.

§ 2.

Let z be some simple function of x and yi , where $i = \sqrt{-1}$, such as $x + yi$; $x(a + yi)$; xe^{yi} ; xe^{a+yi} and others, then if $f(z)$ is such a function that

$$\frac{d}{dy} \left\{ \frac{dz}{dx} f(z) \right\} = \frac{d}{dx} \left\{ \frac{dz}{dy} f(z) \right\}$$

we have, if there is no discontinuous element, the identity

$$\int_{x_0}^{x_n} dx \int_{y_0}^{y_n} dy \frac{d}{dy} \left\{ \frac{dz}{dx} f(z) \right\} = \int_{y_0}^{y_n} dy \int_{x_0}^{x_n} dx \frac{d}{dx} \left\{ \frac{dz}{dy} f(z) \right\}, \quad (3)$$

or

$$\int_{x_0}^{x_n} dx \left[\frac{dz}{dx} f(z) \right]_{y_0}^{y_n} = \int_{y_0}^{y_n} dy \left[\frac{dz}{dy} f(z) \right]_{x_0}^{x_n}. \quad (3')$$

But if $f(z_1) = \infty$ and x_1 and y_1 are the corresponding values of x and y , then if Δx denotes an increment to x_1 we have the following perfect identities for the first case in (2), viz., $x_n > x_1 > x_0$; $y_n > y_1 > y_0$.

$$\int_{x_1 + \Delta x}^{x_n} dx \left[\frac{dz}{dx} f(z) \right]_{y_0}^{y_n} = \int_{y_0}^{y_n} dy \left[\frac{dz}{dy} f(z) \right]_{x_1 + \Delta x}^{x_n},$$

$$\int_{x_0}^{x_1 - \Delta x} dx \left[\frac{dz}{dx} f(z) \right]_{y_0}^{y_n} = \int_{y_0}^{y_n} dy \left[\frac{dz}{dy} f(z) \right]_{x_0}^{x_1 - \Delta x}.$$

Adding these we have

$$\int_{x_0}^{x_n} dx \left[\frac{dz}{dx} f(z) \right]_{y_0}^{y_n} - \int_{x_1 - \Delta x}^{x_1 + \Delta x} dx \left[\frac{dz}{dx} f(z) \right]_{y_0}^{y_n}$$

$$= \int_{y_0}^{y_n} dy \left[\frac{dz}{dy} f(z) \right]_{x_0}^{x_n} - \int_{y_0}^{y_n} dy \left[\frac{dz}{dy} f(z) \right]_{x_1 - \Delta x}^{x_1 + \Delta x}. \quad (4)$$

At the limit when $\Delta x = 0$ we have

$$\int_{x_1-0}^{x_1+0} dx \left[\frac{dz}{dx} f(z) \right]_{y_0}^{y_n} = 0.$$

The term

$$\int_{y_0}^{y_n} dy \left[\frac{dz}{dy} f(z) \right]_{x_1-0}^{x_1+0}$$

would likewise vanish were it not for the discontinuous element

$$dy \left[\frac{dz}{dy} f(z) \right]_{z=z_1}.$$

This term is then the correction for discontinuity, and if z_1 is the only value that makes $f(z) = \infty$ we have the identical equation:

$$\int_{x_0}^{x_n} dx \left[\frac{dz}{dx} f(z) \right]_{y_0}^{y_n} = \int_{y_0}^{y_n} dy \left[\frac{dz}{dy} f(z) \right]_{x_0}^{x_n} - \int_{y_0}^{y_n} dy \left[\frac{dz}{dy} f(z) \right]_{x_1-\Delta x}^{x_1+\Delta x} \quad (\Delta x = 0) \quad (4')$$

If there are other discontinuous elements for $z_2, z_3, \dots, z_r$ we must correct for each and add the corrections.

§ 3.

Let us assume the simplest form for z , viz.,

$$z = x + yi. \quad (5)$$

With this we have in (4'), since $\frac{dz}{dx} = 1$ and $\frac{dz}{dy} = i$,

$$\begin{aligned} \int_{x_0}^{x_n} dx [f(x+yi) - f(x+yi)] &= i \int_{y_0}^{y_n} dy [f(x_n+yi) - f(x_0+yi)] \\ &\quad - i \int_{y_0}^{y_n} dy [f(x_1+\Delta x+yi) - f(x_1-\Delta x+yi)]. \end{aligned} \quad (6)$$

Let the equation

$$[f(z)]^{-1} = 0 \quad (7)$$

have m roots $= z_1^*$ so that we may place

$$f(z) = \frac{\phi(z)}{(z-z_1)^m}. \quad (8)$$

Let us denote the correction of discontinuity for the m -tuple root $z_1 = x_1 + y_1 i$ by $\Delta_{(x_1+y_1 i)}^m$ then we have, using the form (8) in (6),

$$\begin{aligned} \Delta_{(x_1+y_1 i)}^m &= i \int_{y_0}^{y_n} dy \left[\frac{\phi(x_1+\Delta x+yi)}{[\Delta x+(y-y_1)i]^m} - \frac{\phi(x_1-\Delta x+yi)}{[-\Delta x+(y-y_1)i]^m} \right] \\ &= i \int_{y_0}^{y_n} dy \left[\frac{\phi[x_1+y_1 i+\Delta x+(y-y_1)i]}{[\Delta x+(y-y_1)i]^m} - \frac{\phi[x_1+y_1 i-\Delta x+(y-y_1)i]}{[-\Delta x+(y-y_1)i]^m} \right] \end{aligned}$$

*This is not the only supposition that can be made but so far it is the only case Cauchy's method applies.

Developing the numerator of both terms by Taylor's series we have

$$\Delta_{(x_1+y_1i)}^m = i \int_{y_0}^{y_n} dy \sum_{k=1}^{k=\infty} \left[\frac{\phi^{(k-1)}(x_1+y_1i)}{(k-1)!} \left\{ [Ax+(y-y_1)i]^{-(m-k+1)} - [-Ax+(y-y_1)i]^{-(m-k+1)} \right\} \right]$$

and integrating this we obtain

$$\Delta_{(x_1+y_1i)}^m = \sum_{k=1}^{k=\infty} \left[\frac{\phi^{(k-1)}(x_1+y_1i)}{(k-1)!} \cdot \frac{[Ax+(y-y_1)i]^{k-m} - [-Ax+(y-y_1)i]^{k-m}}{k-m} \right]_{y_0}^{y_n}. \quad (9)$$

If we put $Ax=0$ both terms of the general term of this series are perfectly identical and destroy each other as long as k is not equal to m when it becomes indeterminate. The series reduces then to the m th term taken between the limits y_n and y_0 or

$$\begin{aligned} \Delta_{(x_1+y_1i)}^m &= \frac{\phi^{(m-1)}(x_1+y_1i)}{(m-1)!} \left[\frac{[Ax+(y-y_1)i]^{k-m} - [-Ax+(y-y_1)i]^{k-m}}{k-m} \right]_{k=m}^{y_n} \\ &= \frac{\phi^{(m-1)}(x_1+y_1i)}{(m-1)!} \left[\log \frac{Ax+(y-y_1)i}{-Ax+(y-y_1)i} \right]_{y_0}^{y_n}. \end{aligned}$$

Place $\tan \varphi = \frac{y-y_1}{Ax}$ then $\tan(\pi-\varphi) = \frac{y-y_1}{-Ax}$ and

$$Ax+(y-y_1)i = \frac{Ax}{\cos \varphi} e^{\varphi i},$$

$$-Ax+(y-y_1)i = \frac{Ax}{\cos \varphi} e^{(\pi-\varphi)i},$$

$$\varphi_n = \tan^{-1} \frac{y_n-y_1}{Ax} = \tan^{-1}(\infty) = \frac{\pi}{2},$$

$$\varphi_0 = \tan^{-1} \frac{y_0-y_1}{Ax} = \tan^{-1}(-\infty) = -\frac{\pi}{2};$$

therefore

$$\begin{aligned} \Delta_{(x_1+y_1i)}^m &= \frac{\phi^{(m-1)}(x_1+y_1i)}{(m-1)!} \left[\log \frac{e^{\varphi i}}{e^{(\pi-\varphi)i}} \right]_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \\ &= \frac{\phi^{(m-1)}(x_1+y_1i)}{(m-1)!} \left[(2\varphi - \pi)i \right]_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \\ &= 2\pi i \frac{\phi^{(m-1)}(x_1+y_1i)}{(m-1)!}. \end{aligned} \quad (10)$$

If however $y_0 = y_1$ the result is quite different. We have then, replacing y_1 by y_0 in (9),

$$(x_1 + y_0 i)^m = \sum_{k=1}^{k=\infty} \left[\frac{\phi^{(k-1)}(x_1 + y_0 i)}{(k-1)!} \cdot \frac{[Ax + (y - y_1)i]^{k-m} - [-Ax + (y - y_1)i]^{k-m}}{k-m} \right]_{y_0}^{y_n}.$$

are as before all terms of this series vanish for the upper limit except the n th. For the lower limit all terms vanish for which $k - m$ is an even number, zero excepted, and the terms for which $k - m$ is odd and $k < m$ are finite if $Ax = 0$. The m th term is, as we shall see, finite; it must be retained because it is not necessarily homogeneous with the infinite terms.

Let $m = 2n$ then

$$\begin{aligned} A_{(x_1 + y_0 i)^{2n}} &= \frac{2\phi'(x_1 + y_0 i)}{2n-1} Ax^{-2n+1} + \frac{\phi^{(2n-1)}(x_1 + y_0 i)}{(2n-1)!} \\ &\quad \times \left[\log \frac{Ax + (y - y_0)i}{-Ax + (y - y_0)i} \right]_{y_0}^{y_n} \\ &= \frac{2\phi'(x_1 + y_0 i)}{2n-1} Ax^{-2n+1} + \frac{\phi^{(2n-1)}(x_1 + y_0 i)}{(2n-1)!} \left[(2\varphi - \pi)i \right]_0^{\frac{\pi}{2}}, \text{ if } \tan \varphi = \frac{y - y_0}{Ax} \\ &= \frac{2\phi'(x_1 + y_0 i)}{2n-1} \infty^{2n-1} + \pi i \frac{\phi^{(2n-1)}(x_1 + y_0 i)}{(2n-1)!}. \quad (10'_1) \end{aligned}$$

Also if $m = 2n + 1$ we have, since the first term vanishes for the lower limit,

$$\begin{aligned} A_{(x_1 + y_0 i)^{2n+1}} &= \frac{2\phi'(x_1 + y_0 i)}{2n-1} Ax^{-2n+1} + \frac{\phi^{(2n)}(x_1 + y_0 i)}{2n!} \\ &\quad \times \left[\log \frac{Ax + (y - y_0)i}{-Ax + (y - y_0)i} \right]_{y_0}^{y_n} \\ &= \frac{2\phi'(x_1 + y_0 i)}{2n-1} \infty^{2n-1} + \pi i \frac{\phi^{(2n)}(x_1 + y_0 i)}{2n!}. \quad (10'_2) \end{aligned}$$

If $n = 0$ the correction is

$$A_{(x_1 + y_0 i)} = \pi i \phi(x_1 + y_0 i). \quad (10'_3)$$

The form which is given to the first term of (10'₁) and (10'₂) I have preferred to simply writing ∞ , since it shows at least for (10'₂) that this term may yet be zero, viz., if $\phi'(x_1 + y_0 i) = 0$, which is the case if $\phi(z) = \text{const.}$ This term is however omitted by Bierens de Haan in his large work on definite integrals: *Exposée de la théorie des propriétés, des formules de transformation et des méthodes d'évaluation des intégrales définies*, pages 46 and page 666. At the proper place I shall give a test case which shows the necessity of this term.

In a similar manner the corrections for discontinuity for the remaining cases of (2) are found. I shall set them down without analysis:

$$\Delta_{(x_1+y_n i)^{2n}} = -\frac{2\psi(x_1+y_n i)}{2n-1} \infty^{2n-1} + \pi i \frac{\psi^{(2n-1)}(x_1+y_n i)}{(2n-1)!} \quad (10''_1)$$

$$\Delta_{(x_1+y_n i)^{2n+1}} = -\frac{2\psi'(x_1+y_n i)}{2n-1} \infty^{2n-1} + \pi i \frac{\psi^{(2n)}(x_1+y_n i)}{2n!} \quad (10''_2)$$

$$\Delta_{(x_1+y_n i)} = \pi i \psi(x_1+y_n i). \quad (10''_3)$$

To determine the correction if $x_1 = x_0$ or $x_1 = x_n$ we have to divide the limits of the y -integration into two parts viz., $\int_{y_1+\Delta y}^{y_n}$ and $\int_{y_0}^{y_1-\Delta y}$

We shall then find

$$\Delta_{(x_0+y_1 i)^{2n}} = \frac{2i\psi(x_0+y_1 i)}{2n-1} \infty^{2n-1} + \pi i \frac{\psi^{(2n-1)}(x_0+y_1 i)}{(2n-1)!}, \quad (10'''_1)$$

$$\Delta_{(x_0+y_1 i)^{2n+1}} = \frac{2i\psi'(x_0+y_1 i)}{2n-1} \infty^{2n-1} + \pi i \frac{\psi^{(2n)}(x_0+y_1 i)}{2n!}, \quad (10'''_2)$$

$$\Delta_{x_0+y_1 i} = \pi i \psi(x_0+y_1 i), \quad (10'''_3)$$

$$\Delta_{(x_n+y_1 i)^{2n}} = -\frac{2i\psi(x_n+y_1 i)}{2n-1} \infty^{2n-1} + \pi i \frac{\psi^{(2n-1)}(x_n+y_1 i)}{(2n-1)!}, \quad (10'''_1)$$

$$\Delta_{(x_n+y_1 i)^{2n+1}} = -\frac{2i\psi'(x_n+y_1 i)}{2n-1} \infty^{2n-1} + \pi i \frac{\psi^{(2n)}(x_n+y_1 i)}{2n!}, \quad (10'''_2)$$

$$\Delta_{x_n+y_1 i} = \pi i \psi(x_n+y_1 i). \quad (10'''_3)$$

If $x_1 = x_0$ and at the same time $y_1 = y_0$ or $y_1 = y_n$; again if $x_1 = x_n$ and at the same time $y_1 = y_0$ or $y_1 = y_n$ equation (3') is correct because in both members the discontinuous elements are included and not lost. These integrals, though they may be finite or infinite, are identical and need no correction.* We have then

$$\Delta_{(x_0+y_0 i)^m} = \Delta_{(x_0+y_n i)^m} = \Delta_{(x_n+y_0 i)^m} = \Delta_{(x_n+y_n i)^m} = 0. \quad (10''')$$

§ 4.

We are now prepared to make some applications. There are ten special assumptions made originally by Cauchy for the limits x_n, x_0, y_n, y_0 (Bierens de Haan, pages 667, 668). Of these the most useful (so considered by Cauchy himself) is that for which $x_n = \infty$; $x_0 = -\infty$; $y_n = \infty$; $y_0 = 0$. We have then in (6), denoting the sum of all the necessary corrections, as the formulas (10) give them for each special case, simply, by Δ

*Bierens de Haan gives in this case $\pm \infty$ for the correction of discontinuity. Price in his Infinitesimal Calculus Vol. II, does not explicitly treat of this case. (See p. 129 at bottom.)

$$A_{(x_1+y_0i)^m} = \sum_{k=1}^{k=\infty} \left[\frac{\phi^{(k-1)}(x_1+y_0i)}{(k-1)!} \cdot \frac{[Ax+(y-y_1)i]^{k-m} - [-Ax+(y-y_1)i]^{k-m}}{k-m} \right]_{y_0}^{y_n}.$$

Here as before all terms of this series vanish for the upper limit except the m th. For the lower limit all terms vanish for which $k-m$ is an even number, zero excepted, and the terms for which $k-m$ is odd and $k < m$ are infinite if $Ax = 0$. The m th term is, as we shall see, finite; it must be retained because it is not necessarily homogeneous with the infinite terms.

Let $m = 2n$ then

$$\begin{aligned} A_{(x_1+y_0i)^{2n}} &= \frac{2\phi(x_1+y_0i)}{2n-1} Ax^{-2n+1} + \frac{\phi^{(2n-1)}(x_1+y_0i)}{(2n-1)!} \\ &\quad \times \left[\log \frac{Ax+(y-y_0)i}{-Ax+(y-y_0)i} \right]_{y_0}^{y_n} \\ &= \frac{2\phi(x_1+y_0i)}{2n-1} Ax^{-2n+1} + \frac{\phi^{(2n-1)}(x_1+y_0i)}{(2n-1)!} \left[(2\varphi - \pi i) \right]_0^{\frac{\pi}{2}}, \text{ if } \tan \varphi = \frac{y-y_0}{Ax} \\ &= \frac{2\phi(x_1+y_0i)}{2n-1} \infty^{2n-1} + \pi i \frac{\phi^{(2n-1)}(x_1+y_0i)}{(2n-1)!}. \quad (10'_1) \end{aligned}$$

Also if $m = 2n + 1$ we have, since the first term vanishes for the lower limit,

$$\begin{aligned} A_{(x_1+y_0i)^{2n+1}} &= \frac{2\phi'(x_1+y_0i)}{2n-1} Ax^{-2n+1} + \frac{\phi^{(2n)}(x_1+y_0i)}{2n!} \\ &\quad \times \left[\log \frac{Ax+(y-y_0)i}{-Ax+(y-y_0)i} \right]_{y_0}^{y_n} \\ &= \frac{2\phi'(x_1+y_0i)}{2n-1} \infty^{2n-1} + \pi i \frac{\phi^{(2n)}(x_1+y_0i)}{2n!}. \quad (10'_2) \end{aligned}$$

If $n = 0$ the correction is

$$A_{(x_1+y_0i)} = \pi i \phi(x_1+y_0i). \quad (10'_3)$$

The form which is given to the first term of (10'_1) and (10'_2) I have preferred to simply writing ∞ , since it shows at least for (10'_2) that this term may yet be zero, viz., if $\phi'(x_1+y_0i) = 0$, which is the case if $\phi(z) = \text{const.}$ This term is however omitted by Bierens de Haan in his large work on definite integrals: *Exposé de la theorie des propriétés, des formules de transformation et des methodes d'évaluation des integrales definies*, pages 45, 46 and page 666. At the proper place I shall give a test case which proves the necessity of this term.

In a similar manner the corrections for discontinuity for the remaining cases of (2) are found. I shall set them down without analysis:

We have then $x_1 = -a$ and $y_1 = 0$, therefore, since

$$\varpi > -a > -\varpi; \varpi > 0 = 0,$$

we have to use formulæ $(10'_1)$, $(10'_2)$, $(10'_3)$ according as $m = 2n$, $2n+1$. We have then

$$\begin{aligned} \int_{-x}^{\infty} \frac{\varphi(x)}{(a+x)^{2n}} dx &= \frac{2\varphi(-a)}{2n-1} \varpi^{2n-1} + \pi i \frac{\varphi^{(2n-1)}(-a)}{(2n-1)!}, \\ \int_{-x}^{\infty} \frac{\varphi(x)}{(a+x)^{2n+1}} dx &= \frac{2\varphi'(-a)}{2n-1} \varpi^{2n-1} + \pi i \frac{\varphi^{(2n)}(-a)}{2n!}, \\ \int_{-x}^{\infty} \frac{\varphi(x)}{a+x} dx &= \pi i \varphi(-a). \end{aligned}$$

1. Let $\varphi(z) = e^{mzi}$. This becomes $=\varpi$ for $z=0-\varpi i$. Now although the value 0 for x is included within the x -integration, the value $-\varpi$ is not; there is consequently no correction for this value of z . Now

$$\begin{aligned} \frac{e^{b(+or-x+yi)i}}{(a \pm \varpi + yi)^m} &= \frac{e^{-by} [\cos \{ b(\pm \varpi) \} + i \sin \{ b(\pm \varpi) \}]}{(a + \varpi + yi)^m} = 0, \\ \frac{e^{b(x+\varpi i)i}}{(a+x+\varpi i)^m} &= \frac{e^{-bx} (\cos bx + i \sin bx)}{(a+x+\varpi i)^m} = 0 \end{aligned}$$

condition (11) is satisfied and we can safely use formulæ (a_1) , (a_2) and

We have then, since $\varphi^{(m-1)}(-a) = (bi)^{m-1} e^{-abi}$,

$$\begin{aligned} \int_{-x}^{\infty} \frac{e^{bxi} dx}{(a+x)^{2n}} &= \frac{2e^{-abi}}{2n-1} \varpi^{2n-1} + \pi i \frac{(bi)^{m-1}}{(2n-1)!} e^{-abi} \\ &= \frac{2e^{-abi}}{2n-1} \varpi^{2n-1}. \end{aligned}$$

(Omitting here the second term being finite and homogeneous with the which is infinite.)

$$\begin{aligned} \int_{-x}^{\infty} \frac{e^{bxi} dx}{(a+x)^{2n+1}} &= \frac{2bi e^{-abi}}{2n-1} \varpi^{2n-1}, \\ \int_{-x}^{\infty} \frac{e^{bxi} dx}{a+x} &= \pi i e^{-abi}. \end{aligned}$$

Separating the real from the imaginary in (b_3) we have

$$\begin{aligned} \int_{-x}^{\infty} \frac{\cos bx}{a+x} dx &= \pi \sin ab, \\ \int_{-x}^{\infty} \frac{\sin bx}{a+x} dx &= \pi \cos ab. \end{aligned}$$

Placing $b = 0$ in (b_1) , (b_2) , (b'_3) and $a = 0$ in (b''_3) we have

$$\int_{-x}^{\infty} \frac{dx}{(a+x)^{2n}} = \frac{2}{2n-1} \varpi^{2n-1} = \varpi,$$

$$\int_{-\infty}^{\infty} \frac{dx}{(a+x)^{2n-1}} = 0, \quad (b_2''')$$

$$\int_{-\infty}^{\infty} \frac{dx}{a+x} = 0, \quad (b_3''')$$

$$\int_{-\infty}^{\infty} \frac{\sin bx}{x} dx = 2 \int_0^{\infty} \frac{\sin bx}{x} dx = \pi. \quad (b_4''')$$

We can verify (b_1''') , (b_2''') and (b_3''') by common integration. We have

$$\int_q^p \frac{dx}{(a+x)^m} = -\frac{1}{m-1} \left[\frac{1}{(a+p)^{m-1}} - \frac{1}{(a+q)^{m-1}} \right].$$

As long as $p > q > -a$ this integration is correct. But if $p > -a > q$ we have

$$\begin{aligned} \int_q^p \frac{dx}{(a+x)^m} &= \int_{-a+\Delta x}^p \frac{dx}{(a+x)^m} + \int_q^{-a+\Delta x} \frac{dx}{(a+x)^m} \\ &= -\frac{1}{m-1} \left[\frac{1}{(a+p)^{m-1}} - \frac{1}{(\Delta x)^{m-1}} + \frac{1}{(-\Delta x)^{m-1}} - \frac{1}{(a+q)^{m-1}} \right]. \end{aligned}$$

If $m = 2n$ we have

$$\int_q^p \frac{dx}{(a+x)^{2n}} = \frac{2}{2n-1} \cdot \frac{1}{\Delta x^{2n-1}} - \frac{1}{2n-1} \left[\frac{1}{(a+p)^{2n-1}} - \frac{1}{(a+q)^{2n-1}} \right] = \infty,$$

and $\int_{-\infty}^{\infty} \frac{dx}{(a+x)^{2n}} = \frac{2}{2n-1} \cdot \frac{1}{\Delta x^{2n-1}} = \infty$ [the same as (b_1''')].

If $m = 2n+1$ we have

$$\int_q^p \frac{dx}{(a+x)^{2n+1}} = -\frac{1}{2n} \left[\frac{1}{(a+p)^{2n}} - \frac{1}{(a+q)^{2n}} \right], \quad \text{and}$$

$$\int_{-\infty}^{\infty} \frac{dx}{(a+x)^{2n+1}} = 0 \quad [\text{the same as } (b_2''')].$$

Finally if $m = 1$

$$\int_q^p \frac{dx}{a+x} = \left[\frac{(a+p)^{1-m} - (\Delta x)^{1-m} + (-\Delta x)^{1-m} - (a+q)^{1-m}}{1-m} \right]_{m=1}$$

$$= \log \left(\frac{a+p}{\Delta x} \cdot \frac{-\Delta x}{a+q} \right) = \log \left(\frac{a+p}{-(a+q)} \right); \text{ consequently}$$

$$\int_{-\infty}^{\infty} \frac{dx}{a+x} = \log \left(\frac{a+\infty}{-a+\infty} \right) = \log 1 = 0 \quad [\text{the same as } (b_3''')].$$

As stated above, Bierens de Haan omitted the infinite term in formulæ $(10_1'')$, $(10_2'')$. We should then have found according to his formula

$$\int_{-\infty}^{\infty} \frac{dx}{(a+x)^{2n}} = 0,$$

which is absurd for the reason that this integral represents a series of essentially positive terms, the sum of which cannot be zero.

(To be continued.)

ON THE ARITHMETICO-GEOMETRICAL MEAN.

BY PROF. EDGAR FRISBY, U. S. NAVAL OBSERVATORY, WASHINGTON, D.

GAUSS, in his Memoir entitled "Determinatio Attractionis", &c., *Werke* Bd. 3, p. 333, has given a method for finding the value of the elliptic integral

$$\int \frac{d\theta}{\sqrt{(m^2 \cos^2 \theta + n^2 \sin^2 \theta)}},$$

which can be developed in the following manner:

Let m and n be any two real and positive quantities, then if m_1 and n_1 are their Arithmetical and Geometrical means so that

$$m_1 = \frac{m+n}{2}; \quad n_1 = \sqrt{mn};$$

$$m_2 = \frac{m_1+n_1}{2}; \quad n_2 = \sqrt{m_1 n_1}; \quad \&c.,$$

the successive values of m and n will very rapidly converge to a definite limit μ which for simplicity Gauss calls the Arithmetico-Geometrical Mean.

As an example take

$m = 100,$	$n = 1;$
$m_1 = 50.5,$	$n_1 = 10;$
$m_2 = 30.25,$	$n_2 = 22.47221;$
$m_3 = 26.36110,$	$n_3 = 26.07267;$
$m_4 = 26.21689,$	$n_4 = 26.21649;$
$m_5 = 26.21669,$	$n_5 = 26.21669.$

This calculation is made quite readily for each m is the arithmetical mean of the preceding m and n ; also the logarithm of each n is the arithmetical mean of the logarithms of the preceding m and n ; which logarithm will be used in computing the next n , &c.

I. To prove that

$$\int_0^{2\pi} \frac{d\theta}{\sqrt{(m^2 \cos^2 \theta + n^2 \sin^2 \theta)}} = \frac{1}{\mu},$$

suppose the variable θ to be represented as a function of another variable θ_1 so that

$$\sin \theta = \frac{2m \sin \theta_1}{(m+n) \cos^2 \theta_1 + 2m \sin^2 \theta_1}. \quad (1)$$

We know that this is always possible, for the value of θ_1 which makes the right hand member of this equation a maximum or minimum is given by the equation $\cos \theta_1 = 0$; the other equation, $(m-n) \sin^2 \theta_1 = m+n$, is plainly inadmissible. Putting therefore

$$\theta_1 = \frac{1}{2}\pi \text{ or } \frac{3}{2}\pi$$

we find that this expression always lies between +1 and -1; θ and θ_1 will always have the same sign and the limits will be the same from 0 to $\frac{1}{2}\pi$, π , $\frac{3}{2}\pi$ or 2π .

Substituting $1 - \sin^2\theta_1$ for $\cos^2\theta_1$ in (1) we have

$$\left. \begin{aligned} \sin \theta &= \frac{2m \sin \theta_1}{m+n+(m-n)\sin^2\theta_1}, \\ \text{or } (m-n) \sin \theta \sin^2\theta_1 - 2m \sin \theta_1 + (m+n) \sin \theta &= 0, \end{aligned} \right\} \quad (2)$$

and solving this equation for θ_1 we have

$$(m-n) \sin \theta \sin \theta_1 = m \pm \sqrt{(m^2 \cos^2 \theta + n^2 \sin^2 \theta)},$$

$$\text{or } \sqrt{(m^2 \cos^2 \theta + n^2 \sin^2 \theta)} = m - (m-n) \sin \theta \sin \theta_1. \quad (3)$$

Differentiating equation (2) under the following form,

$$\frac{\sin \theta}{\sin \theta_1} = \frac{m+n+(m-n)\sin^2\theta_1}{2m},$$

we have

$$\sin \theta \cos \theta_1 d\theta_1 - \sin \theta_1 \cos \theta d\theta = \frac{m-n}{m} \sin \theta_1 \cos \theta_1 \sin^2 \theta d\theta_1,$$

or

$$\frac{d\theta}{m-(m-n)\sin \theta \sin \theta_1} = \frac{d\theta_1}{m \cot \theta \tan \theta_1} = \text{from (3)} \frac{d\theta}{\sqrt{(m^2 \cos^2 \theta + n^2 \sin^2 \theta)}}. \quad (4)$$

Expressing the denominator $m \cot \theta \tan \theta_1$ as a function of m_1, n_1 and θ_1 we have

$$m^2 \cot^2 \theta \tan^2 \theta_1 = m^2 \tan^2 \theta_1 \left(\frac{1}{\sin^2 \theta} - 1 \right)$$

$$\text{from (2),} \quad = m^2 \tan^2 \theta_1 \left[\frac{(m+n)^2}{4m^2 \sin^2 \theta_1} + \frac{(m-n)^2}{4m^2} \sin^2 \theta_1 + \frac{m^2 - n^2}{2m^2} - 1 \right].$$

On substituting $(m+n)^2 = 4m_1^2$, $(m-n)^2 = 4(m_1^2 - n_1^2)$ and $\frac{1}{2}(m^2 + n^2) = 2m_1^2 - n_1^2$ this equation becomes

$$\begin{aligned} &= \frac{1}{\cos^2 \theta_1} \left[m_1^2 (\cos^2 \theta_1 + \sin^2 \theta_1)^2 + (m_1^2 - n_1^2) \sin^4 \theta_1 \right. \\ &\quad \left. - (2m_1^2 - n_1^2) \sin^2 \theta_1 (\cos^2 \theta_1 + \sin^2 \theta_1) \right] \\ &= m_1^2 \cos^2 \theta_1 + n_1^2 \sin^2 \theta_1. \end{aligned} \quad (5)$$

Equation (4) becomes

$$\frac{d\theta}{\sqrt{(m^2 \cos^2 \theta + n^2 \sin^2 \theta)}} = \frac{d\theta_1}{\sqrt{(m_1^2 \cos^2 \theta_1 + n_1^2 \sin^2 \theta_1)}}, \quad (6)$$

and therefore

$$\int_0^{2\pi} \frac{d\theta}{2\pi \sqrt{(m^2 \cos^2 \theta + n^2 \sin^2 \theta)}} = \int_0^{2\pi} \frac{d\theta_1}{2\pi \sqrt{(m_1^2 \cos^2 \theta_1 + n_1^2 \sin^2 \theta_1)}};$$

and proceeding in the same way with $\theta_2, m_2, n_2, \&c.$, we find it to be ultimately equal to

$$\int_0^{2\pi} \frac{d\theta}{2\pi\sqrt{\mu^2\cos^2\theta + \mu'^2\sin^2\theta}} = \frac{1}{\mu}. \quad (7)$$

II. From this value many more interesting integrals may be found. Resuming equations (3) and (5) we have

$$\sqrt{m^2\cos^2\theta + n^2\sin^2\theta} = m - (m-n)\sin\theta\sin\theta_1,$$

$$\sqrt{m_1^2\cos^2\theta_1 + n_1^2\sin^2\theta_1} = m \cot\theta \tan\theta_1;$$

whence we find

$$\begin{aligned} \sin\theta\sin\theta_1\sqrt{m^2\cos^2\theta + n^2\sin^2\theta} - \cos\theta\cos\theta_1\sqrt{m_1^2\cos^2\theta_1 + n_1^2\sin^2\theta_1} \\ = m\sin\theta\sin\theta_1 - (m-n)\sin^2\theta\sin^2\theta_1 - m\cos\theta\cot\theta\sin\theta_1 \\ = m\sin\theta\sin\theta_1(1-\cot^2\theta) - (m-n)\sin^2\theta\sin^2\theta_1 \\ = -m\frac{\sin\theta_1}{\sin\theta}(\cos^2\theta - \sin^2\theta) - (m-n)\sin^2\theta\sin^2\theta_1 \\ = -\frac{1}{2}(\cos^2\theta - \sin^2\theta)[m+n+(m-n)\sin^2\theta_1] - (m-n)\sin^2\theta\sin^2\theta_1, \text{ by (2)} \\ = -\frac{1}{2}(m+n)(\cos^2\theta - \sin^2\theta) - \frac{1}{2}(m-n)(1-2\sin^2\theta)\sin^2\theta_1 - (m-n)\sin^2\theta\sin^2\theta_1 \\ = -\frac{1}{2}(m+n)(\cos^2\theta - \sin^2\theta) - \frac{1}{2}(m-n)\sin^2\theta_1; \text{ therefore} \end{aligned}$$

$$\begin{aligned} \sin\theta\sin\theta_1\sqrt{m^2\cos^2\theta + n^2\sin^2\theta} + \frac{1}{2}(m+n)(\cos^2\theta - \sin^2\theta) \\ = \cos\theta\cos\theta_1\sqrt{m_1^2\cos^2\theta_1 + n_1^2\sin^2\theta_1} - \frac{1}{2}(m-n)\sin^2\theta_1. \end{aligned}$$

Multiplying this equation by each member of (6) we have

$$\frac{(m+n)(\cos^2\theta - \sin^2\theta)d\theta}{2\sqrt{m^2\cos^2\theta + n^2\sin^2\theta}} = -\frac{(m-n)\sin^2\theta_1 d\theta_1}{2\sqrt{m_1^2\cos^2\theta_1 + n_1^2\sin^2\theta_1}} + d\sin\theta_1\cos\theta. \quad (8)$$

Multiplying this equation by $(m-n)\div\pi$ and substituting $(m-n)^2 = 4(m_1^2 - n_1^2)$, $\sin^2\theta_1 = \frac{1}{2} - \frac{1}{2}(\cos^2\theta_1 - \sin^2\theta_1)$ and integrating from 0 to 2π we have

$$\begin{aligned} (m^2 - n^2) \int_0^{2\pi} \frac{(\cos^2\theta - \sin^2\theta)d\theta}{2\pi\sqrt{m^2\cos^2\theta + n^2\sin^2\theta}} = -\frac{2(m_1^2 - n_1^2)}{\mu} \\ + 2(m_1^2 - n_1^2) \int_0^{2\pi} \frac{(\cos^2\theta_1 - \sin^2\theta_1)d\theta_1}{2\pi\sqrt{m_1^2\cos^2\theta_1 + n_1^2\sin^2\theta_1}}; \end{aligned}$$

and by successive substitutions in the same way it becomes

$$\begin{aligned} \int_0^{2\pi} \frac{(\cos^2\theta - \sin^2\theta)d\theta}{2\pi(m^2\cos^2\theta + n^2\sin^2\theta)} = \frac{2(m_1^2 - n_1^2) + 4(m_2^2 - n_2^2) + 8(m_3^2 - n_3^2) + \&c.}{(m^2 - n^2)\mu} \\ = -\frac{\nu}{\mu}. \quad (9) \end{aligned}$$

If we now put $\frac{1}{2}\sqrt{m^2 - n^2} = \lambda$; $\frac{1}{2}\sqrt{m_1^2 - n_1^2} = \lambda_1$; $\frac{1}{2}\sqrt{m_2^2 - n_2^2} = \lambda_2$, &c., we find that

$$\lambda_1 = \frac{\lambda^2}{m_1}; \quad \lambda_2 = \frac{\lambda_1^2}{m_2} \quad \&c.; \quad \nu = \frac{2\lambda_1^2 + 4\lambda_2^2 + 8\lambda_3^2 + \&c.}{\lambda^2}.$$

If we now put $\omega = \sqrt{(m^2 \cos^2 \theta + n^2 \sin^2 \theta)}$ we find, from (7) and (9),

$$\int_0^{2\pi} \frac{\cos^2 \theta d\theta}{2\pi\omega} = \frac{1-\nu}{2\mu}; \quad \int_0^{2\pi} \frac{\sin^2 \theta d\theta}{2\pi\omega} = \frac{1+\nu}{2\mu}. \quad (10)$$

If we put

$$P = \int_0^{2\pi} \frac{\cos^2 \theta d\theta}{2\pi\omega^3}; \quad Q = \int_0^{2\pi} \frac{\sin^2 \theta d\theta}{2\pi\omega^3},$$

we can find their values in the following manner:

$$\begin{aligned} d. \frac{\sin \theta \cos \theta}{\pi\omega} &= \frac{(\cos^2 \theta - \sin^2 \theta) d\theta}{\pi\omega} + \frac{(m^2 - n^2) \sin^2 \theta \cos^2 \theta d\theta}{\pi\omega^3} \\ &= \frac{(\cos^2 \theta - \sin^2 \theta) d\theta}{\pi\omega} + \left[\frac{(m^2 \cos^2 \theta - n^2 \sin^2 \theta)(\cos^2 \theta + \sin^2 \theta)}{2\pi\omega^3} \right. \\ &\quad \left. - \frac{(m^2 \cos^2 \theta + n^2 \sin^2 \theta)(\cos^2 \theta - \sin^2 \theta)}{2\pi\omega^3} \right] d\theta \\ &= \frac{(\cos^2 \theta - \sin^2 \theta) d\theta}{2\pi\omega} + \left[\frac{m^2 \cos^2 \theta - n^2 \sin^2 \theta}{2\pi\omega^3} \right] d\theta. \end{aligned} \quad (11)$$

Integrating this equation between the limits 0 and 2π , we have

$$-\frac{\nu}{\mu} + m^2 P - n^2 Q = 0, \text{ and from (7) } -\frac{1}{\mu} + m^2 P + n^2 Q = 0;$$

$$\therefore P = \frac{1+\nu}{2m^2\mu}; \quad Q = \frac{1-\nu}{2n^2\mu}; \text{ and } \int_0^{2\pi} \frac{d\theta}{2\pi\omega^3} = P + Q. \quad (12)$$

It is evident from equations (8) and (11) that the integrals from 0 to $\frac{1}{2}\pi$, π , $\frac{3}{2}\pi$ and 2π will be proportional to these values, for $\sin \theta$, $\cos \theta$ and $\sin \theta \cos \theta$ will vanish at both limits.

NOTE.—The indefinite integral

$$\int \frac{d\zeta^*}{\sqrt{(m^2 \cos^2 \zeta + n^2 \sin^2 \zeta)}} = \frac{\theta}{\mu} \text{ from (7).}$$

Multiplying (8) by $2(m-n)$ we find

$$\int \frac{(\cos^2 \zeta - \sin^2 \zeta) d\zeta}{\sqrt{(m^2 \cos^2 \zeta + n^2 \sin^2 \zeta)}} = -\frac{\nu\theta}{\mu} + \frac{\lambda_1 \cos \zeta \sin \zeta_1 + 2\lambda_2 \cos \zeta_1 \sin \zeta_2 + \&c.}{\lambda_2},$$

where θ is the limiting value of ζ if ζ_2 is determined from ζ_1 , m_1 and n_1 in the same manner that ζ_1 is determined from m , n , ζ , &c.

Collecting all these results and putting, for the sake of brevity,

$$\omega = \sqrt{(m^2 \cos^2 \zeta + n^2 \sin^2 \zeta)}$$

we have

$$\begin{aligned} \int_0^{2\pi} \frac{d\zeta}{2\pi\omega} &= \frac{1}{\mu}, \\ \int_0^{2\pi} \frac{(\cos^2 \zeta - \sin^2 \zeta) d\zeta}{2\pi\omega} &= -\frac{\nu}{\mu}, \end{aligned}$$

*For want of sorts, ζ is here, and through the succeeding part of this Art., used for θ .

$$\begin{aligned}\int_0^{2\pi} \frac{\cos^2 \zeta d\zeta}{2\pi\omega} &= \frac{1-\nu}{2\mu}; & \int_0^{2\pi} \frac{\sin^2 \zeta d\zeta}{2\pi\omega} &= \frac{1+\nu}{2\mu}; \\ \int_0^{2\pi} \frac{\cos^2 \zeta d\zeta}{2\pi\omega^3} &= \frac{1+\nu}{2m^2\mu}; & \int_0^{2\pi} \frac{\sin^2 \zeta d\zeta}{2\pi\omega^3} &= \frac{1-\nu}{2n^2\mu}; \\ \int_0^{2\pi} \frac{d\zeta}{2\pi\omega^3} &= \frac{1+\nu}{2m^2\mu} + \frac{1-\nu}{2n^2\mu} = \frac{m^2+n^2-(m^2-n^2)\nu}{2m^2n^2\nu};\end{aligned}$$

and the integrals from 0 to $\frac{1}{2}\pi$, π , $\frac{3}{2}\pi$ and 2π are always proportional to them.

The indefinite integrals

$$\begin{aligned}\int \frac{d\zeta}{\omega} &= \frac{\theta}{\mu}; \\ \int \frac{(\cos^2 \zeta - \sin^2 \zeta) d\zeta}{\omega} &= -\frac{\nu\theta}{\mu} + \frac{\lambda_1 \cos \zeta \sin \zeta_1 + \lambda_2 \cos \zeta \sin \zeta_2 + \&c.}{\lambda^2} = -\frac{\nu\theta}{\mu} + A; \\ \int \frac{\cos^2 \zeta d\zeta}{\omega} &= \frac{(1-\nu)\theta}{2\mu} + \frac{A}{2}; & \int \frac{\sin^2 \zeta d\zeta}{\omega} &= \frac{(1+\nu)\theta}{2\mu} - \frac{A}{2}; \\ \int \frac{\cos^2 \zeta d\zeta}{\omega^3} &= \frac{(1+\nu)\theta + \left(\frac{\sin 2\zeta}{\omega} - A\right)\mu}{2m^2\mu}; & \int \frac{\sin^2 \zeta d\zeta}{\omega^3} &= \frac{(1-\nu)\theta - \left(\frac{\sin 2\zeta}{\omega} - A\right)\mu}{2n^2\mu}; \\ \int \frac{d\zeta}{\omega^3} &= \frac{(m^2+n^2-(m^2-n^2)\nu)\theta - (m^2-n^2)\left(\frac{\sin 2\zeta}{\omega} - A\right)\mu}{2m^2n^2\nu};\end{aligned}$$

where ν , A , &c., have the values given above.

THE SUMMATION OF SERIES WHOSE COEFFICIENTS FOR M AN ARITHMETICAL PROGRESSION OF ANY ORDER.

BY PROF. J. SCHEFFER, MERCERSBERG, PA.

LET the series be represented by $ax+bx^2+cx^3+dx^4+\dots$ whose sum we denote by S . Putting $x = y/(1+y)$, we have

$$S = a\left(\frac{y}{1+y}\right) + b\left(\frac{y}{1+y}\right)^2 + c\left(\frac{y}{1+y}\right)^3 + d\left(\frac{y}{1+y}\right)^4 + \dots$$

Developing each of these fractions by the Binomial Theorem, we have
 $S = ay - (a-b)y^2 + (a-2b+c)y^3 - (a-3b+3c-d)y^4 + (a-4b+6c-4d+e)y^5 - \dots$

We see that the coefficients of this series are the successive finite differences, which we shall denote by Δa , $\Delta^2 a$, $\Delta^3 a$, ..., consequently

$$S = ay + \Delta a.y^2 + \Delta^2 a.y^3 + \Delta^3 a.y^4 + \dots \text{ or}$$

$$S = a\left(\frac{x}{1-x}\right) + \Delta a\left(\frac{x}{1-x}\right)^2 + \Delta^2 a\left(\frac{x}{1-x}\right)^3 + \Delta^3 a\left(\frac{x}{1-x}\right)^4 \dots$$

As the coefficients form an arithmetical progression, one of these differences and all the following will become = 0.

This method is very useful in finding the sum of a great many series.

For instance; let $S = x + 3x^2 + 5x^3 + 7x^4 + 9x^5 + \dots$ to infinity. Here $a = 1$, $\Delta a = 2$, $\Delta^2 a = \Delta^3 a = \dots = 0$. Hence

$$S = \frac{x}{1-x} + 2\left(\frac{x}{1-x}\right)^2 = \frac{x(1+x)}{(1-x)^2}.$$

In the series

$$S = x + 4x^2 + 9x^3 + 16x^4 + 25x^5 + \dots \text{ ad inf,}$$

we have $a = 1$, $\Delta a = 3$, $\Delta^2 a = 2$, $\Delta^3 a = \Delta^4 a = \dots = 0$. Therefore

$$S = \frac{x}{1+x} + 3\left(\frac{x}{1+x}\right)^2 + 2\left(\frac{x}{1+x}\right)^3 = \frac{x(1+x)}{(1-x)^3}.$$

According to this method we can also find the sum of a given number of terms. Putting

$$S' = ax + bx^2 + cx^3 + dx^4 + \dots + lx^n + px^{n+1} + qx^{n+2} + \dots$$

and $S = ax + bx^2 + cx^3 + dx^4 + \dots + lx^n$, we have

$$\begin{aligned} S &= S' - x^n(px + qx^2 + rx^3 + \dots) \\ &= S' - x^n \left[p\left(\frac{x}{1-x}\right) + \Delta p\left(\frac{x}{1-x}\right)^2 + \Delta^2 p\left(\frac{x}{1-x}\right)^3 + \dots \right], \end{aligned}$$

but $S' = a\left(\frac{x}{1-x}\right) + \Delta a\left(\frac{x}{1-x}\right)^2 + \Delta^2 a\left(\frac{x}{1-x}\right)^3 + \dots$; therefore

$$S = (a - x^n p) \frac{x}{1-x} + (\Delta a - x^n \Delta p) \left(\frac{x}{1-x}\right)^2 + (\Delta^2 a - x^n \Delta^2 p) \left(\frac{x}{1-x}\right)^3 + \dots$$

Ex. 1. In the common geometrical progression, we have

$$ar + ar^2 + ar^3 + \dots + ar^n;$$

where $a = a$, $\Delta a = 0$; $p = a$, $\Delta p = 0$; hence

$$S = (a - ar^n) \frac{r}{1-r}.$$

Ex. 2. $S = x + 2x^2 + 3x^3 + 4x^4 + \dots + nx^n$. $a = 1$, $\Delta a = 1$, $p = n + 1$, $\Delta p = 1$; hence

$$S = [1 - (n+1)x^n] \frac{x}{1-x} + (1-x^n) \left(\frac{x}{1-x}\right)^2 = \frac{x - (n+1)x^{n+1} + nx^{n+2}}{(1-x)^2}.$$

Ex. 3. $S = x + 4x^2 + 9x^3 + 16x^4 + \dots + n^2 x^n$. $a = 1$, $\Delta a = 3$, $\Delta^2 a = 2$, $p = (n+1)^2$, $\Delta p = 2n+3$, $\Delta^2 p = 2$; therefore

$$S = [1 - (n+1)^2 x^n] \frac{x}{1-x} + [3 - (2n+3)x^n] \left(\frac{x}{1-x}\right)^2 + (2 - 2x^n) \left(\frac{x}{1-x}\right)^3$$

$$= \frac{-x^{n+1}[n(x-1)-1]^2 + x(x+1) - x^{n+2}}{(1-x)^3}.$$

Putting $-x$ in place of x , we have

$$\begin{aligned} S &= ax - bx^2 + cx^3 - dx^4 + \dots \\ &= a \left[\frac{x}{1+x} \right] - 4a \left[\frac{x}{1+x} \right]^2 + 4^2 a \left[\frac{x}{1+x} \right]^3 - \dots \end{aligned}$$

Putting $x = 1$, we have

$$a - b + c - d + e - f + \dots \text{ ad inf. } = \frac{1}{2}a - \frac{1}{4}4a + \frac{1}{8}4^2a - \frac{1}{16}4^3a + \dots$$

Thus we find, $1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + \dots \text{ ad inf. } = \frac{1}{2},$

$$1 - 2 + 3 - 4 + 5 - 6 + 7 - 8 + \dots \text{ ad inf. } = \frac{1}{4},$$

$$1 - 3 + 5 - 7 + 9 - 11 + 13 - \dots \text{ ad inf. } = 0,$$

$$1 - 4 + 9 - 16 + 25 - 36 + 49 - \dots \text{ ad inf. } = 0.$$

The formula $ax - bx^2 + cx^3 - dx^4 + \dots$

$$= a \left[\frac{x}{1+x} \right] - 4a \left[\frac{x}{1+x} \right]^2 + 4^2 a \left[\frac{x}{1+x} \right]^3 - \dots$$

is also very useful, sometimes, to change a series into one that converges more rapidly. For instance,

$$\log(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \dots$$

$$= \frac{x}{1+x} \left\{ 1 + \frac{1}{2} \left[\frac{x}{1+x} \right] + \frac{1}{3} \left[\frac{x}{1+x} \right]^2 + \frac{1}{4} \left[\frac{x}{1+x} \right]^3 + \dots \right\}.$$

$$\tan^{-1}x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \dots = \frac{1}{x} (x^2 - \frac{1}{3}x^4 + \frac{1}{5}x^6 - \frac{1}{7}x^8 + \dots)$$

$$= \frac{x}{1+x^2} \left\{ 1 + \frac{2}{3} \left[\frac{x^2}{1+x^2} \right] + \frac{2 \cdot 4}{3 \cdot 5} \left[\frac{x^2}{1+x^2} \right]^2 + \frac{2 \cdot 4 \cdot 6}{3 \cdot 5 \cdot 7} \&c. \right\}.$$

The above method of the summation of series is by *Euler*. He published it in his *Inst. Calc. Diff.*

QUERY BY E. HANCOCK, ELGIN, ILL. — If a wheel of 60 teeth work in a pinion of 10 leaves, the distance between their centres being .455 in., the diameters of the pitch circles will be .78 in., = a , and .13 in., = b , respectively. Allowing one-half for space between the teeth, each tooth will occupy a central angle of 3° , = c . Suppose the teeth to terminate at the intersection of two similar epicycloidal curves the diameter of whose generating circle is .065, = $\frac{1}{2}b$; what must be the addendum to the wheel outside of the pitch circle, i. e., what is the length of the teeth, outside of the pitch circle?

SOLUTION OF PROBLEM 226.

BY PROF. W. W. JOHNSON, ST. JOHN'S COLLEGE, ANNAPOLIS, MD.

PUTTING $\frac{2k\pi}{n} = \theta$, $x = \cos \theta$ in the first of the required equations, and $x = \sin \theta$ in the second. In either case, $\sin n\theta = 0$, and $\cos n\theta = 1$; hence if we can express either of these quantities in a finite number of terms involving powers of $\cos \theta$ we shall have the first of the required equations, and if we express either of them in terms of $\sin \theta$, we have the second equation. Now $\sin n\theta$ cannot be expressed in a finite number of terms involving $\cos \theta$; but for $\cos n\theta$ we have (See Chauvenet's Trig., eq. 460, page 138),

$$\cos n\theta = \cos \frac{n\pi}{2} \left[1 - \frac{n^2}{2} \cos^2 \theta + \frac{n^2(n^2-2^2)}{1.2.3.4} \cos^4 \theta + \dots \right] \\ + \sin \frac{n\pi}{2} \left[n \cos \theta - \frac{n(n^2-1^2)}{1.2.3} \cos^3 \theta + \frac{n(n^2-1^2)(n^2-3^2)}{5!} \cos^5 \theta \dots \right].$$

Substituting $\cos \theta = x$ we have, when n is an odd integer,

$$0 = -1 + (-1)^{\frac{n-1}{2}} \left[nx - \frac{n(n^2-1^2)}{1.2.3} x^3 + \frac{n(n^2-1^2)(n^2-3^2)}{5!} x^5 - \dots \right. \\ \left. + (-1)^{\frac{n-1}{2}} \frac{n(n^2-1^2) \dots [n^2-(n-2)^2]}{n!} x^n \right], \quad (1)$$

but when n is even we have

$$0 = -1 + (-1)^{\frac{n}{2}} \left[1 - \frac{n^2}{2} x^2 + \frac{n^2(n^2-2^2)}{4!} x^4 - \dots \right. \\ \left. + (-1)^{\frac{n}{2}} \frac{n^2(n^2-2^2) \dots [n^2-(n-2)^2]}{n!} x^n \right]. \quad (2)$$

Equations (1) and (2) are therefore the equations whose roots are

$$x = \cos \frac{2k\pi}{n}.$$

For the second equation $x = \sin \theta$, and we are required to express either $\cos n\theta$ or $\sin n\theta$ in terms of $\sin \theta$. We have

$$\cos n\theta = 1 - \frac{n^2}{2} \sin^2 \theta + \frac{n^2(n^2-2^2)}{1.2.3.4} \sin^4 \theta - \frac{n^2(n^2-2^2)(n^2-4^2)}{6!} \sin^6 \theta + \dots \\ \sin n\theta = n \sin \theta - \frac{n(n^2-1^2)}{1.2.3} \sin^3 \theta + \frac{n(n^2-1^2)(n^2-3^2)}{5!} \sin^5 \theta - \dots$$

When n is odd, the expression for $\sin n\theta$ terminates, and we have by substitution

$$0 = x - \frac{n^2-1^2}{1.2.3} x^3 + \frac{(n^2-1^2)(n^2-3^2)}{5!} x^5 - \dots \\ + (-1)^{\frac{n-1}{2}} \frac{(n^2-1^2)(n^2-3^2) \dots [n^2-(n-2)^2]}{n!} x^n, \quad (3)$$

but when n is even the expression for $\cos \theta$ terminates, and we have, on substitution,

$$0 = x^2 - \frac{n^2-2^2}{3.4} x^4 + \frac{(n^2-2^2)(n^2-4^2)}{3.4.5.6} x^6 - \dots \\ - (-1)^{\frac{n}{2}} \frac{(n^2-2^2) \dots [n^2-(n-2)^2]}{3.4.5 \dots n} x^n. \quad (4)$$

Equations (3) and (4) are therefore those whose roots are

$$x = \sin \frac{2k\pi}{n}.$$

The several roots are in each case found by giving to k the values $0, 1, 2, \dots, n-1$. Putting $k=0$ we see that $x=1$ is a root of equations (1) and (2), whence we have the numerical series, when n is odd,

$$(-1)^{\frac{n-1}{2}} = n - \frac{n(n^2-1^2)}{1.2.3} + \frac{n(n^2-1^2)(n^2-3^2)}{5!} - \dots$$

and when n is even,

$$(-1)^{\frac{n}{2}} = 1 - \frac{n^2}{2} + \frac{n^2(n^2-2^2)}{4!} - \frac{n^2(n^2-2^2)(n^2-4^2)}{6!} + \dots$$

Since $\cos \frac{2k\pi}{n} = \cos \frac{2(n-k)\pi}{n}$ the roots (exclusive of unity) of eq. (1) occur in equal pairs, and the equation must be of the form

$$(x-1)[F_1(x)]^2 = 0,$$

and for a like reason equation (2) must have the form

$$(x^2-1)[F_2(x)]^2 = 0.$$

The equations $F_1(x) = 0$ and $F_2(x) = 0$ are therefore those whose roots are the *different* values of x , exclusive of 1 and -1 , which are represented by $\cos(2k\pi/n)$.

I proceed to find the form of the functions $F_1(x)$ and $F_2(x)$. Observe in the first place that we may, in any one of the equations, change n to $-n$, in each of the coefficients; and also that, since in (1) and (2) the degree of each term in n is the same as its degree in x , the same will be true in the quotients when we divide these equations by $-1+x$ or $-1+x^2$, and also in the square roots of the results, that is in the functions F_1 and F_2 .

Assuming n to have the form $4n'+1$ in eq. (1) we may give to n the successive values $1, -3, 5, -7, 9$ etc., and by changing the sign of n we shall

have the form which the function takes when n is of the form $4n' + 3$, which form may be denoted by F_3 , the admissible values of n being $-1, 3, -5, 7$ etc. It is obvious that the first term of $F_1(x)$ is 1; and, since the expression must reduce to its first term when $n = 1$, every term after the first must contain the factor $n-1$. In like manner all terms after the second must contain the factor $n+3$, since when $n = -3$ all terms after the second must vanish; and so on. Thus we may assume

$$F_1(x) = 1 + A(n-1)x + B(n-1)(n+3)x^2 + \dots$$

in which A, B , etc., are numerical factors, since the degree of each term in n is to be the same as its degree in x . Then the expression $F_3(x)$, found by changing n to $-n$, is

$$F_3(x) = 1 - A(n+1)x + B(n+1)(n-3)x^2 - \dots$$

In either case the last term contains $x^{\frac{n-1}{2}}$, and $\frac{n-1}{2}$ factors which are successive multiples of 4, the least being 4; hence, denoting the numerical factor by L , the value of this term is

$$L \cdot 4^{\frac{n-1}{2}} \left(\frac{n-1}{2}\right)! x^{\frac{n-1}{2}}.$$

Now this term is the square root of the last term of $[F_1(x)]^2$ which is the result of dividing the second member of (1) by $-1+x$. The latter term may be written in the form

$$\frac{(2n-2)(2n-4) \dots (n+1)(n-1) \dots 2}{(n-1)!} x^{n-1},$$

$$= 2^{n-1} x^{n-1},$$

the square root of which is $\pm 2^{\frac{n-1}{2}} x^{\frac{n-1}{2}}$. Therefore

$$L \cdot 4^{\frac{n-1}{2}} \left(\frac{n-1}{2}\right)! x^{\frac{n-1}{2}} = \pm 2^{\frac{n-1}{2}} x^{\frac{n-1}{2}},$$

whence

$$L = \pm \frac{1}{2^{\frac{n-1}{2}} \left(\frac{n-1}{2}\right)!},$$

which gives the value of each coefficient, since any term may be made the last term in one or other of the expressions $F_1(x)$ and $F_3(x)$. To determine the signs of the terms, observe that the coefficient of the term preceding the last in $[F_1(x)]^2$ is the same as that of the last term, and is therefore positive; hence the last two terms of $F_1(x)$ or $F_3(x)$ have the same sign. Then since $F_1(x)$ contains an odd number of terms A and B must have the same sign, and so must C and D , E and F , etc., and since $F_3(x)$ contains an even number of terms, 1 and $-A$, B and $-C$, etc., must have the same

sign. Thus we have when $n = 4n' + 1$

$$F_1(x) = 1 - \frac{n-1}{2}x - \frac{(n-1)(n+3)}{2^2 \cdot 2!}x^2 + \frac{(n-1)(n+3)(n-5)}{2^3 \cdot 3!}x^3 + \dots$$

and when $n = 4n' + 3$

$$F_3(x) = 1 + \frac{n+1}{2}x - \frac{(n+1)(n-3)}{2^2 \cdot 2!}x^2 - \frac{(n+1)(n-3)(n+5)}{2^3 \cdot 3!}x^3 + \dots$$

Eq. (2) takes different forms when n is of the form $4n' + 2$, and when it is of the form $4n'$. In the first case, all the pairs of equal roots occur in couples having numerically equal positive and negative values, so that the equation may be written in the form $(x^2 - 1)[F_2(x^2)]^2 = 0$; but in the second case there is a single pair of zero roots, so that the form of the equation is $x^2(x^2 - 1)[F_4(x^2)]^2 = 0$.

When $n = 4n' + 2$, eq. (2) becomes

$$0 = -1 + \frac{n^2}{2 \cdot 2}x^2 - \frac{n^2(n^2 - 2^2)}{2 \cdot 4!}x^4 + \dots + \frac{n^2(n^2 - 2^2) \dots [n^2 - (n-2)^2]}{2 \cdot n!}x^n$$

in which n admits of the values $\pm 2, \pm 6, \pm 10$, etc. The first term of $F_2(x^2)$ is unity; and, since the expression must reduce to its first term when $n = \pm 2$, all the terms beyond the first must contain $n^2 - 2^2$ as a factor. In like manner, all the terms beyond the second must contain $n^2 - 6^2$, etc. Thus we may assume

$$F_2(x^2) = 1 + A(n^2 - 2^2)x^2 + B(n^2 - 2^2)(n^2 - 6^2)x^4 + \dots + L(n^2 - 2^2) \dots \times [n^2 - (n-4)^2]x^{\frac{n-2}{2}},$$

in which A, B , etc., have numerical values. The last term may be written in the form

$$L \cdot (2n-4)(2n-8) \dots (n+2)(n-2) \dots 4x^{\frac{n-2}{2}} = L \cdot 4^{\frac{n-2}{2}} \left(\frac{n-2}{2}\right)! x^{\frac{n-2}{2}}.$$

But the last term of the quotient, when the above form of eq. (2) is divided by $-1 + x^2$, is

$$\frac{(2n-2)(2n-4) \dots (n+2)n(n-2) \dots 2}{2 \cdot (n-1)!} x^{n-2} = 2^{n-2} x^{n-2},$$

the square root of which is $\pm 2^{\frac{n-2}{2}} x^{\frac{n-2}{2}}$. Hence

$$L \cdot 4^{\frac{n-2}{2}} \left(\frac{n-2}{2}\right)! x^{\frac{n-2}{2}} = \pm 2^{\frac{n-2}{2}} x^{\frac{n-2}{2}},$$

and

$$L = \pm \frac{1}{2^{\frac{n-2}{2}} \left(\frac{n-2}{2}\right)!},$$

which gives the form of the numerical factors. It is easily shown that the last term but one of $F_2(x^2)$ is negative, hence the last two terms of $F_2(x^2)$ are always of opposite signs, and we have

$$F_2(x^2) = 1 - \frac{n^2-2^2}{2^2 \cdot 2!} x^2 + \frac{(n^2-2^2)(n^2-6^2)}{2^4 \cdot 4!} x^4 - \dots$$

When n is of the form $4n'$, eq. (2) becomes

$$0 = -x^2 + \frac{n^2-2^2}{3 \cdot 4} x^4 - \frac{(n^2-2^2)(n^2-4^2)}{3 \cdot 4 \cdot 5 \cdot 6} x^6 + \dots + \frac{2(n^2-2^2) \dots [n^2-(n-2)^2]}{n!} x^n,$$

in which n admits of the values $\pm 4, \pm 8$, etc., the form of the equation being $x^2(x^2-1)[F_4(x^2)]^2 = 0$, so that the degree of each term of $F_4(x^2)$ in n will still be the same as the degree in x . The first term of $F_4(x^2)$ is unity, and it may be assumed in the form

$$F_4(x^2) = 1 + A(n^2-4^2)x^2 + B(n^2-4^2)(n^2-8^2)x^4 + \dots + L(n^2-4^2) \dots \times [n^2-(n-4)^2] x^{\frac{n-4}{2}},$$

A, B , etc., having numerical values. In this case, the last term is

$$L(2n-4)(2n-8) \dots (n+4)(n-4) \dots 4 x^{\frac{n-4}{2}} = L \frac{4^{\frac{n-2}{2}} \left(\frac{n-2}{2}\right)!}{n} x^{\frac{n-4}{2}}.$$

But the last term of $[F_4(x)]^2$ may be written

$$\begin{aligned} & \frac{2(2n-2)(2n-4) \dots (n+2)n(n-2) \dots 2}{n \cdot n!} x^{n-4} \\ &= \frac{2^n(n-1)!}{n^2(n-1)!} x^{n-4} = \frac{2^n}{n^2} x^{n-4}, \end{aligned}$$

the square root of which is $\pm \frac{2^{\frac{1}{2}n}}{n} x^{\frac{n-4}{2}}$. Hence

$$L \frac{4^{\frac{n-2}{2}} \left(\frac{n-2}{2}\right)!}{n} x^{\frac{n-4}{2}} = \pm \frac{2^{\frac{1}{2}n}}{n} x^{\frac{n-4}{2}}, \text{ and } L = \pm \frac{1}{2^{\frac{n-4}{2}} \left(\frac{n-2}{2}\right)!}.$$

The terms are of alternate signs, for the reason already given in the case of F_2 , hence we have

$$F_4(x^2) = 1 - \frac{n^2-4^2}{2^2 \cdot 3!} x^2 + \frac{(n^2-4^2)(n^2-8^2)}{2^4 \cdot 5!} x^4 - \dots$$

Of equation (3) we can only infer, what is obvious on inspection, that it takes the form $xF(x^2) = 0$; but eq. (4) takes the form $x^2[F(x^2)]^2 = 0$ when n has the form $4n'+2$, and the form $x^2(x^2-1)[F(x^2)]^2 = 0$, when n has the form $4n'$. In the latter case, the equation becomes identical with eq. (2) in the like case, and in the former case, we find by the method pursued above

$$F(x^2) = 1 - \frac{n^2-2^2}{2^2 \cdot 3!} x^2 + \frac{(n^2-2^2)(n^2-6^2)}{2^4 \cdot 5!} x^4 - \dots$$

ANSWER TO QUERY AT PAGE 143, VOL. V.

BY GEORGE EASTWOOD, SAXONVILLE, MASS.

Analysis.—Suppose the construction effected, and that the required circles O, O', O'' , are drawn so as to touch each other and the given circles Q and Q' , as in the annexed diagram.

Join the centers O and O'' ; O' and O'' ; O and Q' ; and O' and Q' . Produce OO'' to L , and OQ' to H , and upon AD demit the perpendiculars $O''K$ and $Q'M$.

By a property of the Arbelos (See Leybourn's Math. Rep., pp. 207–209), $O''K = 2FO''$, $Q'M = 4EQ'$, etc. [See ANALYST, p. 171, Vol. I.]

From the triangle $O'O''O$, we have
 $(O'F + FO'')^2 = O'O^2 + O''O^2 + 2O'O.OK$

$$\begin{aligned} &= (AO - AO')^2 + (OL - O''L)^2 + 2(AO - AO')OK \\ &= (AO - AO')^2 + (AO - FO'')^2 + 2(AO - AO')OK; \end{aligned}$$

whence
$$OK = \frac{(AO + AO')FO''}{AO - AO'} - AO.$$

Make $AO = R$, $AO' = R'$, $BQ = R''$, $FO'' = r$, and $EQ' = r'$; then, since $R = R' + R''$, we have

$$R + R' = 2R' + R'', \quad R - R' = R'',$$

and therefore
$$OK = \frac{(2R' + R'')r}{R''} - (R' + R''). \quad (A)$$

But $OK^2 = OO''^2 - O''K^2 = OO''^2 - (2FO'')^2 = (R' + R'' - r)^2 - 4r^2. \quad (B)$

Putting the square of (A) in (B), we deduce

$$r = \frac{(R' + R'')R'R''}{R'^2 + (R' + R'')R'}. \quad (C)$$

By a similar process we deduce

$$r' = \frac{(R' + R'')R'R''}{4R'^2 + (R' + R'')R'}. \quad (D)$$

By the question, $r' = EQ' = Q'H$, is a given line. Therefore, solving the symbolical equation (D) for R' , we readily find

$$R' = \frac{1}{2}R''[\sqrt{(R'' + 15r') - 1}],$$

$$R = \frac{1}{2}R''[\sqrt{(R'' + 15r') + 1}],$$

$$r = \frac{R''(R'' + 15r' - 1)}{R'' + 15r' + 3}.$$



Furthermore, adding AO to OK we find

$$AK = \frac{(AO + AO')FO''}{AO - AO'} = \frac{(2R' + R'')r}{R''} = \frac{R''(R'' + 15r')^{\frac{1}{2}}(R'' + 15r' - 1)}{R'' + 15r' + 3},$$

and, similarly,

$$AM = \frac{(AO + AO')EQ'}{AO - AO'} = \frac{(2R' + R'')r'}{R''} = r'\sqrt{(R'' + 15r')}. \quad \text{Hence this}$$

Construction.—On the indefinite straight line AX , set off

$$AB = 2AO' = \frac{1}{2}R''[\sqrt{(R'' + 15r')} - 1],$$

$$AD = 2AO = \frac{1}{2}R''[\sqrt{(R'' + 15r')} + 1],$$

$$BQ = R''.$$

Bisect AB in O' , and AD in O : then, with centers O' , O and Q , and with radii AO' , AO and BQ , describe the circles AEB , $AHLD$ and BGD . These will obviously touch each other, two and two.

Again, on AD set off

$$AK = \frac{R''\sqrt{(R'' + 15r')}(R'' + 15r' - 1)}{R'' + 15r' + 3},$$

and

$$AM = r'\sqrt{(R'' + 15r')}. \quad \text{At } K, \text{ erect the perpend. } KO'' \\ = \frac{2R''(R'' + 15r' - 1)}{R'' + 15r' + 3},$$

and at M erect the perpendicular $MQ' = 4r'$. Finally, with centers O'' and Q' , and radii $FO'' = R''(R'' + 15r' - 1) \div (R'' + 15r' + 3)$ and $EQ' = 4r'$, describe the circles FIL , EIH and it is done.

[The foregoing solution, it will be seen, applies only to a special case of the question quoted at p. 143, Vol. V, as the relative position of the given circles cannot be arbitrarily taken. No general solution has been rec'd.]

NOTE ON THE DIFFERENTIATION OF EXPONENTIAL AND TRIGONOMETRICAL FUNCTIONS.

BY ARTEMAS MARTIN, M. A., ERIE, PA.

THE following methods may not be new, but I have not met with them in any of the many works on the calculus to which I have access.

1. Differentiate $y = a^x$.

We have, by the Exponential Theorem,

$$y = a^x = 1 + x \log a + \frac{x^2(\log a)^2}{1.2} + \frac{x^3(\log a)^3}{1.2.3} + \text{etc};$$

$$\begin{aligned}\therefore dy &= d(a^x) = (\log a)dx + x(\log a)^2 dx + \frac{x^2(\log a)^3 dx}{1 \cdot 2} + \text{etc.}, \\ &= \log a \left[1 + x \log a + \frac{x^2(\log a)^2}{1 \cdot 2} + \frac{x^3(\log a)^3}{1 \cdot 2 \cdot 3} + \text{etc.} \right] dx, \\ &= a^x \log a dx.\end{aligned}$$

2. Differentiate $y = \sin x$.

We have, by Analytical Trigonometry,

$$\begin{aligned}y = \sin x &= x - \frac{x^3}{1 \cdot 2 \cdot 3} + \frac{x^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} - \frac{x^7}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} + \text{etc.}, \\ \cos x &= 1 - \frac{x^2}{1 \cdot 2} + \frac{x^4}{1 \cdot 2 \cdot 3 \cdot 4} - \frac{x^6}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} + \text{etc.}; \\ \therefore dy &= d(\sin x) = dx - \frac{x^2 dx}{1 \cdot 2} + \frac{x^4 dx}{1 \cdot 2 \cdot 3 \cdot 4} - \frac{x^6 dx}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} + \text{etc.}, \\ &= \left[1 - \frac{x^2}{1 \cdot 2} + \frac{x^4}{1 \cdot 2 \cdot 3 \cdot 4} - \frac{x^6}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} + \text{etc.} \right] dx, \\ &= \cos x dx.\end{aligned}$$

EVALUATION OF THE EULERIAN INTEGRAL $u = \int_0^{\frac{\pi}{2}} \log(\sin x) dx$, By
PROF. E. J. EDMUNDS, NEW ORLEANS, LA.—We have

$$u = \lim_{n \rightarrow \infty} \frac{\pi}{2n} \left[\log \sin \left(\frac{1}{n} \cdot \frac{\pi}{2} \right) + \log \sin \left(\frac{2}{n} \cdot \frac{\pi}{2} \right) - \dots + \log \sin \left(\frac{n-1}{n} \cdot \frac{\pi}{2} \right) \right];$$

hence,

$$u = \lim_{n \rightarrow \infty} \frac{\pi}{2n} \cdot \log \left[\sin \left(\frac{1}{n} \cdot \frac{\pi}{2} \right) \sin \left(\frac{2}{n} \cdot \frac{\pi}{2} \right) \dots \sin \left(\frac{n-1}{n} \cdot \frac{\pi}{2} \right) \right],$$

n increasing to infinity.

But by Cotes' Theorem we have

$$\frac{z^{2n}-1}{z^n-1} = \left(z^2 - 2z \cos \frac{\pi}{n} + 1 \right) \left(z^2 - 2z \cos \frac{2\pi}{n} + 1 \right) \dots \left(z^2 - 2z \cos \frac{(n-1)\pi}{n} + 1 \right).$$

For $z = 1$, this equation becomes,

$$n = 2^{n-1} \sin^2 \left(\frac{1}{n} \cdot \frac{\pi}{2} \right) \sin^2 \left(\frac{2}{n} \cdot \frac{\pi}{2} \right) \dots \sin^2 \left(\frac{n-1}{n} \cdot \frac{\pi}{2} \right); \text{ therefore}$$

$$\frac{\log n - 2(n-1) \log 2}{2} = \log \sin \left(\frac{1}{n} \cdot \frac{\pi}{2} \right) \dots \log \sin \left(\frac{n-1}{n} \cdot \frac{\pi}{2} \right).$$

Multiplying both members by $\pi \div 2n$ and passing to limits we finally obtain

$$u = \frac{1}{2} \pi \cdot \log \frac{1}{2},$$

which is the value of the integral sought.

SOLUTIONS OF PROBLEMS IN NUMBER SIX, VOL. V.

SOLUTIONS of problems in No. 6, Vol. V, have been received as follows:

From R. J. Adcock, 239; Marcus Baker, 234; Prof. P. E. Chase, 239; George Eastwood, 232; F. H. Hoffstrom, 234; W. E. Heal, 234; Prof. W. W. Johnson, 236, 240; Chas. H. Kummell, 237, 239; Prof. Joseph H. Kershner, 232, 233, 234, 235, 237, 239; Prof. D. J. Mc Adam, 234, 237; Artemas Martin, 238; Prof. Orson Pratt, Sen., 231; P. Richardson, 234; E. B. Seitz, 232, 234, 236, 237, 238, 240; Prof. J. Scheffer, 232, 233, 234, 237; Prof. M. C. Stevens, 234.

231. "What is the sum expressed in terms of m , of the values of all the determinants, from the second to the n th orders inclusive, which can be formed from the m -gonal series of numbers, represented by $1, a_2, a_3, a_4, \dots, a_m$, the arrangement of the constituents of the respective determinants being after the following form:

$$\begin{vmatrix} 1, & a_2 \\ a_3, & a_4 \end{vmatrix} + \begin{vmatrix} 1, & a_2, & a_3 \\ a_4, & a_5, & a_6 \\ a_7, & a_8, & a_9 \end{vmatrix} + \begin{vmatrix} 1, & a_2, & a_3, & a_4, \\ a_5, & a_6, & a_7, & a_8, \\ a_9, & a_{10}, & a_{11}, & a_{12}, \\ a_{13}, & a_{14}, & a_{15}, & a_{16}, \end{vmatrix} + \&c."$$

SOLUTION BY PROF. ORSON PRATT, SEN., SALT LAKE CITY, UTAH.

The value of the determinant of the second order, in terms of m , is

$$\begin{vmatrix} 1, & m \\ 3m-3, & 6m-8 \end{vmatrix} = -3m^2 + 9m - 8.$$

That of the third order is

$$\begin{vmatrix} 1, & m, & 3m-3 \\ 6m-8, & 10m-15, & 15m-24 \\ 21m-35, & 28m-48, & 36m-63 \end{vmatrix} = -3^3(m-2)^3.$$

All determinants of higher orders vanish; therefore the sum of these two values is the solution required.

232. "From two given points on a circle to draw straight lines through a point C in the circumference so they shall form with a line MN , given in position, a triangle CMN of given area."

SOLUTION BY E. B. SEITZ, GREENVILLE, OHIO.

Let A and B be the two given points, O the center of the circle. Draw OH , AS , BR and CK perpendicular to MN .

Let $OH = a$, $OA = b$, $OB = c$, $OC = r$, $\angle AOH = \beta$, $\angle BOH = \gamma$, $\angle COH = \theta$. Then is $AS = a - b \cos \beta$, $BR = a - c \cos \gamma$, $CK = a - r \cos \theta$, $SH = b \sin \beta$, $RH = c \sin \gamma$, $KH = r \sin \theta$.

By similar triangles we have

$$AS - CK : KH + SH :: CK : MK = \frac{(a - r \cos \theta)(r \sin \theta + b \sin \beta)}{r \cos \theta - b \cos \beta},$$

$$BR - CK : KH - RH :: CK : NK = \frac{(a - r \cos \theta)(r \sin \theta - c \sin \gamma)}{r \cos \theta - c \cos \gamma}.$$

$$\therefore MN = \frac{(a - r \cos \theta)(r \sin \theta + b \sin \beta)}{r \cos \theta - b \cos \beta} - \frac{(a - r \cos \theta)(r \sin \theta - c \sin \gamma)}{r \cos \theta - c \cos \gamma},$$

and area $CMN = \frac{1}{2}MN \times CK$

$$= \frac{(a - r \cos \theta)^2 (r \sin \theta + b \sin \beta)}{2(r \cos \theta - b \cos \beta)} - \frac{(a - r \cos \theta)^2 (r \sin \theta - c \sin \gamma)}{2(r \cos \theta - c \cos \gamma)} = m. \quad (1)$$

Reducing (1), we get an equation of the sixth degree in $\cos \theta$.

[Professors Scheffer and Kershner, and Mr. Eastwood, each locate the points A and B on the *circumference* of the circle, and thereby effect the solution by an equation of the fourth degree; but this is a restriction of the language in which the question is announced, and cannot be regarded as a general solution; for the circumference of a circle is not the circle, but is a *line* which envelopes the circle.—Ed.]

233. "If $ABCD$ be a spherical quadrilateral whose sides AB , DC are produced to meet at P , and AD , BC , to meet at Q , and whose diagonals AC , BD intersect at R , then

$$\sin AB \sin CD \cos P - \sin AD \sin BC \cos Q = \pm \sin AC \sin BD \cos R."$$

SOLUTION BY PROF. J. SCHEFFER, MERCERSBERG, PA.

Denoting, for the sake of brevity, AB , BC , CD , DA , AC , BD , respectively, by a , b , c , d , e , f ; and by A , B , C , D , the four angles of the quadrilateral, we have, in the $\triangle ABC$,

$$\sin a : \sin e = \sin ACB : \sin B,$$

and in $\triangle BCD$, $\sin c : \sin f = \sin DBC : \sin C$; $\therefore$ by multiplying,

$$\sin a \sin c \sin B \sin C = \sin e \sin f \sin ACB \sin DBC. \quad (1)$$



Again, in the triangle ABC , we have

$$\sin b : \sin e = \sin CAB : \sin B,$$

and in the triangle ABD we have

$$\sin d : \sin f = \sin ABD : \sin A. \therefore$$

$$\sin b \sin d \sin A \sin B = \sin e \sin f \sin CAB \times \sin ABD. \quad (2)$$

But, in the triangle BCP ,

$$\sin B \sin C = \frac{\cos P + \cos B \cos C}{\cos b}, \quad (3)$$

$$\text{and, in } \triangle ABQ, \quad \sin A \sin B = \frac{\cos Q + \cos A \cos B}{\cos a}, \quad (4)$$

$$\text{in } \triangle BCR, \quad \sin ACB \sin DBC = \frac{\cos R + \cos ACB \cos DBC}{\cos b}, \quad (5)$$

where $\angle BRC = R$; and, in the triangle ABR we have

$$\sin CAB \sin ABD = \frac{-\cos R + \cos CAB \cos ABD}{\cos a}. \quad (6)$$

Substituting these values in (1) and two, we obtain

$$\sin a \sin c (\cos P + \cos B \cos C) = \sin e \sin f (\cos R + \cos ACB \cos DBC), \quad (7)$$

$$\sin b \sin d (\cos Q + \cos A \cos B) = \sin e \sin f (-\cos R + \cos CAB \cos ABD). \quad (8)$$

Substituting in (8) from (7), we get

$$\begin{aligned} \sin a \sin c \cos P - \sin b \sin d \cos Q &= 2 \sin e \sin f \cos R \\ &+ \sin e \sin f (\cos ACB \cos DBC - \cos CAB \cos DBA) \\ &- \sin a \sin c \cos B \cos C + \sin b \sin d \cos A \cos B. \end{aligned} \quad (9)$$

Substituting in (9)

$$\cos ACB = \frac{\cos a - \cos b \cos e}{\sin b \sin e}, \quad \cos DBC = \frac{\cos e - \cos b \cos f}{\sin b \sin f},$$

$$\cos CAB = \frac{\cos b - \cos a \cos e}{\sin a \sin e}, \quad \cos DBA = \frac{\cos d - \cos a \cos f}{\sin a \sin f};$$

$$\cos A = \frac{\cos f - \cos a \cos d}{\sin a \sin d}, \quad \cos B = \frac{\cos e - \cos a \cos b}{\sin a \sin b},$$

$$\cos C = \frac{\cos f - \cos b \cos e}{\sin b \sin c};$$

and we get, after some transformations that present no difficulty,

$$\sin a \sin c \cos P - \sin b \sin d \cos Q = 2 \sin e \sin f \cos R + \cos a \cos c - \cos b \cos d. \quad (10)$$

But

$$\cos a = \cos AR \cos BR - \sin AR \sin BR \cos R,$$

$$\cos c = \cos DR \cos CR - \sin DR \sin CR \cos R,$$



$$\begin{aligned}\cos b &= \cos BR \cos CR + \sin BR \sin CR \cos R, \\ \cos d &= \cos AR \cos DR + \sin AR \sin DR \cos R.\end{aligned}$$

Consequently

$$\begin{aligned}\cos a \cos c - \cos b \cos d &= -(\sin AR \cos CR + \sin CR \cos AR) \\ &\quad \times (\sin BR \cos DR + \sin DR \cos BR) \cos R \\ &= -\sin (AR + CR) \sin (BR + DR) \cos R \\ &= -\sin e \sin f \cos R.\end{aligned}$$

Therefore, $\sin a \sin c \cos P - \sin b \sin d \cos Q = \pm \sin e \sin f \cos R$,
the double sign referring to the case of R being acute or obtuse.

234. " $ABCD$ is a trapezium, $AB = a$, $CD = b$, $AD = c$; angle ABC is a right angle, and E is a point on AD such that angle BCE is a right angle and $CE = CD = b$."

It is required to find BC , DE and AE .

SOLUTION BY PROF. M. C. STEVENS, LA FAYETTE, INDIANA.

Construction.—On AF take $AG = a - b$, $GF = 2b$ and $AD = c$. On AF and AD describe semicircles. Draw GH perpendicular to AF meeting the circumference on AF in H . Draw HI parallel to AF meeting the semicircle on AD in I . Let fall IE perpendicular to AD . On AE construct the triangle AEL right angled at L , with $AL = AG$. Complete the parallelogram $LBCE$, and join CD ; then is $ABCD$ the trapezium required.



Demonstration.—Bisect ED in K and join CK . By construction, $AG \times GF = GH^2 = EI^2 = AE \times ED$; $\therefore GF : ED :: AE : AG$; or $\frac{1}{2}GF : \frac{1}{2}ED :: AE : AG$. Since $AB = a$ and $AL = AG = a - b$, $\therefore LB = EC = b$. Also $\frac{1}{2}GF = b$. Whence the above proportion may be written $CE : EK :: AE : AL$. But the angle $CEK = LAE$, $\therefore$ the triangle CEK is similar to EAL . Hence, the angle CKE is a right angle and $CD = CE = b$.

Calculation.— $OE = \sqrt{[\frac{1}{4}c^2 - 2b(a - b)]}$, $\therefore AE = \frac{1}{2}c \pm \sqrt{[\frac{1}{4}c^2 - 2b(a - b)]}$ and $DE = \frac{1}{2}c \mp \sqrt{[\frac{1}{4}c^2 - 2b(a - b)]}$; $BC = EL$

$$= \sqrt{\left\{ \left(\frac{1}{2}c \pm \sqrt{[\frac{1}{4}c^2 - 2b(a - b)]} \right)^2 - (a - b)^2 \right\}}.$$

235. "Find the condition that the general equation of the n th degree may have q equal roots."

SOLUTION BY PROF. J. H. KERSHNER, MERCERSBERG, PA.

The absolute term is the product of all the roots with their signs changed, the last coefficient is the sum of the products of every $n-1$ roots with their signs changed, etc. Therefore, if a be one of the q equal roots, the absolute term must be divisible by a^q , the last coefficient must be divisible by a^{q-1} , etc. The form of the general equation is as follows:

$$x^q + p_1 x^{q-1} + \dots m_1 a x^{q-1} + m_2 a^2 x^{q-2} + \dots m_{q-1} a^{q-1} x + m_q a^q = 0.$$

236. "If $A_1 A_2, B_1 B_2, C_1 C_2$ are three lines which meet in a point O , then there are four different ways in which the points $A_1, B_1, C_1, A_2, B_2, C_2$, can be combined into two homologous triangles, and for each combination there is a different axis of homology. Show that these four axes form a complete quadrilateral whose diagonals intersect each other on the lines $A_1 A_2, B_1 B_2, C_1 C_2$."

SOLUTION BY PROF. JOHNSON.

The fundamental property of homologous triangles is demonstrated in my solution of problem 186, ANALYST, Vol. V, No. 1, by a method which is readily applied to the present problem. Thus, let the line OA_1, OB_1, OC_1 be at first supposed not in the same plane; then a being the intersection of $B_1 C_1$ with $B_2 C_2$ is at once in the plane $A_1 B_1 C_1$ and in the plane $A_2 B_2 C_2$, hence it is in the intersection of these planes; for a similar reason b and c are in the intersection of the same planes, therefore abc is a straight line, and in the projection of the fig-



ure upon any plane abc is likewise a straight line. It may be remarked that the figure in space is simply that formed by the intersections of five planes, and hence may be called a *complete pentahedron*. The full and broken lines in the figure represent a complete pentahedron; it has ten edges and ten vertices each of which is the intersection of three planes which cut

out two triangles on the other two planes, and in any projection of the figure, the triangles become homologous triangles, the point in which the three planes meet becomes the centre, and the intersection of the planes of the triangles the axis of homology.

Now let c' be the intersection of A_1B_2 with A_2B_1 and let a' and b' be determined in like manner; then will $a'b'c$, $a'bc'$ and $ab'c'$ be straight lines, the six points $a b c a' b' c'$ are in a plane and form the vertices of a complete quadrilateral of which aa' , bb' and cc' are the diagonals. But cc' is in the plane OA_1B_1 and therefore intersects OA_1 , and bb' is in the plane OA_1C_1 and therefore intersects OA_1 , hence cc' and bb' meet in the point in which the plane of the complete quadrilateral cuts the line A_1A_2 . Hence also in the plane figure which is the projection of the figure in space, the diagonals bb' and cc' meet on the line A_1A_2 .

237. "Prove that

$$\int_0^\pi \frac{\cos \frac{1}{2}(n-1)(\theta+\pi) \sin \frac{1}{2}[n(\theta+\pi)]}{\sin \frac{1}{2}(\theta+\pi)} d\theta = \pi."$$

SOLUTION BY PROF. D. J. MC ADAM, WASHINGTON, PA.

In Todhunter's Plane Trigonometry, page 245, 4th Ed., we have

$$\frac{\cos [a + \frac{1}{2}(n-1)(\beta+\pi)] \sin \frac{1}{2}n(\beta+\pi)}{\sin \frac{1}{2}(\beta+\pi)} = \cos a - \cos (a+\beta) + \cos (a+2\beta) \\ - \dots + (-)^{n-1} \cos [a + (n-1)\beta].$$

Let $a = 0$, $\beta = \theta$; then

$$\frac{\cos [\frac{1}{2}(n-1)(\theta+\pi)] \sin \frac{1}{2}n(\theta+\pi)}{\sin \frac{1}{2}(\theta+\pi)} = \cos 0 - \cos \theta + \cos 2\theta - \&c. \\ \therefore \int_0^\pi \frac{\cos [\frac{1}{2}(n-1)(\theta+\pi)] \sin \frac{1}{2}n(\theta+\pi)}{\sin \frac{1}{2}(\theta+\pi)} d\theta \\ = \int_0^\pi (\cos 0 - \cos \theta + \cos 2\theta - \&c.) d\theta \\ = \left[\theta - \sin \theta + \frac{1}{2} \sin 2\theta - \&c. \right]_0^\pi \\ = \pi.$$

[The solutions of problems 238, 239 and 240, will be published in No. 2 as follows:—Solution of 238, by Artemas Martin, 239, a solution by Chas. H. Kummell and also a solution by R. J. Adcock, 240, solution by E. B. Seitz..]

PROBLEMS.

241. BY P. RICHARDSON.—From any point within a parallelogram to draw three straight lines which shall divide the parallelogram into three equal parts.

242. BY G. J. HENDRICKS, EDGERTON, OHIO. — The difference between the areas of the inscribed and circumscribed circles of an equilateral triangle is a . Required the side of the triangle.

243. BY PROF. KERSHNER. — Adapt $\sqrt{a^2 + b^2}$, and $\sqrt{(a + b) + (a - b)}$, to computation by the usual logarithmic tables.

244. BY MARCUS BAKER.—Prove that

$$1^9 + 2^9 + 3^9 + 4^9 + \dots + n^9 = \frac{1}{10} [16s^5 - 20s^4 + 12s^3 - 3s^2],$$

where $s = 1 + 2 + 3 + 4 + \dots + n$.

245. BY PROF. SCHEFFER.—The three transversals AA' , BB' , CC' of triangle ABC intersect each other in one point, cutting the sides of the triangle at A' , B' , C' . If the angles BAA' , ACC' and CBB' are equal, compute the transversals AA' , BB' , CC' by means of the sides a , b , c , of the triangle.

246. BY ARTEMAS MARTIN, M. A.—Show that the sum of five consecutive integral square numbers cannot be a square.

247. E. B. SEITZ. — Three points are taken at random on the surface of a circle; find the chance that the triangle formed by joining them is acute.

248. BY PROF. HALL.—If in the plane of xy the directions of the forces P and P' make with the axis of x the angles a and a' , and the direction of their resultant the angle A , and if we denote by p , p' and r the perpendiculars from the origin on these directions, we have the relation,

$$r \sin (a' - a) + p \sin (A - a') + p' \sin (a - A) = 0.$$

249. BY REQUEST. — Representing the amount of an insolvent estate by d , A has a claim of a , B a claim of b and C a claim of c against it. A's claim has a legal priority over B's, B's a priority over C's and C's a priority over A's. How should the estate be divided? Consider the several cases when the estate is insufficient to pay any two of the claims, when it is sufficient to pay any two, etc.

NOTES OF TRAVEL, BY THE EDITOR.—Though we do not usually occupy the pages of the ANALYST with notes of travel, yet as many of our readers will, at some time, either for health or recreation, visit the sublime and beautiful scenery and salubrious climate of Colorado, we insert a few notes of a hasty trip to Colorado Springs and vicinity, made during the latter part of Oct., '78.

We spent the summer of 1861, as U. S. Deputy Surveyor, within sight of Pike's Peak and therefore know something by experience of the congeniality of the climate and purity of the atmosphere in that part of Colorado, but of that we do not now propose to speak. We again visited Denver and Georgetown, by way of the U. P. R. R., in the summer of 1875, and intended to make a third trip to the State during July last, to witness the total eclipse of the sun from the summit of Pike's Peak, but the state of our health not permitting at that time, we consummated our third visit to the State, during the latter part of Oct. last, in a hasty trip to Colorado Springs and vicinity, by way of the Atchison, Topeka and Santa Fe R. R.; and it is for the purpose of calling the attention of our readers to this route that these notes of the route through Southern Kansas are inserted.

Passengers arrive at Topeka, by trains from Atchison and Kansas City, at about half-past 2 o'clock in the afternoon, and about 30 minutes later, the daily train starts from Topeka to Pueblo, on the Atchison, Topeka and Santa Fe R. R. The train that leaves Topeka contains, besides the cars that go through to Pueblo, cars for the several branches that diverge to the south at Florence, Newton, &c.

No finer land for the production of wheat, corn, grass, &c., and fruits of all varieties suited to this latitude, can be found anywhere in America, than is seen in this part of southern Kansas. Large tracts of winter wheat, already (Oct. 31st) quite green, are seen as far as 400 miles west of the Missouri river.

The time occupied in going from Topeka to Pueblo is about 24 hours, so that passengers arrive at Pueblo about 3 o'clock in the afternoon of the next day after leaving Topeka, and passengers for Colorado Springs have their choice of the evening train on the Denver and Rio Grand R. R., or the 6 o'clock train next morning; and should they choose the morning train, and if the sky is clear, as it generally is in that country, they will have a magnificent view of mountain scenery during a two hours ride, up the valley of the Fountain que Bouille, to Colorado Springs.

From Colorado Springs, an hour's ride in a stage coach, at a cost of 75 cents, places the traveler at Manitou, where he is in the immediate vicinity of the soda springs, and the "Garden of the Gods".

ERRATA.

On page 5, line 1, substitute y_0 for y_1 .

" " 6, " 11, for ψ , read ψ' .

" " 7, last line, for $z = \infty$, read, $z = a$.

" " 8, line 8, for e^{mi} , read, e^{bi} .

" " 8, " 11, for $+\infty$, read, $\pm\infty$.

" " 8, " 15, for $(bi)^{m-1}$, read, $(bi)^{2n-1}$.

" " 10, " 8, from bottom, insert 2π before the radical in the denom.

" " 11, " 12, for $\frac{\sin \theta}{\sin \theta_1}$, read, $\frac{\sin \theta_1}{\sin \theta}$.

" " 12, " 17, insert parenthesis before the first minus sign.

" " 13, " 9, for $n^2 \cos^2 \theta$, read, $n^2 \sin^2 \theta$.

" " 14, " 7, for $\lambda_2 \cos \zeta \sin \zeta_2$, read, $\lambda_2 \cos \zeta_1 \sin \zeta_2$.

THE ANALYST.

VOL. VI.

MARCH, 1879.

No. 2.

STELLAR PARALLAX.

BY PROF. ASAPH HALL, NAVAL OBSERVATORY, WASHINGTON, D. C.

(1). THE annual parallax of a star is the angle at the star subtended by the mean distance of the Earth from the Sun.

Imagine a right line drawn from the Earth to a star. As the Earth moves around the Sun this line will generate the surface of a cone having an elliptical base, and an observer on the Earth will always see the star on the surface of this cone. Generally therefore the small annual parallax will cause the star to describe an ellipse on the surface of the heavens. At the pole of the ecliptic this ellipse will be similar to the orbit of the Earth, while if the star is in the plane of the ecliptic it will move backward and forward in a right line. Since the Earth's orbit is nearly circular we see, from geometrical considerations, that the maximum value of the parallax of the star in longitude will occur when the longitudes of the earth and the star differ by a quadrant, and the maximum value of the parallax in latitude will occur when the star is in opposition to the Sun.

Let Δ be the distance of the star from the Sun, and denote by λ and β its longitude and latitude when seen from the Sun. If we take the plane of the ecliptic as the plane of xy the values of the rectangular coordinates are

$$x = \Delta \cos \beta \cos \lambda: \quad y = \Delta \cos \beta \sin \lambda: \quad z = \Delta \sin \beta.$$

Denoting by Δ' , λ' , β' the corresponding quantities seen from the Earth we have

$$x' = \Delta' \cos \beta' \cos \lambda': \quad y' = \Delta' \cos \beta' \sin \lambda': \quad z' = \Delta' \sin \beta'.$$

The ecliptic coordinates of the Sun referred to the Earth are

$$X = R \cos \theta: \quad Y = R \sin \theta: \quad Z = 0,$$

where R is the radius vector of the Earth, and θ is the longitude of the Sun seen from the Earth. We have therefore the equations

$$\left. \begin{aligned} \Delta' \cos \beta' \cos \lambda' &= \Delta \cos \beta \cos \lambda + R \cos \theta, \\ \Delta' \cos \beta' \sin \lambda' &= \Delta \cos \beta \sin \lambda + R \sin \theta, \\ \Delta' \sin \beta' &= \Delta \sin \beta. \end{aligned} \right\} \quad (1)$$

If we multiply the first of these equations by $\sin \lambda$, the second by $\cos \lambda$, and subtract the products we shall have

$$\Delta' \cos \beta' \sin (\lambda' - \lambda) = -R \sin (\lambda - \theta).$$

In the present case we have $\Delta = \Delta'$, and $\beta = \beta'$, very nearly, and $\lambda' - \lambda$ is so small that we may put the arc for its sine. Also if we take the distance of the Earth from the Sun as our unit and denote the annual parallax of the star by π we have,

$$\sin \pi = \frac{1}{\Delta},$$

or in seconds of arc, since the parallax is small,

$$\pi = \frac{206264.8}{\Delta}.$$

Hence we have for the parallax in longitude,

$$\lambda' - \lambda = -\pi.R \sin (\lambda - \theta) \sec \beta.$$

Again if we multiply the first of equations (1) by $\cos \lambda'$, the second by $\sin \lambda'$ and add the products we have

$$\Delta' \cos \beta' = \Delta \cos \beta \cos (\lambda' - \lambda) + R \cos (\lambda' - \theta).$$

If now we put $\cos (\lambda' - \lambda) = 1$, we have from this equation and the third of equations (1)

$$\begin{aligned} \Delta' \cos \beta' &= \Delta \cos \beta + R \cos (\lambda' - \theta), \\ \Delta' \sin \beta' &= \Delta \sin \beta. \end{aligned}$$

Multiplying the first of these equations by $\sin \beta$, the second by $\cos \beta$ and subtracting the products we have as before

$$\beta' - \beta = -\pi.R \cos (\lambda - \theta) \sin \beta.$$

Hence the formulæ for the parallax of a star in longitude and latitude are

$$\left. \begin{aligned} d\lambda &= -\pi.R \sin (\lambda - \theta) \sec \beta, \\ d\beta &= -\pi.R \cos (\lambda - \theta) \sin \beta. \end{aligned} \right\} \quad (2)$$

(2). Since at the present time in all astronomical observations the plane of the equator is taken as the plane of reference, or as that of xy , we need expressions for the parallax in right ascension and declination. We

may obtain these by means of equations (2) and the differential formulæ between the right ascension and declination and the longitude and latitude, but it is better to start directly from the fundamental equations.

Let $\alpha, \delta, \alpha', \delta'$ be the right ascensions and declinations of the star seen from the Sun and the Earth, and denote by ϵ the obliquity of the ecliptic, the values of the coordinates referred to the plane of the equator are

$$\begin{aligned} x &= \Delta \cos \delta \cos \alpha, & y &= \Delta \cos \delta \sin \alpha, & z &= \Delta \sin \delta, \\ x' &= \Delta' \cos \delta' \cos \alpha', & y' &= \Delta' \cos \delta' \sin \alpha', & z' &= \Delta' \sin \delta', \\ X &= R \cos \theta, & Y &= R \sin \theta \cos \epsilon, & Z &= R \sin \theta \sin \epsilon. \end{aligned}$$

Hence we have the equations,

$$\left. \begin{aligned} \Delta' \cos \delta' \cos \alpha' &= \Delta \cos \delta \cos \alpha + R \cos \theta, \\ \Delta' \cos \delta' \sin \alpha' &= \Delta \cos \delta \sin \alpha + R \sin \theta \cos \epsilon, \\ \Delta' \sin \delta' &= \Delta \sin \delta + R \sin \theta \sin \epsilon. \end{aligned} \right\} \quad (3)$$

Multiplying the first of these equations by $\sin \alpha$, the second by $\cos \alpha$, and subtracting the products we have

$$\Delta' \cos \delta' \sin (\alpha' - \alpha) = R (\sin \theta \cos \epsilon \cos \alpha - \cos \theta \sin \alpha).$$

Multiplying the first of equations (3) by $\cos \alpha'$, the second by $\sin \alpha'$ and adding the products we have, if we put $\cos (\alpha' - \alpha) = 1$,

$$\Delta' \cos \delta' = \Delta \cos \delta + R (\cos \theta \cos \alpha' + \sin \theta \cos \epsilon \sin \alpha').$$

If we multiply this equation by $\sin \delta$, and the third equation of (3) by $\cos \delta$, and subtract the products, we shall have

$$\Delta' \sin (\delta' - \delta) = R \sin \theta \sin \epsilon \cos \delta - R (\cos \theta \cos \alpha' \sin \delta + \sin \theta \cos \epsilon \sin \alpha' \sin \delta).$$

Hence writing these in the usual form, and putting α for α' , on the right hand side of the last equation, we have

$$\left. \begin{aligned} \alpha' - \alpha &= -\pi R [\cos \theta \sin \alpha - \sin \theta \cos \epsilon \cos \alpha] \sin \delta, \\ \delta' - \delta &= -\pi R [\cos \epsilon \sin \alpha \sin \delta - \sin \epsilon \cos \delta] \sin \theta - \pi R \cos \alpha \sin \delta \cos \theta. \end{aligned} \right\} \quad (4)$$

If now we put

$$\begin{aligned} a \sin A &= -\cos \epsilon \cos \alpha \\ a \cos A &= -\sin \alpha \\ b \sin B &= \cos \epsilon \sin \alpha \sin \delta - \sin \epsilon \cos \delta \\ b \cos B &= -\cos \alpha \sin \delta, \end{aligned}$$

we have for the values of the parallaxes in right ascension and declination

$$\left. \begin{aligned} d\alpha &= \pi R a \cos (\theta + A) \sec \delta \\ d\delta &= \pi R b \cos (\theta + B). \end{aligned} \right\} \quad (5)$$

The above simple method of determining the parallax of a body by means of its rectangular coordinates referred to the different points appears to have been given first by Lagrange in his memoir on the transit of Venus, June 3, 1769. (*Oeuvres de Lagrange*, Tome II, p. 335.)

If we wish to observe a star which has a large parallax in right ascension we must find the maximum value of a in equations (5), and if we wish a large parallax in declination we have to find the maximum value of b in the same equations. From the values of $a \sin A$, $a \cos A$, $b \sin B$, $b \cos B$ we have

$$\begin{aligned} a^2 &= 1 - \cos^2 \delta \sin^2 \varepsilon, \\ b^2 &= 1 - (\cos \delta \cos \varepsilon + \sin \delta \sin \varepsilon \sin \alpha)^2, \end{aligned}$$

and for the maximum values the last terms of these equations must be zero. Now from the spherical triangle between the pole of the equator, the pole of the ecliptic and the star, we have, if we denote by η the parallactic angle at the star,

$$\left. \begin{aligned} \cos \beta \sin \eta &= \cos \alpha \sin \varepsilon \\ \cos \beta \cos \eta &= \cos \delta \cos \varepsilon + \sin \delta \sin \varepsilon \sin \alpha. \end{aligned} \right\} \quad (6)$$

These equations show that for the two maxima we have

$$\eta = 0; \quad \text{and} \quad \eta = 90^\circ:$$

or in the first case the circles of declination and latitude coincide, and in the second case they are perpendicular to each other. The coefficients a and b have simple geometrical meanings.

(3). The preceding formulæ (5) give the absolute parallax of a star in right ascension and declination, but the determination of the absolute position of a star is a difficult work, and generally in the case of stellar parallax it is better to proceed by differential methods. In this way the star is compared with one or two small stars near it, but which have no physical connection with it, and which may be assumed to be so much more distant that their parallaxes are insensible. This method gives us indeed only the relative parallax of the bright star, but it has the great advantage of permitting accurate differential observations, which will be very nearly independent of the position of the instrument. We may observe either differences of right ascension or of declination, or both, but generally it is better to observe the polar coordinates, that is the angle of position and the distance. To be adapted to this case the formulæ (5) need a transformation.

If we denote by s the distance, and by p the angle of position, and assume that s is so small that we may write the arc for its sine, we have

$$\begin{aligned}s \sin p &= (\alpha - \alpha') \cos \delta \\ s \cos p &= \delta - \delta',\end{aligned}$$

and by differentiating

$$\begin{aligned}ds &= \sin p \cos \delta \cdot d(\alpha - \alpha') + \cos p \cdot d(\delta - \delta') \\ s \cdot dp &= \cos p \cos \delta \cdot d(\alpha - \alpha') - \sin p \cdot d(\delta - \delta').\end{aligned}$$

The values of the differentials $d(\alpha - \alpha')$ and $d(\delta - \delta')$ are the values of $-(\alpha' - \alpha)$ and $-(\delta' - \delta)$, given by equations (4). Substituting these values we have

$$ds = \pi R \cdot \left[\cos \theta \sin \alpha \sin p - \sin \theta \cos \varepsilon \cos \alpha \sin p + \cos \varepsilon \sin \alpha \sin \delta \sin \theta \cos p - \sin \varepsilon \cos \delta \sin \theta \cos p + \cos \theta \cos \alpha \sin \delta \cos p \right],$$

$$dp = \frac{\pi R}{s} \cdot \left[\cos \theta \sin \alpha \cos p - \sin \theta \cos \varepsilon \cos \alpha \cos p - \cos \varepsilon \sin \alpha \sin \delta \sin \theta \sin p + \sin \varepsilon \cos \delta \sin \theta \sin p - \cos \theta \cos \alpha \sin \delta \sin p \right].$$

If now we introduce the auxiliary quantities,

$$m \sin M = -\cos \varepsilon \cos \alpha \sin p + \cos \varepsilon \sin \alpha \sin \delta \cos p - \sin \varepsilon \cos \delta \cos p$$

$$m \cos M = \sin \delta \cos \alpha \cos p + \sin \alpha \sin p$$

$$m' \sin M' = \frac{1}{s} \left[-\cos \varepsilon \cos \alpha \cos p - \cos \varepsilon \sin \alpha \sin \delta \sin p + \sin \varepsilon \cos \delta \sin p \right]$$

$$m' \cos M' = \frac{1}{s} \left[-\sin \delta \cos \alpha \sin p + \sin \alpha \cos p \right],$$

we have the following formulæ for computing the effect of the parallax in distance and angle of position,

$$\left. \begin{aligned}ds &= \pi R \cdot m \cos(\theta - M) \\ dp &= \pi R \cdot m' \cos(\theta - M').\end{aligned} \right\} \quad (7)$$

These are the formulæ used by Bessel in his determination of the parallax of 61 Cygni. (*Astronomische Nachrichten*, Band 16, p. 83.)

The equations (6) give

$$\cos \beta \cos(\gamma + p) = \cos \varepsilon \cos \delta \cos p + \sin \varepsilon \sin \delta \sin \alpha \cos p - \cos \alpha \sin \varepsilon \sin p,$$

$$\cos \beta \sin(\gamma + p) = \cos \varepsilon \cos \delta \sin p + \sin \varepsilon \sin \delta \sin \alpha \sin p + \cos \alpha \sin \varepsilon \cos p.$$

If now we square and add the values of $m \sin M$, $m \cos M$, $m' \sin M'$, and $m' \cos M'$ we shall find

$$\begin{aligned}m^2 &= 1 - \cos^2 \beta \cos^2(\gamma + p), \\ s^2 m'^2 &= 1 - \cos^2 \beta \sin^2(\gamma + p).\end{aligned}$$

These equations show that the coefficients m and m' have their maximum values, and that the effect of the parallax on the distance and in the angle is greatest when the great circle joining the stars is perpendicular to the circle of latitude or coincides with it.

In order to render the formulæ for the auxiliaries m, m', M, M' convenient for calculation we may put with Brünnow,

$$f \sin F = \sin \delta \cos \alpha, \quad g \sin G = \sin \delta \sin \alpha, \quad h \sin H = g \sin(G + \epsilon),$$

$$f \cos F = \sin \alpha, \quad g \cos G = -\cos \delta, \quad h \cos H = -\cos \alpha \cos \epsilon,$$

and we shall have,

$$m \sin M = h \sin(H + p), \quad m' \sin M' = \frac{h}{s} \cos(H + p),$$

$$m \cos M = f \sin(F + p), \quad m' \cos M' = \frac{f}{s} \cos(F + p).$$

By means of these formulæ the coefficients are easily computed.

The observations should extend over a year, so that the coefficients of the parallax may attain their maximum and minimum values with opposite signs. The observations must be corrected for refraction, which in the case of differential observations is a simple work, and then reduced to a chosen epoch by applying the necessary corrections for precession, nutation, aberration and proper motion. This reduction is also easily made in the case of differential observations. The observations are then ready to form the equations of condition.

If we denote by p_0 and s_0 assumed values of the angle of position and distance at the epoch t_0 ; by dp_0 and ds_0 the corrections to these assumed values; by dp' and ds' the corrections to the assumed annual variations of p and s produced by the proper motion, and designate by a and b the coefficients for the parallax, each complete observation gives two equations of condition of the form,

$$dp_0 + (t - t_0).dp' + a\pi + n = 0,$$

$$ds_0 + (t - t_0).ds' + b\pi + n' = 0,$$

where n and n' are the residuals found by comparing the computed with the observed place. Professor Brünnow, to whom we owe the latest and some of the best determinations of stellar parallax, assumes further that the constant of aberration may be different for the two stars, and introduces into the equations a term to correct the assumed value of this constant. The coefficients for this term are easily computed, for we have only to put $\theta = 90^\circ$, for θ , in the formulæ for the coefficients of the parallax.

(4). If we determine the parallax of a binary star whose orbit is known, and assume that the orbital motion of the stars in the distant spaces are governed by the law of gravitation, we shall have the data for a determination of the sum of the masses of the two stars. From the theory of elliptic motion we have the known equation,

$$\mu \tau^2 = \left(\frac{a}{\pi}\right)^3 :$$

where μ is the sum of the masses of the stars, the mass of our Sun being the unit of mass, τ the time of their revolution, expressed in sidereal years, a the angle subtended by the semi-major axis of their orbit, and π the annual parallax. It is remarkable that Bessel in 1812, nearly thirty years before his determination of the parallax of 61 Cygni, estimated by means of the above equation the annual parallax of this star to be very nearly what the later determinations have given. Bessel, after his discussion of the proper motion of this star, assumed

$$a = 25''; \quad \tau = 400 \text{ years,}$$

and put $\mu =$ the mass of the Sun $= 1$.

The equation gives

$$\pi = \frac{a}{\tau^{\frac{2}{3}} \mu^{\frac{1}{3}}} = 0''.4604.$$

In 1840 Bessel found from his observations the value

$$\pi = 0''.3483 \pm 0''.0095.$$

Besides the preceding method of estimating the distances of the fixed stars by assuming a knowledge of their masses, Savary, in an interesting paper on the orbits of double stars, published in the *Connaissance des Temps* for 1830, has remarked that the aberration of light may give us some idea of these distances. If the plane of the orbit of the double star were perpendicular to the line of sight, aberration would produce only insensible changes in the motions; but if this plane be much inclined to the line of sight, the time which light requires to pass over the radius of the orbit will produce apparent variations in the angular movements of the stars, and if these could be determined by observation they would give the distance of the stars. In order to show the maximum effect of this variation let us suppose with Struve that the major axis of the orbit is directed toward the Sun, and that the star is at its perihelion. The time which light requires to pass over the axis of the orbit will be

$$z = 0.00003155 \cdot \frac{a}{\pi},$$

where the unit of z is one year. In this time the apparent angular motion of the double star seen from the Earth will be

$$\gamma = \frac{z}{\tau} \cdot \frac{3.1416 \cdot 2a}{1 - e},$$

where e is the excentricity of its orbit. From the value of z we have

$$\gamma = \frac{0.0001982 \cdot a^2}{\pi(1-e)\tau};$$

or since

$$a = \pi \cdot \tau^{\frac{3}{2}} \cdot \mu^{\frac{1}{2}}; \quad \text{we have the ratio of } \gamma \text{ and } \pi,$$

$$\frac{\gamma}{\pi} = \frac{0.0001982 \cdot \tau^{\frac{3}{2}} \cdot \mu^{\frac{1}{2}}}{1 - e}.$$

This equation shows that if the masses of the stars are not much greater than the mass of our Sun then γ is much smaller than π , unless the periodic time is very great. Thus if we suppose the mass of the two stars to be one thousand times as great as the mass of our Sun, it will be seen that τ must still be very great in order to make γ as large as π . Moreover this inequality would probably be connected with the equation of centre in such a way as to make its determination nearly impossible. The idea of Savary therefore, although ingenious and founded on a real phenomenon, does not seem to promise any practical result.

(5). The only practicable method of determining the distance of a star appears to result from the determination of its annual parallax by observation, and I wish to call the attention of American astronomers to this interesting problem. The theory is simple, the reductions are easy, and the final calculation of the equations of condition and their solution by the method of least squares presents no great difficulties. In this problem the astronomer has the satisfaction of knowing that the value of the result lies wholly in his own hands. It will depend on the care and thoroughness with which he has studied and mastered his instrument, and on his skill in making the observations. As the quantity to be determined is small, and has a period of a year, it may easily be masked by the yearly change of temperature, or a habit of observing differently on different sides of the meridian, so that there is ample room for thoughtful observation. An equatorial telescope, having an objective of ten or twelve inches aperture, and provided with a good filar micrometer and a driving clock, is sufficient. The excellent objectives made by Alvan Clark and Sons are well suited to this work, since they give clear, sharp images of the stars. Many of these telescopes are now established in different parts of our country, and perhaps a few of them might be used for observations of stellar parallax.

AN ACCOUNT OF CAUCHY'S "CALCUL DES RESIDUS."

BY CHAS. H. KUMMELL, ASSISTANT U. S. LAKE SURVEY, DETROIT, MICH.

(Continued from page 9.)

2. Let $\varphi(z) = z^{p-1}$ where $m+1 > p > 1$. We have then

$$f(\infty + 0i) = \infty.$$

But since the value ∞ of x agrees with the upper limit of the x -integration and the value 0 of y with the lower limit of the y -integration, there is no corresponding correction by virtue of (10^v)*. Since

$$\frac{(\pm\infty + yi)^{p-1}}{(a \pm \infty + yi)^m} = \frac{(x + \infty i)^{p-1}}{(a + x + \infty i)^m} = 0,$$

if $m+1 > p > 1$ condition (11) is fulfilled and we have by formulas (10₁'), (10₂') and (10₃')

$$\int_{-\infty}^{\infty} \frac{x^{p-1} dx}{(a+x)^m} = \frac{2(-a)^{p-1}}{2n-1} \infty^{2n-1} + \pi i \frac{p-1}{1} \cdot \frac{p-2}{2} \dots \frac{p-2n+1}{2n-1} \times (-a)^{p-2n}, \quad (c_1)$$

$$\int_{-\infty}^{\infty} \frac{x^{p-1} dx}{(a+x)^{2n+1}} = \frac{2(p-1)(-a)^{p-2}}{2n-1} \infty^{2n-1} + \pi i \frac{p-1}{1} \cdot \frac{p-2}{2} \dots \frac{p-2n}{2n} \times (-a)^{p-2n-1}, \quad (c_2)$$

$$\int_{-\infty}^{\infty} \frac{x^{p-1} dx}{a+x} = \pi i (-a)^{p-1}. \quad (c_3)$$

In order to separate the real from the imaginary we have to transform these integrals by (13). We have then

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{x^{p-1} dx}{(a+x)^m} &= \int_0^{\infty} x^{p-1} dx \left[\frac{1}{(a+x)^m} + \frac{(-1)^{p-1}}{(a-x)^m} \right] \\ &= \int_0^{\infty} \frac{x^{p-1} dx}{(a^2-x^2)^m} \left[(a-x)^m + e^{(p-1)\pi i} (a+x)^m \right]. \end{aligned}$$

We have then

$$\begin{aligned} \int_0^{\infty} \frac{x^{p-1} dx}{(a^2-x^2)^{2n}} \left[(a-x)^{2n} - e^{p\pi i} (a+x)^{2n} \right] &= -\frac{2}{2n-1} a^{p-1} e^{p\pi i} \infty^{2n-1} \\ &\quad + \pi i \frac{p-1}{1} \cdot \frac{p-2}{2} \dots \frac{p-2n+1}{2n-1} a^{p-2n} e^{p\pi i}, \end{aligned}$$

consequently, separating the possible from the impossible part

*Bierens de Haan gives, in such a case, $+\infty$ for the correction for discontinuity.

$$\int_0^{\infty} \frac{x^{p-1} dx}{(a^2 - x^2)^{2n}} \left[(a-x)^{2n} - \cos p\pi (a+x)^{2n} \right] = -\frac{2}{2n-1} a^{p-1} \infty^{2n-1} \cos p\pi$$

$$- \pi \frac{p-1}{1} \cdot \frac{p-2}{2} \dots \frac{p-2n+1}{2n-1} a^{p-2n} \sin p\pi, (c'_1)$$

$$\int_0^{\infty} \frac{x^{p-1} dx}{(a-x)^{2n}} = \frac{2}{2n-1} a^{p-1} \infty^{2n-1} - \pi \frac{p-1}{1} \cdot \frac{p-2}{2} \dots \frac{p-2n+1}{2n-1} a^{p-2n} \cot p\pi$$

$$= \infty. (c'_1)$$

The integral (c'_1) is likewise ∞ unless $p = \frac{3}{2}$ when we have

$$\int_0^{\infty} \frac{x^{\frac{1}{2}} dx}{(a+x)^{2n}} = \pi \cdot \frac{1}{2} \cdot \frac{1}{4} \cdot \frac{3}{6} \dots \frac{3-4n+2}{4n-2} a^{\frac{1}{2}-2n}$$

$$= \pi a^{\frac{1}{2}-2n} \frac{1}{2} \cdot \frac{1}{4} \cdot \frac{3}{6} \cdot \frac{5}{8} \dots \frac{4n-5}{4n-2}.$$

Eliminating the integral $\int_0^{\infty} \frac{x^{p-1} dx}{(a-x)^{2n}}$ we obtain

$$\int_0^{\infty} \frac{x^{p-1} dx}{(a+x)^{2n}} = -\frac{\pi a^{p-2n}}{\sin p\pi} \frac{p-1}{1} \cdot \frac{p-2}{2} \dots \frac{p-2n+1}{2n-1}. (c''_1)$$

In Γ functions we have

$$\int_0^{\infty} \frac{x^{p-1} dx}{(1+x)^{2n}} = \frac{\Gamma(p)\Gamma(2n-p)}{\Gamma(2n)} = \Gamma(p)\Gamma(1-p) \frac{2n-p-1}{2n-1} \cdot \frac{2n-p-2}{2n-2}$$

$$\dots \frac{p-2}{2} \cdot \frac{p-1}{1}.$$

But

$$\int_0^{\infty} \frac{x^{p-1} dx}{(a+x)^{2n}} = a^{p-2n} \int_0^{\infty} \frac{x^{p-1} dx}{(1+x)^{2n}}$$

and by the second theorem of Γ functions

$$\Gamma(p)\Gamma(1-p) = \frac{\pi}{\sin p\pi}, \text{ consequently } (c''_1) \text{ proves cor-}$$

rect.

By a similar transformation we have from (c_2)

$$\int_0^{\infty} \frac{x^{p-1} dx}{(a^2 - x^2)^{2n+1}} \left[(a-x)^{2n+1} - e^{p\pi i} (a+x)^{2n+1} \right] = \frac{2(p-1)}{2n-1} a^{p-2} e^{p\pi i} \infty^{2n-1}$$

$$- \pi i a^{p-2n-1} e^{p\pi i} \frac{p-1}{1} \cdot \frac{p-2}{2} \dots \frac{p-2n}{2n},$$

and, separating possible from impossible parts,

$$\int_0^{\infty} \frac{x^{p-1} dx}{(a^2 - x^2)^{2n+1}} \left[(a-x)^{2n+1} - \cos p\pi (a+x)^{2n+1} \right] = \frac{2(p-1)}{2n-1} a^{p-2} \cos p\pi \infty^{2n-1}$$

$$+ \pi a^{p-2n-1} \sin p\pi \frac{p-1}{1} \cdot \frac{p-2}{2} \dots \frac{p-2n}{2n}, (c'_2)$$

$$\begin{aligned}\int_0^{\infty} \frac{x^{p-1} dx}{(a-x)^{2n+1}} &= -\frac{2(p-1)}{2n-1} a^{p-2} \infty^{2n-1} \\ &\quad + \pi a^{p-2n-1} \cot p\pi \frac{p-1}{1} \cdot \frac{p-2}{2} \dots \frac{p-2n}{2n} (c_2') \\ &= \infty \text{ if } p > 1 \\ &= 0 \text{ if } p = 1. \end{aligned} \quad (c_2'')$$

Eliminating the latter integral we have also

$$\int_0^{\infty} \frac{x^{p-1} dx}{(a+x)^{2n+1}} = \frac{\pi}{\sin p\pi} a^{p-2n+1} \frac{p-1}{1} \cdot \frac{p-2}{2} \dots \frac{p-2n}{2n}. \quad (c_2'')$$

This and formula (c_1'') are comprised in the following

$$\int_0^{\infty} \frac{x^{p-1} dx}{(a+x)^m} = \frac{\pi}{\sin p\pi} a^{p-m} \frac{p-1}{1} \cdot \frac{p-2}{2} \dots \frac{p-m+1}{m-1}. \quad (c''')$$

Finally we have from (c_3)

$$\int_0^{\infty} \frac{x^{p-1} dx}{a^2 - x^2} \left[a - x - e^{p\pi i} (a + x) \right] = -\pi i a^{p-1} e^{p\pi i},$$

$$\therefore \int_0^{\infty} \frac{x^{p-1} dx}{a+x} - \cos p\pi \int_0^{\infty} \frac{x^{p-1} dx}{a-x} = \pi a^{p-1} \sin p\pi, \quad (c'_3)$$

$$\int_0^{\infty} \frac{x^{p-1} dx}{a-x} = \pi a^{p-1} \cot p\pi. \quad (c'_3)$$

Eliminating $\int_0^{\infty} \frac{x^{p-1} dx}{a-x}$ we have

$$\int_0^{\infty} \frac{x^{p-1} dx}{a+x} = \frac{\pi a^{p-1}}{\sin p\pi}. \quad (c''')$$

Ex. 2. Required $\int_{-\infty}^{\infty} \frac{\varphi(x) dx}{\pm a^2 + 2bx + x^2}$.

We suppose

$$\frac{\varphi(\pm \infty + yi)}{\pm a^2 + 2b(\pm \infty + yi) + (\pm \infty + yi)^2} = \frac{\varphi(x + \infty i)}{\pm a^2 + 2b(x + \infty i) + (x + \infty i)^2} = 0,$$

also that $\varphi(z)$ does not become infinite for any value between $\infty + \infty i$ and $-\infty + 0i$. The element function can become infinite only if

$$\pm a^2 + 2bz + z^2 = 0,$$

viz., if

$$z_1 = -b + \sqrt{(b^2 \mp a^2)},$$

$$z_2 = -b - \sqrt{(b^2 \mp a^2)}.$$

1. For the upper sign let $b < a$, then

$$z_1 = -b + i\sqrt{(a^2 - b^2)}$$

is the only value of z coming into consideration for which $\frac{\varphi(z)}{a^2 + 2bz + z^2} = \infty$, because $y_1 = +\sqrt{(a^2 - b^2)}$ is comprised between the limits of the y -integration, viz., ∞ and 0. We have then, by (12),

$$\int_{-\infty}^{\infty} \frac{\varphi(x)dx}{a^2+2bx+x^2} = \Delta_{-b+i\sqrt{(a^2-b^2)}}.$$

To obtain this correction we have to use (10) which becomes here, since $m = 1$,

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{\varphi(x)dx}{a^2+2bx+x^2} &= \Delta_{-b+i\sqrt{(a^2-b^2)}} = 2\pi i \psi[-b+i\sqrt{(a^2-b^2)}] \\ &= 2\pi i \left[\frac{z+b-i\sqrt{(a^2-b^2)}}{a^2+2bz+z^2} \varphi(z) \right]_{z=-b+i\sqrt{(a^2-b^2)}} \\ &= 2\pi i \cdot \frac{\varphi(-b+i\sqrt{(a^2-b^2)})}{2i\sqrt{(a^2-b^2)}} \\ &= \frac{\pi}{\sqrt{(a^2-b^2)}} \varphi[-b+i\sqrt{(a^2-b^2)}]. \end{aligned} \quad (d)$$

2. Let for the upper sign $b = a$ then the proposed integral becomes

$$\int_{-\infty}^{\infty} \frac{\varphi(x)dx}{(a+x)^2} = \infty \text{ [by } (a_1)]. \quad (d')$$

3. If for the upper sign $b > a$ we have two unequal real roots. The same is the case for the lower sign whatever the value of a or b . We have then two discontinuous elements, viz., if

$$\begin{aligned} z_1 &= -b + \sqrt{(b^2 \mp a^2)} + 0.i, \\ z_2 &= -b - \sqrt{(b^2 \mp a^2)} + 0.i, \end{aligned}$$

because x_1 as well as x_2 are comprised within the limits $\pm \infty$ of the x -integration and $y_1 = y_2 = 0$ = lower limit of the y integration. We have then to use formula (10') and obtain

$$\begin{aligned} \int_{-\infty \pm a^2}^{\infty} \frac{\varphi(x)dx}{a^2+2bx+x^2} &= \pi i \left[\frac{\varphi[-b+\sqrt{(b^2 \mp a^2)}]}{2\sqrt{(b^2 \mp a^2)}} + \frac{\varphi[-b-\sqrt{(b^2 \mp a^2)}]}{-2\sqrt{(b^2 \mp a^2)}} \right] \\ &= \frac{\pi i}{2\sqrt{(b^2 \mp a^2)}} \left[\varphi[-b+\sqrt{(b^2 \mp a^2)}] - \varphi[-b-\sqrt{(b^2 \mp a^2)}] \right]. \end{aligned} \quad (d'')$$

Corollaries. Let $b = 0$ in (d), then

$$\int_{-\infty}^{\infty} \frac{\varphi(x)dx}{a^2+x^2} = \frac{\pi}{a} \varphi(ai); \quad (d_a)$$

also let $b = 0$ in (d'') for the lower sign only, then

$$\int_{-\infty}^{\infty} \frac{\varphi(x)dx}{-a^2+x^2} = \frac{\pi i}{2a} \left[\varphi(a) - \varphi(-a) \right]; \quad (d''_a)$$

and if we put $a = 0$ in (d'') we have

$$\int_{-\infty}^{\infty} \frac{\varphi(x)dx}{x(2b+x)} = \frac{\pi i}{2b} \left[\varphi(0) - \varphi(-2b) \right]. \quad (d''_b)$$

These formulæ may be used for any special form of $\varphi(x)$ and after properly separating the real from the imaginary we obtain a great variety of integrals as in Ex. 1.

§ 5.

Another useful formula is obtained by assuming in (6) $z_n = \omega = y_n$ and $z_0 = 0 = y_0$. We have then

$$\int_0^\infty dx [f(x + \omega i) - f(x + 0i)] = i \int_0^\infty dy [f(\omega + yi) - f(0 + yi)] - \Delta,$$

and if $f(x + \omega i) = f(\omega + yi) = 0$, (11')

$$\int_0^\infty dx f(x) = i \int_0^\infty dy f(yi) + \Delta. \quad (12')$$

Since both integrals have the same range of integration we can write this equation thus:

$$\int_0^\infty dx [f(x) - i f(xi)] = \Delta. \quad (12'_1)$$

If the element function is par we have also

$$\int_0^\infty dx [f(x^2) - i f(-x^2)] = \Delta, \quad (12'_2)$$

and if it is par par we have

$$\int_0^\infty dx [f(x^4) - i f(x^4)] = \Delta,$$

or
$$\int_0^\infty dx f(x^4) = \frac{\Delta}{1-i} = \frac{\Delta}{2}(1+i). \quad (12'_3)$$

If the element function is of the form $x^{p-1} f(x^4)$, where p is any number between such limits as condition (11') requires, we have

$$\int_0^\infty dx [x^{p-1} f(x^4) - i^p x^{p-1} f(x^4)] = \Delta,$$

or
$$\int_0^\infty x^{p-1} dx f(x^4) = \frac{\Delta}{1-i^p} = \frac{\Delta}{1-e^{i\frac{1}{2}p\pi}} = -\frac{\Delta e^{-(p+2)\frac{1}{4}\pi i}}{2 \sin \frac{1}{4} p\pi} \quad (12'_4)$$

$$= \frac{\Delta e^{(2-p)\frac{1}{4}\pi i}}{2 \sin \frac{1}{4} p\pi}.$$

This latter formula reduces to (12'_3) if $p = 1$.

Ex. 4. Evaluate $\int_0^\infty \frac{dx}{a^4 - x^4} \varphi(x^4)$ if $\varphi(x^4)$ does not become infinite for any value of x between the limits and $\frac{\varphi(x^4)}{a^4 - x^4}$ satisfies (11').

The roots of the equation

$$a^4 - z^4 = 0$$

are

$$z_1 = ae^0; \quad z_3 = ae^{\pi i};$$

$$z_2 = ae^{i\frac{1}{2}\pi}; \quad z_4 = ae^{i\frac{3}{2}\pi}.$$

Only the roots $z_1 = a(1+0i)$; $z_2 = a(0+i)$ need to be considered since x and y must be positive. We have by (10'_3)

$$\Delta_{a(1+0i)} = \pi i \varphi(a^4) \left[\frac{z-a}{a^4-z^4} \right]_{z=a} = -\frac{\pi i}{4a^3} \varphi(a^4),$$

and by (10'''_3)

$$\Delta_{a(0+i)} = \pi i \varphi(a^4) \left[\frac{z-ai}{a^4-z^4} \right]_{z=ai} = -\frac{\pi i}{4a^3 i} \varphi(a^4) = \frac{\pi}{4a^3} \varphi(a^4);$$

$$\therefore \Delta = \frac{\pi}{4a^3} \varphi(a^4) (1-i).$$

Hence by (12'₃)

$$\int_0^\infty \frac{dx}{a^4 - x^4} \varphi(x^4) = \frac{\pi}{4a^3} \varphi(a^4). \quad (e)$$

Let $\varphi(x^4) = e^{-bx^4}$ then $\frac{e^{-b(\infty+yi)^4}}{a^4 - (\infty+yi)^4} = \frac{e^{-b(x+\infty i)^4}}{a^4 - (x+\infty i)^4} = 0$, and therefore

$$\int_0^\infty \frac{dx}{a^4 - x^4} e^{-bx^4} = \frac{\pi}{4a^3} e^{-a^4 b}. \quad (e')$$

Ex. 5. Evaluate $u = \int_0^\infty \frac{dx}{a+x} x^{p-1} \varphi(x)$.—Replace x by x^4 then

$$u = 4 \int_0^\infty \frac{x^{4p-1} dx}{a+x^4} \varphi(x^4).$$

This is of the form (12'₄) and if $\varphi(x^4)$ does not become infinite between the limits ∞ and 0 of x and y , the points of discontinuity are given by the eq'

$$a + z^4 = 0,$$

which has the four roots

$$\begin{aligned} z_1 &= a^{\frac{1}{4}} e^{\frac{1}{4}\pi i}; & z_3 &= a^{\frac{1}{4}} e^{\frac{3}{4}\pi i}; \\ z_2 &= a^{\frac{1}{4}} e^{\frac{5}{4}\pi i}; & z_4 &= a^{\frac{1}{4}} e^{\frac{7}{4}\pi i}. \end{aligned}$$

Of these only $z_1 = a^{\frac{1}{4}} (\cos \frac{1}{4}\pi + i \sin \frac{1}{4}\pi)$ falls within the limits, since x as well as y must be positive. By formula (10) we have then

$$\begin{aligned} \Delta_{a^{\frac{1}{4}} e^{\frac{1}{4}\pi i}} &= 2\pi i a^{p-\frac{1}{4}} e^{(p-\frac{1}{4})\pi i} \varphi(-a) \left[\frac{z - a^{\frac{1}{4}} e^{\frac{1}{4}\pi i}}{a + z^4} \right]_{z=a^{\frac{1}{4}} e^{\frac{1}{4}\pi i}} \\ &= \frac{1}{2}\pi i a^{p-1} e^{(p-1)\pi i} \varphi(-a). \end{aligned}$$

This being used for Δ in (12'₄) we obtain

$$\begin{aligned} u &= \int_0^\infty \frac{x^{p-1} dx}{a+x} \varphi(x) = 4 \int_0^\infty \frac{x^{4p-1} dx}{a+x^4} \varphi(x^4) = 4 \cdot \frac{1}{2}\pi i a^{p-1} e^{(p-1)\pi i} \varphi(-a) \\ &\quad \times \frac{e^{-(p-\frac{1}{4})\pi i}}{2 \sin p\pi} \\ &= \frac{\pi a^{p-1}}{\sin p\pi} \varphi(-a). \end{aligned} \quad (f)$$

Let $\varphi(x) = 1$, then if $2 > p > 1$ (11') is satisfied and we obtain the familiar integral already given, § 4, Ex. 1 and 2,

$$\int_0^\infty \frac{x^{p-1} dx}{a+x} = \frac{\pi a^{p-1}}{\sin p\pi}. \quad (f')$$

[Mr. Kummell has contributed another § to this paper but for want of suitable type we are not able to insert it at present, but hope to be able to do so before the close of the present volume.]

SOLUTION OF PROBLEMS IN NUMBER SIX VOL. FIVE.

(Continued from page 30.)

SOLUTION OF 237 BY CHAS. H. KUMMELL, DETROIT, MICHIGAN.

We have

$$u = \int_0^\pi \left[\frac{\sin(n - \frac{1}{2})(\theta + \pi)}{\sin(\frac{1}{2}(\theta + \pi))} + 1 \right] d\theta = u' + \frac{1}{2}\pi. \quad (1)$$

Placing $\theta = 2\varphi$ we have

$$u' = \int_0^{\frac{1}{2}\pi} \frac{\sin(2n - 1)(\frac{1}{2}\pi + \varphi)}{\sin(\frac{1}{2}\pi + \theta)} d\varphi. \quad (2)$$

We have $\sin nx = \text{real part of } (\cos x + i \sin x)^n \div i$

$$= \sin x \left[n \cos^{n-1} x - \frac{n(n-1)(n-3)}{1 \cdot 2 \cdot 3} \cos^{n-3} x \sin^2 x + \dots \right].$$

Applying this formula we have in (2)

$$u' = \int_0^{\frac{1}{2}\pi} d\varphi \left[(2n-1) \sin^{2n-2} \varphi - \frac{2n-1}{1} \cdot \frac{2n-2}{2} \cdot \frac{2n-3}{3} \sin^{2n-4} \varphi \cos^2 \varphi + \frac{2n-1}{1} \dots \frac{2n-5}{3} \sin^{2n-6} \varphi \cos^4 \varphi - \dots \right]. \quad (3)$$

We have

$$\int_0^{\frac{1}{2}\pi} d\varphi \sin^{p-1} \varphi \cos^{q-1} \varphi = \frac{1}{2} \cdot \frac{\Gamma(\frac{p}{2}) \Gamma(\frac{q}{2})}{\Gamma(\frac{p+q}{2})}. \quad (4)$$

Applying this formula term by term to (3) we obtain

$$\begin{aligned} u' &= (n - \frac{1}{2}) \left[\frac{\Gamma(n - \frac{1}{2}) \Gamma(\frac{1}{2})}{\Gamma(n)} - \frac{n-1}{1} \cdot \frac{2n-3}{3} \cdot \frac{\Gamma(n - \frac{3}{2}) \Gamma(\frac{3}{2})}{\Gamma(n)} \right. \\ &\quad \left. + \frac{n-1}{1} \cdot \frac{2n-3}{3} \cdot \frac{n-2}{2} \cdot \frac{2n-5}{5} \cdot \frac{\Gamma(n - \frac{5}{2}) \Gamma(\frac{5}{2})}{\Gamma(n)} - \dots \right] \\ &= \frac{\Gamma(n + \frac{1}{2}) \pi^{\frac{1}{2}}}{\Gamma(n)} \left[1 - \frac{n-1}{1} \cdot \frac{1}{3} + \frac{n-1}{1} \cdot \frac{n-2}{2} \cdot \frac{1}{5} - \dots \right] \\ &= \frac{\Gamma(n + \frac{1}{2}) \pi^{\frac{1}{2}}}{\Gamma(n)} \int_0^1 dx \left[1 - \frac{n-1}{1} x^2 + \frac{n-1}{1} \cdot \frac{n-2}{2} x^4 - \dots \right] \\ &= \frac{\Gamma(n + \frac{1}{2}) \pi^{\frac{1}{2}}}{\Gamma(n)} \int_0^1 dx (1 - x^2)^{n-1} \\ &= \frac{\Gamma(n + \frac{1}{2}) \pi^{\frac{1}{2}}}{\Gamma(n)} \int_0^{\frac{1}{2}\pi} d\varphi \cos^{2n-1} \varphi \quad (x = \sin \varphi) \\ &= \frac{\Gamma(n + \frac{1}{2}) \pi^{\frac{1}{2}}}{\Gamma(n)} \times \frac{1}{2} \frac{\Gamma(\frac{1}{2}) \Gamma(n)}{\Gamma(n + \frac{1}{2})} \quad [\text{by (4)}] \\ &= \frac{1}{2} \pi. \end{aligned}$$

We have then, by (1), $u = u' + \frac{1}{2}\pi = \pi$.

238. "Two points are taken at random in the surface of a circle, radius r , one of them being confined to a given radius, and a chord drawn through the points. Find (1) the chance that the length of the chord does not exceed $2a$, and (2) the chance that it does not exceed the radius of the circle."

SOLUTION BY ARTEMAS MARTIN, M. A., EDITOR MATH. VISITOR, ERIE, PA.

1. Let O be the center of the circle, and P one of the random points. Through P draw two chords, length $2a$. If the other point is taken in the surface APC , or in the surface BPD , the chord drawn through them will not exceed the length $2a$.

Let $OP = x$, $S = \text{surface } APC = \text{surface } BPD$, $m = \sqrt{(r^2 - a^2)} = OE$, and $p_1 = \text{the chance required}$.

We readily find $S = r^2 \cos^{-1} \left(\frac{m}{r} \right) - m(x^2 - m^2)^{\frac{1}{2}}$.

Hence
$$p_1 = \frac{\int_m^r 2Sdx}{\int_0^r \pi r^2 dx},$$

$$\begin{aligned} &= \frac{2}{\pi r^3} \int_m^r \left[r^2 \cos^{-1} \left(\frac{m}{r} \right) - m \sqrt{(x^2 - m^2)} \right] dx, \\ &= \frac{1}{\pi} \left[2 \cos^{-1} \left(\frac{m}{r} \right) - \frac{m \sqrt{(r^2 - m^2)}}{r^2} - \frac{m(2r^2 - m^2)}{r^3} \log \left(\frac{r + \sqrt{(r^2 - m^2)}}{m} \right) \right], \\ &= \frac{1}{\pi} \left[2 \sin^{-1} \left(\frac{a}{r} \right) - \frac{a \sqrt{(r^2 - a^2)}}{r^2} - \frac{(r^2 + a^2) \sqrt{(r^2 - a^2)}}{2r^3} \log \left(\frac{r + a}{r - a} \right) \right]. \end{aligned}$$

2. Taking $a = \frac{1}{2}r$, we have

$$p_2 = \frac{1}{3} - \frac{1}{\pi} \left(\frac{\sqrt{3}}{4} - \frac{5\sqrt{3}}{16} \log 3 \right).$$



239. "A can plant thirty-six per cent of his arrows within a circular target ten inches in diameter at the distance of one hundred yards; B can plant sixty-four per cent of his arrows within a circle thirteen and one third inches in diameter at the same distance.—Prove that B's skill is greater than A's."

SOLUTION BY CHAS. H. KUMMELL.

Assuming that the marks made in shooting the arrows are distributed according to the laws of probability, we have, if h_1 is the skill of A and h_2 that of B, and d any given distance of a mark from the centre:

$$\frac{h_1}{\sqrt{\pi}} \int_{-10}^{10} d\Delta e^{-h_1^2 \Delta^2} = 0.36 = \frac{2h_1}{\sqrt{\pi}} \int_0^{10} d\Delta e^{-h_1^2 \Delta^2}; \quad \frac{2h_2}{\sqrt{\pi}} \int_0^{13\frac{1}{2}} d\Delta e^{-h_2^2 \Delta^2} = 0.64.$$

Place $h_1 \Delta = \theta$ and also $h_2 \Delta = \theta$, that is, consider θ the variable of the integration instead of Δ , then we have

$$\frac{2}{\sqrt{\pi}} \int_0^{h_1 10} d\theta e^{-\theta^2} = 0.36; \quad \frac{2}{\sqrt{\pi}} \int_0^{h_2 13\frac{1}{2}} d\theta e^{-\theta^2} = 0.64.$$

There are tables of these integrals, for instance, Table IX in Chauvenet's Method of Least Squares. From this we find

$$h_1 10 = 0.33071, \therefore h_1 = 0.033071;$$

$$h_2 13\frac{1}{2} = 0.64727, \therefore h_2 = 0.048545.$$

Therefore $h_2 = 1.4679h_1$, or B shoots 1.4679 times better than A.

SOLUTION BY R. J. ADCOCK, MONMOUTH, ILLINOIS.

Since A can plant $\frac{3.6}{100}$ of his arrows within 5 inches of the centre, the probability that he can shoot the whole number, n , within that distance is $\frac{2^5}{25}$. Hence in eq'n (4), page 21, Vol. V, where $l = \sqrt{pS(d_1^2) + n}$, making $l = 5$, and $p = \frac{9}{25}$, there results

$$\bar{5} = \frac{3}{5} \sqrt{\left[\frac{S(d_1^2)}{n} \right]} \text{ or } \sqrt{\left[\frac{S(d_1^2)}{2n} \right]} = \frac{25}{3\sqrt{2}} = \frac{25\sqrt{2}}{6} \text{ inches,}$$

which, by eq'n (6) on the same page (where the sign of equality is wanting after l through a mistake in printing), is the probable error of a shot by A.

In the same way we find that the probable error of one of B's shots is $(25\sqrt{2}) \div 6$. Therefore they must shoot with an equal degree of accuracy.

SOLUTION BY PROF. P. E. CHASE.

The diameters of targets in which there would be an equal chance to hit or miss, are

$$\text{for A, } 10 \sqrt{\left[-\frac{\log 2}{\log(1-.36)} \right]} = 12.46;$$

$$\text{for B, } \frac{40}{3} \sqrt{\left[-\frac{\log 2}{\log(1-.64)} \right]} = 10.98.$$

B's skill is therefore $(12.46 \div 10.98) = 1.13 \times \text{A's}$.

The above is based on Herschel's formula

$$M = \left(\frac{1}{2} \right) \frac{r^2}{a^2},$$

in which M = probability of missing, in any single shot, a circular area of radius r at whose centre the shot is aimed; a = radius of circular area within which his skill would, on the average of an infinite number of shots, enable him to plant half the total number discharged. (Familiar Lectures on Scientific Subjects. London and New York, 1867, p. 495.)

[If O is the origin and x, y rectangular coordinates, the equation

$$y = ce^{-h^2x^2}$$

will represent the curve $mB'C Bn$, which is called the *probability curve*; and, assuming the whole area $= 1$, if the area $OA-BC = OA'B'C = \frac{1}{4}$, OA is call'd the *probable error*.



In the case under consideration, suppose O to represent the center of the target at which the marksman aims, and let Oa denote the radius of the target, then the probability of hitting the target is $2OabC$; and, for the marksman A, there are give $Oa = 5$ inches and area $2OabC (= \text{probability of hitting the target}) = 0.36$; and for the marksman B, $Oa = 6\frac{2}{3}$ inches and area $2OabC = 0.64$. But the area $2OabC$ is represented by the int'gl

$$\frac{2}{\sqrt{\pi}} \int_0^{hx} e^{-h^2x^2} dx,$$

in which h is the precision of the marksman and is $= 0.4769 \div r$, r being the probable error OA ; and in the table referred to by Mr. Kummell in the foregoing solution, the values of hx are calculated for the different values of P . Taking from the table the values corresponding to $P_1 = 0.36$, and $P_2 = 0.64$, and dividing respectively by 5 and $6\frac{2}{3}$, we find the precision of A and B as determined by Mr. Kummell, and by Professor Kershner whose solution is substantially the same as Mr. Kummell's.

The first of the foregoing solutions is therefore correct if the hypothesis upon which it is based is correct; i. e., if the equation

$$y = ce^{-h^2x^2} \tag{1}$$

represents the relation between an error and its probability.

In the second solution, given above, the result obtained is at variance with the first. It will be seen, however, that the hypothesis upon which Mr. Adcock's solution is based virtually assumes that all possible errors are equally probable. A glance at the probability curve shows, however, that, if equation (1) represents the law of facility of error, small errors occur more frequently than large ones. Moreover *our* experience, and probably the experience of most persons, is confirmatory of that fact.

That Mr. Adcock assumes all possible errors equally probable, appears not only from his discussion at pages 21 and 22 of Vol. V, upon which the above solution is based, but is also deducible from his result here obtained.

If we suppose a circular target of radius $8\frac{1}{2}$ inches to have 100 points uniformly distributed over its area, 36 of these points will be within a concentric circle of 5 in. radius (A's target) and 64 within a circle of $6\frac{3}{4}$ in. radius (B's target); which proves that, if A's skill is equal to B's, great and small errors must be equally probable.

In the third solution given above the greater frequency of small than of large errors is recognized in the use of eq'n (1), but by assuming that the degree of the error varies with x [a fact included in (1)] and multiplying the differential of P by this factor a result is obtained which differs from both the others. The reason for multiplying the diff. of P by x is not apparent.

After the foregoing was written, we rec'd from Mr. Adcock the subjoined Revision of his paper on Limits of Errors and his solution of 239.—Ed.]

REVISION OF "LIMITS OF ERRORS", etc. (P. 21, VOL. V.)—Since " $p = n \div 4\pi m S(d_1^2) \dots$ (1) is the probability that any point of the total number $4\pi m S(d_1^2)$ shall be at or on any point line or surface," it is, when $S(d_1^2)$ is a minimum, the probability that the point, line or surface is in its true position. For any other position of the point, line or surface, where the sum of the squares of the normals = $S(\delta^2)$, the probability is $y = n \div 4\pi m S(\delta^2)$.

$$\text{Hence} \quad \frac{y}{p} = \frac{S(d_1^2)}{S(\delta^2)}. \quad (2)$$

Let x equal the distance between these two positions measured from the first, then by my proposition, 200 (pp. 64 and 91, Vol. V), and because that proposition may be extended to include the line and surface,

$$x^2 = [S(\delta^2) - S(d_1^2)] \div n.$$

$$\text{Hence } y = \frac{p S(d_1^2)}{S(d_1^2) + n x^2} = \frac{p}{1 + [n x^2 \div S(d_1^2)]} = \frac{p}{1 + c^2 x^2}. \quad (3)$$

The curve represented by (3) is the true "probability curve", being that imposed by the theory of probabilities. The ordinate y is the probability, frequency or density of the number of normals for the abscissa x . The area is

$$\int_0^x y dx = \frac{p}{c} \tan^{-1} cx.$$

Let z equal the number of errors not greater than x , and n the number not greater than l , then

$$\frac{\tan^{-1} cx}{\tan^{-1} cl} = \frac{z}{n}, \quad (4)$$

equals the probability that an error shall not exceed x . Eq'n (4) satisfactorily agrees with the number and magnitude of errors in actual observations, as far as has been examined by me.—[Revised solution, as follows:]

$$\int_0^x y dx = \frac{p}{c} \tan^{-1} cx = \frac{n}{4\pi m S(d_1^2)} \sqrt{\left(\frac{S(d_1^2)}{n}\right)} \tan^{-1} \frac{x}{\sqrt{[S(d_1^2) \div n]}};$$

$$\frac{1}{4\pi m} \sqrt{\left(\frac{n}{S(d_1^2)}\right)} \tan^{-1} \frac{5}{\sqrt{[S(d_1^2) \div n]}} = \frac{9}{25};$$

$$\frac{1}{4\pi m} \sqrt{\left(\frac{n}{S(d^2)}\right)} \tan^{-1} \frac{\frac{1}{3}(20)}{\sqrt{[S(d^2) \div n]}} = \frac{16}{25};$$

$$\therefore \frac{z \tan^{-1} 5z}{y \tan^{-1} \frac{1}{3}(20)y} = \frac{9}{16}.$$

Hence $y > z$, and $\therefore$ B's skill is greater than A's.

240. "If the angles θ , φ and ψ are connected by a certain relation any two of them may be the oblique angles of a spherical right triangle, and the third will be the complement of the perpendicular from the right-angle to the hypotenuse. Give a geometrical construction of the three triangles thus connected, and find the relations that exist between their sides."

SOLUTION BY E. B. SEITZ, GREENVILLE, OHIO.

Let ABC be a tri-rectangular triangle. With any point O in ABC as a pole describe a great circle, cutting the sides of ABC produced in D, E, F, G, H, K . Draw the quadrants OM, ON, OP through A, B, C . Then the right triangles ADE, BFG, CHK are the required triangles.



For let arc $AO = \theta$, arc $BO = \varphi$, and arc $CO = \psi$. Then since CEF is the polar triangle of AOB , we have $\angle AED = 180^\circ - CEF = \varphi$, and $\angle BFG = 180^\circ - CFE = \theta$. For like reasons, $\angle BGF = \psi$, $\angle CHK = \varphi$, $\angle KCA = \theta$, and $\angle ADE = \psi$. We also have arc $AM = 90^\circ - \theta$, $BN = 90^\circ - \varphi$, and $CP = 90^\circ - \psi$. Hence the triangles ADE, BFG and CHK have the required relation. To find the relation of the angles θ, φ and ψ , let $EAM = A$; then $\cos \varphi = \sin \theta \sin A$, and $\cos \psi = \sin \theta \cos A$, whence, by squaring, adding and reducing, we obtain

$$\cos^2 \theta + \cos^2 \varphi + \cos^2 \psi = 1.$$

Because $DABG, FBCK$ and $HCAE$ are semi-circumferences, and AB, BC and CA are quadrants, we have $AD + BG = BF + CK = CH + AE =$ a quadrant, for the relation between the sides about the right angles of the triangles.

Again, since $DE = GH, FG = KD$ and $HK = EF$, we have $DE + FG + HK =$ a semi-circumf., for the relation between the hypotenuses.

THE TANGENCY PROBLEMS.

BY PROF. JOSEPH H. KERSHNER, MERCERSBERG, PA.

WE are indebted to the Mathematical Collections of Pappus of Alexandria for a general account of the Course in Geometry prescribed, and partly created by Apollonius of Perga for his pupils, and universally adopted by the Ancients.

The curriculum commenced with the *Elements* and *Data*, and closed with the *Porisms*, each originally arranged by Euclid. After the *Data*, followed the Books of Apollonius in order thus: The *Section of Ratio*, *Section of Space*, *Determinate Section*, *Inclinations*, *Tangencies*, *Plane Loci*, divided into two Books each. Of all these, except a few meagre fragments, only the *Elements* and *Data* have escaped the tooth of Time. Attempts at restoration, by the greatest Geometers of later times, have met with more or less success. The two Books of Tangencies were devoted to the *Tangency Problem* of Apollonius: "*Of points, straight lines and circles, any three whatever being given in position, to draw a circle through each given point, which, if possible, shall also touch the given lines.*" The Tangency Problem of Pappus, designed as an introduction to these, is thus stated: "*Of points, straight lines and circles, any two whatever being given in position in the same plane, to draw a circle of given radius, which shall touch the given points, straight lines and circles.*"

Fermat has skirmished across into the hill-country of Solid Geometry with Tangencies: "*Of points, planes and spheres, some four being given in position, any two of the same, and any other two of any kind whatever, to draw a sphere which shall touch the four points, planes and spheres.*"

These problems come down to us with the accents of the ages. We think of them reverently when we consider that a large majority of all the great and good men the world has seen have labored at some, or all, of the original Tangency Problems.

NOTE.—It has been suggested that we call the attention of the readers of the ANALYST to such problems and queries as have been published therein and to which no solutions, or answers, have been given, so that persons not caring to look over all that has been published may know just where to find acceptable work in that line.

According to our recollection, attempts have been made, and published, to solve all the problems and answer all the queries, except prob. 68, p. 63, V. II, and the "Query by request" on p. 96, Vol. V. An answer to this query is specially solicited; and, also, further discussion of problems 47, 68, 96, 120, 132 and 169 would be acceptable.

CORRESPONDENCE.—* * “I was much interested in your note on the Correspondence of Material Forms with Mathematical Relations, published in the ANALYST of last September. It gives a mathematical expression to a theory of the constitution of matter that I have long held, and seems to be a necessary corollary from the doctrine of the infinity of space.

“Conceive the smallest atom known to Chemists, this may be, and probably is, composed of suns at immense distances from each other, when compared with their own size; these suns are probably surrounded by planets, on these planets organized beings with all the phenomena of civilizations may be conceived to exist, and what would be known as matter to these beings might, in turn, have its atoms composed in a similar manner, and so on in an infinite series downward. On the other hand, all the star clusters and distant nebulae known to astronomers may make up but an atom, in a portion of organic, or inorganic, matter belonging to a system of a higher order. Pursuing this line of thought we might go on in an infinite series upward. The subject opens up a vast field for speculation, and will, I think, in the future, more and more engage the attention of science and philosophy.”

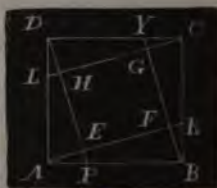
[The foregoing extract is from a letter by J. M. Arnold, of Boston, Mass., dated Jan. 17, 1879, and, in a subsequent letter, he refers to ex-Pres. Hill, of Harvard University, Prof. De Morgan, and others, as entertaining similar views, of the constitution of matter, to those expressed above.

The very interesting paper read by J. Norman Lockyer, and published in the American Journal of Science and Arts for Feb., 1879, has, it seems to us, an important bearing on this subject. —Ed.]

NOTE, BY PROF. M. L. COMSTOCK, GALESBURG, ILL.—In Runkle's Math. Monthly, Vol. II, page 45, there is given a collection of demonstrations of the 47th Prop. of the first Book of Euclid, 28 in all. I append two, which I take to be original.

Let AFB be the right angled triangle, $ACBD$ a sq're described on AB ; then, from similar triangles, the following relations will appear.

$$\begin{aligned}
 1. \quad AB^2 &= 4AEP + 4EFBP + EF^2 \\
 &= 2AE \cdot EP + 2EF(EP + BF) + EF^2 \\
 &= 2AE \cdot EP + 2EF \cdot EP + 2EF \cdot BF + EF^2 \\
 &= 2AF \cdot FR + 2EF \cdot BF + EF^2, \text{ since } EP = FR, \\
 &= 2BF^2 + 2EF \cdot BF + EF^2 \\
 &= BF^2 + AE^2 + 2EF \cdot AE + EF^2, \text{ since } BF = AE, \\
 &= BF^2 + (AE + EF)^2 = BF^2 + AF^2.
 \end{aligned}$$



$$\begin{aligned} 2. \quad AB^2 &= 2ADP + BPDY = AE \cdot DP + EF \cdot DP \\ &= AF \cdot DP = AF(DE + EP) = AF(AF + EP) \\ &= AF^2 + AF \cdot EP = AF^2 + AF \cdot FR = AF^2 + BF^2. \end{aligned}$$

In Todhunter's Spher. Trig., p. 109, Ex. 16, it is stated that, "If O be any point within any spherical triangle ABC , the product of the sines of any two sides and the sine of the included angle is

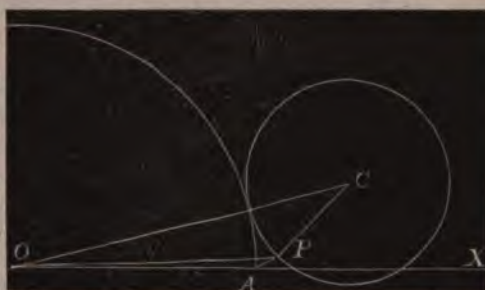
$$\sin AO \sin BO \sin CO (\cot AO \sin BOC + \cot BO \sin COA + \cot CO \sin AOB)."$$

How is this demonstrated?

ANSWER TO QUERY AT P. 16, BY PROF. C. M. WOODWARD, WASHINGTON UNIV., ST. LOUIS, MO.—Let the origin of coördinates be the centre of the largest wheel, and let the axis of X pass through the point where the face of a tooth meets the pitch circle.

Let P be the tracing point, and C the centre of the rolling circle.

The equations of the face of the tooth are



$$x = (R+r) \cos \theta - r \cos \left(\frac{R+r}{r} \theta \right), \quad y = (R+r) \sin \theta - r \sin \left(\frac{R+r}{r} \theta \right),$$

in which θ is the angle between the line of centres and the axis of X .

If we designate by β the angle between the radius vector to the tracing point and the axis of X , the triangle OCP readily gives

$$\frac{\sin(\theta - \beta)}{\sin[\theta(R+r) \div r - \beta]} = \frac{r}{R+r},$$

which gives the relation between θ and β .

It is obvious that the addendum required is given by the expression

$$\sqrt{x^2 + y^2} - R.$$

In the given example, $R = 0.39$ in., $r = 0.0325$ in., $(R+r) \div r = 13$.

The equation of relation is

$$\frac{\sin(\theta - 1^\circ.5)}{\sin(13\theta - 1^\circ.5)} = \frac{1}{13},$$

which gives, by approximation, $\theta = 5^\circ 42' 48''$.

Making the substitutions, I get

$$\sqrt{x^2 + y^2} - R = 0.0218 \text{ in., the addendum.}$$

SOLUTION OF A PROBLEM.

BY GEORGE H. HARVILL, BONNER, LOUISIANA.

Problem.—What must be the size of an auger that will cut away just half of a solid ball 3 inches in diameter, by boring two holes, perpendicular to each other, through its center?

Solution.—Let r = radius of ball and x = radius of the auger. The volume of the ball is $\frac{4}{3}\pi r^3$. The part bored away will consist of the volumes of two cylinders, $ABDE$ and $GKIH$ minus the volume cut from one of the cylinders by the other plus the four segments at the ends of the two cylinders, which must = $\frac{1}{2}$ of $\frac{4}{3}\pi r^3 = \frac{2}{3}\pi r^3$. The volume of each cylinder = $2\pi x^2 \times \sqrt{r^2 - x^2}$, $\therefore$ that of the two cylinders = $4\pi x^2 \sqrt{r^2 - x^2}$. The volume cut from one cylinder by the other is $\frac{2}{3}$ the cube of the diameter of its base = $\frac{2}{3}(2x)^3 = \frac{16}{3}x^3$. The volume of each segment = $\frac{2}{3}\pi r^2[r - \sqrt{r^2 - x^2}] - \frac{1}{3}\pi x^2 \sqrt{r^2 - x^2}$, hence that of the four segments = $\frac{8}{3}\pi r^2[r - \sqrt{r^2 - x^2}] - \frac{4}{3}\pi x^2 \sqrt{r^2 - x^2}$. Hence, putting $r=1$ for convenience, we have the eqⁿ

$$4\pi x^2 \sqrt{1-x^2} + \frac{8}{3}\pi - \frac{8}{3}\pi [\sqrt{1-x^2}] - \frac{4}{3}\pi x^2 \sqrt{1-x^2} - \frac{16}{3}\pi x^3 = \frac{2}{3}\pi;$$

whence by reduction we find

$$8x^3 + 4\pi \sqrt{1-x^2} = 3\pi.$$

Solving this equation by trial, I find $x = .474595$, or $x = .474595r$.

For a sphere 3 inches in diameter, $x = .711892$ inches.

If the holes are cut with a chisel, instead of an auger, what must be the width of the chisel?

SOLUTIONS OF PROBLEMS IN NUMBER ONE.

SOLUTIONS of problems in No. 1, have been received as follows:

From Prof. L. G. Barbour, 245, 246, 248; Prof. P. E. Chase, 241, 242, 243, 246, 247, 249; Prof. W. P. Casey, 241, 242, 243, 245; Geo. M. Day, 244, 246, 248; Prof. E. J. Edmunds, 242, 243, 244, 246; Prof. A. B. Evans, 247; Geo. Eastwood, 248, 249; Henry Gunder, 242, 244, 246; W. E. Heal, 242, 243, 244, 246; J. H. Hoffstrom, 242; G. J. Hendricks, 242; Prof. A. Hall, 248; Dr. David S. Hart, 246; Prof. J. H. Kershner, 241, 242, 243, 244, 245, 246, 248, 249; Prof. D. J. Mc Adam, 242, 248; Artemas Martin, 246; E. B. Opdycke, 242; K. S. Putnam, 242; P. Richardson, 241, 242, 245, 246; Prof. M. C. Stevens, 241, 242, 246; E. B. Seitz, 241, 242, 243, 244, 245, 246, 247, 248; Prof. J. Scheffer, 242, 243, 244, 245, 246, 248.

241. "From any point within a parallelogram to draw three straight lines which shall divide the parallelogram into three equal parts."

CONSTRUCTION BY PROF. W. P. CASEY, SAN FRANCISCO, CAL.

Let O be the given point and through it draw hg parallel to AD , and draw On , Om trisecting the parallelogram $hBCg$; and also, Or , Op trisecting the parallelogram $AhgD$, by a well known problem; then draw Ox , making the triangle $Omz =$ the triangle Orp ; then the three lines On , Ox , Or trisect the parallelogram. If the triangle OmC be less than Orp , make the triangle $OCx =$ their difference; then the line Ox will intersect the line CD .



[Prof. Stevens gives a very elegant, but more complicated, construction. Mr. Seitz shows that the question is indeterminate, and therefore admits of any number of *different* constructions.]

242. "The difference between the areas of the inscribed and circumscribed circles of an equilateral triangle is a . Required the side of the triangle."

SOLUTION BY K. S. PUTNAM, ROME, NEW YORK.

The area of a ring $= \pi(R^2 - r^2)$. $R = AE$, $r = DE$ and $AE^2 - DE^2 = AD^2$.

We therefore have

$$\pi \cdot AD^2 = a;$$

$$AD = \sqrt{\frac{a}{\pi}};$$

and

$$AB = 2AD = 2\sqrt{\frac{a}{\pi}}.$$



243. "Adapt $\sqrt{a^2 + b^2}$ and $\sqrt{a+b} + \sqrt{a-b}$, to computation by the usual logarithmic tables."

SOLUTION BY W. E. HEAL, WHEELING, IND.

1. We have $\sqrt{a^2 + b^2} = a\sqrt{1 + (b^2/a^2)}$
 $\sqrt{1 + (b^2/a^2)} = \sqrt{1 + \tan^2 \theta} = \sec \theta;$

if $b/a = \tan \theta$ Then
 $\log a + \log \sec \theta;$

2. We have $\sqrt{a+b} + \sqrt{a-b} = \sqrt{a} [\sqrt{1+(b/a)} + \sqrt{1-(b/a)}]$. Put $b/a = \cos \varphi$. Then $\sqrt{1+(b/a)} + \sqrt{1-(b/a)} = \sqrt{1+\cos \varphi} + \sqrt{1-\cos \varphi} = (\cos \frac{1}{2}\varphi + \sin \frac{1}{2}\varphi) \cdot \sqrt{2} = 2 \sin (45^\circ - \frac{1}{2}\varphi)$. (Chauvenet's Trig., Equations 138, 139 and 149.)

244. "Prove that $1^9 + 2^9 + 3^9 + 4^9 + \dots + n^9 = \frac{1}{5}(16s^5 - 20s^4 + 12s^3 - 3s^2)$, where $s = 1 + 2 + 3 + 4 + \dots + n$."

SOLUTION BY E. B. SEITZ.

Let $S = 1^9 + 2^9 + 3^9 + 4^9 + \dots + n^9 \dots (1)$. Assume
 $S = As(s-1)(s-3)(s-6)(s-10) + Bs(s-1)(s-3)(s-6)$
 $\quad \quad \quad + Cs(s-1)(s-3) + Ds(s-1) + s. \quad (2)$
 When $n = 2, s = 3$, and $6D + 3 = 1^9 + 2^9 = 513; \quad \therefore D = 85.$
 " $n = 3, s = 6$, and $90C + 30D + 6 = 1^9 + 2^9 + 3^9 = 20196;$
 $\quad \quad \quad \therefore C = 196.$
 " $n = 4, s = 10$, and $2520B + 630C + 90D + 10 = 282340;$
 $\quad \quad \quad \therefore B = 60.$
 " $n = 5, s = 15$, and $113400A + 22680B + 2520C + 210D + 15$
 $\quad \quad \quad = 2235465; \therefore A = 3\frac{1}{5}.$

Hence by substitution and reduction (2) becomes

$$S = \frac{1}{5}[16s^5 - 20s^4 + 12s^3 - 3s^2].$$

SOLUTION BY HENRY GUNDER, NORTH MANCHESTER, IND.

Assume $1^9 + 2^9 + 3^9 + \dots + n^9 = a + bn + cn^2 + dn^3 + \dots + ln^{10}. \quad (1)$
 Substitute $n+1$ for n , and subtract, and we get
 $(n+1)^9 = b + c[(n+1)^2 - n^2] + d[(n+1)^3 - n^3] + \dots + l[(n+1)^{10} - n^{10}]. \quad (2)$
 Expanding (2) and equating the coefficients of like powers of n , we find
 $l = \frac{1}{10}, k = \frac{1}{2}, i = \frac{3}{2}, h = 0, g = -\frac{7}{10}, f = 0, e = \frac{1}{2}, d = 0, c = -\frac{3}{20},$
 $b = 0$; therefore

$$1^9 + 2^9 + 3^9 + \dots + n^9 = a - \frac{3}{20}n^2 + \frac{1}{2}n^4 - \frac{7}{10}n^6 + \frac{3}{2}n^8 + \frac{1}{2}n^{10} + \frac{1}{10}n^{10}. \quad (3)$$

Putting $n = 1$ in (3), $a = 0$. The second member is easily put under the form $\frac{1}{5}[16s^5 - 20s^4 + 12s^3 - 3s^2]$, where $s = 1 + 2 + 3 \dots + n = \frac{1}{2}n(n+1)$.

245. "The three transversals $AA' BB' CC'$ of a triangle ABC intersect each other in *one* point, cutting the sides of the the angle at $A' B', C'$. If the angles BAA', ACC' and CBB' are equal, compute the transversals AA', BB', CC' by means of the sides a, b, c , of the triangle."

SOLUTION BY P. RICHARDSON, BROOKLYN, NEW YORK.

Let ABC represent the triangle and O the point where the transversals meet. Produce the transversals to meet the circumscribing circle in D, E, F ; Join AD, BE and CF . Let the sides opposite the angles A, B, C , be, respectively, a, b, c .

Put $BO = x, CO = y, AO = z$. It is easily shown that angle $BOC = A + B$, $AOB = B + C$, $COA = C + A$. The double area of triangle is, $xy \sin C + xz \sin B + yz \sin A = bc \sin A$. But $\sin B = (b \div a) \times \sin A$, $\sin C = (c \div a) \sin A$; hence, by subst'n and red., $cxy + bxz + ayz = abc$. (1)



The double area of BOC is $xy \sin C = A'O(x \sin B + y \sin A)$, hence, by substitution, &c., as above,

$$AO' = \frac{cxy}{bx + ay}, \text{ hence } AA' = z + A'O = z + \frac{cxy}{bx + ay} = \frac{cxy + bxz + ayz}{bx + ay} = \frac{abc}{bx + ay}.$$

Similarly, $BB' = \frac{abc}{bz + cy}$, and $CC' = \frac{abc}{cx + az}$. (2)

The similar triangles ABC, AOD, BOE and COF , give $AD = (cy \div a)$, $BE = (az \div b)$, $CF = (bx \div c)$. Hence (because $AD = BE = CF$), $(cy \div a) = (az \div b) = (bx \div c)$. Hence, by substitution (1) gives

$$x = \frac{ac^2}{\sqrt{(a^2b^2 + a^2c^2 + b^2c^2)}}, y = \frac{a^2b}{\sqrt{(a^2b^2 + a^2c^2 + b^2c^2)}}, z = \frac{b^2c}{\sqrt{(a^2b^2 + a^2c^2 + b^2c^2)}}$$

Substituting these values for x, y, z , in (2) we get

$$AA' = \frac{c\sqrt{(a^2b^2 + a^2c^2 + b^2c^2)}}{a^2 + c^2},$$

$$BB' = \frac{a\sqrt{(a^2b^2 + a^2c^2 + b^2c^2)}}{a^2 + b^2},$$

$$CC' = \frac{b\sqrt{(a^2b^2 + a^2c^2 + b^2c^2)}}{b^2 + c^2}.$$

246. "Show that the sum of five consecutive integral square numbers cannot be a square."

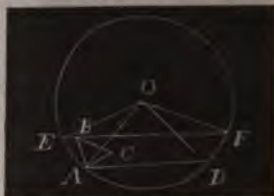
SOLUTION BY PROF. M. C. STEVENS, LA FAYETTE, INDIANA.

Let $(a-2)^2$, $(a-1)^2$, a^2 , $(a+1)^2$ and $(a+2)^2$ be any five consecutive square numbers; then $5(a^2+2)$ is their sum. Now as 5 is a factor of this number, if it is a square number it must end in either 00 or 25, and consequently a^2+2 must end in either 0 or 5. But a^2 must end in either 0, 1, 4, 5, 6 or 9; hence a^2+2 cannot end in either 0 or 5, and hence $5(a^2+2)$ cannot be a square number.

247. "Three points are taken at random on the surface of a circle; find the chance that the triangle formed by joining them is acute."

SOLUTION BY PROF. A. B. EVANS, LOCKPORT, NEW YORK.

It can easily be shown that the probability required will remain unchanged if one of the three points be restricted to a fixed point in the circumference of a circle whose radius is unity, and the other two points, to the surface of such circle.



In any position of the three points, let r represent the distance from the centre of the given circle of the point farthest therefrom; the other two points will evidently be confined to the area of a concentric circle whose radius is r . If p be the probability of an acute triangle when one point is thus confined to the circumference of a circle, radius r , while the other two are within that circle, the value of p will evidently be independent of r . Making therefore the farthest point move over every point of the area of the given circle and, in each of its positions, giving every possible position to each of the other two points, we find, for each position of the farthest point, the probability of an acute triangle to be p . The total probability is therefore evidently p .

Let one of the points, as A , be fixed on the circumference of a circle, radius unity, concentric with the given circle, and let the other two points, B and C , be taken at random within said circumference. Let p_a , p_b , p_c be the separate probabilities that A , B , C are obtuse; then, since the triangle can have only one obtuse angle, $p = 1 - (p_a + p_b + p_c)$.

To find the chance that A is obtuse fix B and draw AD perpendicular to AB ; then the chance required is

$$\frac{\text{area of segment } AHD}{\text{area of circle}}.$$

Let $AO = 1$, $\rho = AB$, $\theta = \angle OAB$, and $\varphi = \frac{1}{2} \angle EOF$; then the segment $AHD = \theta - \sin \theta \cos \theta$, and the chance $(\theta - \sin \theta \cos \theta) \div \pi$.

Now supposing B to move over the whole area of the circle, we have for the probability that A is obtuse

$$\begin{aligned} p_a &= \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \int_0^{2 \cos \theta} (\theta - \sin \theta \cos \theta) \rho d\theta d\rho \\ &= \frac{4}{\pi^2} \int_0^{\frac{\pi}{2}} \theta (\cos^2 \theta - \sin \theta \cos^3 \theta) d\theta \\ &= \frac{4}{\pi^2} \left(\frac{\pi^2}{16} - \frac{1}{2} \right) = \frac{1}{4} - \frac{2}{\pi^2}. \end{aligned}$$

To find the chance that B is obtuse, fix B and draw EBF perpendicular to AB ; then the chance that B is acute is

$$\frac{\text{area of segment } EHF}{\text{area of circle}}.$$

Now, area of segment $EHF = \theta - \sin \theta \cos \theta$, and chance $= (\theta - \sin \theta \cos \theta) \div \pi$. Hence the total probability that B is acute,

$$1 - p_b = \frac{2}{\pi^2} \int_0^{\frac{\pi}{2}} \int_0^{2 \cos \theta} (\theta - \sin \theta \cos \theta) \rho d\theta d\rho.$$

Since $\cos \varphi = \cos \theta - \rho$, $d\rho = -\sin \varphi d\varphi$ and $\rho d\rho = (\cos \theta - \cos \varphi) \sin \varphi d\varphi$;

$$\begin{aligned} 1 - p_b &= \frac{2}{\pi^2} \int_0^{\frac{\pi}{2}} \int_{\theta}^{\pi-\theta} (\theta - \sin \theta \cos \theta) \sin \varphi (\cos \theta - \cos \varphi) d\varphi d\theta \\ &= \frac{2}{\pi^2} \left(\frac{9\pi^2}{32} + \frac{1}{2} \right) = \frac{9}{16} + \frac{1}{\pi^2}; \\ \therefore p_b &= \frac{7}{16} - \frac{1}{\pi^2}. \end{aligned}$$

The probability that C is obtuse is also $\frac{7}{16} - (1 \div \pi^2)$; for p_b and p_c are obviously equal. Now, $p_a + p_b + p_c = \frac{3}{8} - (4 \div \pi^2)$; therefore

$$p = 1 - \left(\frac{9}{8} - \frac{4}{\pi^2} \right) = \frac{4}{\pi^2} - \frac{1}{8} = \text{the probability that the}$$

triangle ABC is acute.

NOTE.—If three points be taken at random in a *given plane*, the probability that they will be the vertices of an acute triangle is also $(4 \div \pi^2) - \frac{1}{8}$.

248. "If in the plane of xy the directions of the forces P and P' make with the axis of x the angles a and a' , and the direction of their resultant the angle A , and if we denote by p , p' and r the perpendiculars from the origin on these directions, we have the relation,

$$r \sin (a' - a) + p \sin (A - a') + p' \sin (a - A) = 0."$$

SOLUTION BY WALTER SIVERLY, OIL CITY, PA.

Let R = the resultant; then $Pp + P'p' = Rr \dots (1)$. Also, resolving forces parallel and perpendicular to the resultant,

$$P \cos(a-A) + P' \cos(A-a') = R \dots (2), \quad P \sin(a-A) - P' \sin(A-a') = 0. \quad (3)$$

Elim'g R from (1), (2), $P \cos(a-A) + P' \cos(A-a') = (Pp + P'p') \div r$. (4)

From (3), $P = P' \sin(A-a') \div \sin(a-A)$. Substituting in (4), observing that $\sin(A-a') \cos(a-A) + \cos(A-a') \sin(a-A) = \sin(a-a') = -\sin(a'-a)$,

$$r \sin(a'-a) + p \sin(A-a') + p' \sin(a-A) = 0.$$

SOLUTION BY PROF. ASAPH HALL.

Let L be the distance from the origin to the intersection of the directions of P and P' , and α the angle which this line makes with the axis of x .

$$\text{We have } p = L \sin(a-\alpha): p' = L \sin(\alpha-a'): r = L \sin(\alpha-A).$$

If we put $x = L \sin \alpha$: $y = L \cos \alpha$: these equations take the form

$$p = x \cos \alpha - y \sin \alpha: p' = x \cos \alpha' - y \sin \alpha': r = x \cos A - y \sin A.$$

The elimination of x and y from these equations gives the relation sought.

249. "Representing the amount of an insolvent estate by d , A has a claim of a , B a claim of b and C a claim of c against it. A's claim has a legal priority over B's, B's a priority over C's and C's a priority over A's. How should the estate be divided? Consider the several cases when the estate is insufficient to pay any two of the claims, when it is sufficient to pay any two, etc."

SOLUTION BY GEORGE EASTWOOD, SAXONVILLE, MASS.

A can claim the order of priority a, b, c ; let a', b', c' denote the parts of the estate d found on that supposition. Then

$$\text{B can claim } b'', c'', a''; \quad \text{C can claim } c''', a''', b''.$$

$$\begin{aligned} \text{Now} \quad & a' + a'' + a''' = \text{A's relative claim;} \\ & b' + b'' + b''' = \text{B's " " "}; \\ & c' + c'' + c''' = \text{C's " " "}. \end{aligned}$$

The sum of these is obviously equal to $3d$. Hence

$$\frac{1}{3}(a' + a'' + a''') = \text{A's portion}, \quad (1)$$

$$\frac{1}{3}(b' + b'' + b''') = \text{B's " " "}, \quad (2)$$

$$\frac{1}{3}(c' + c'' + c''') = \text{C's " " "}. \quad (3)$$

Ex. Suppose A's claim (a) = \$500; B's claim (b) = \$300; C's claim (c) = \$200, and the estate (d) = \$400. Then A can claim

$$\left\{ \begin{array}{ccc} a' & b' & c' \\ 400 & 0 & 0 \end{array} \right\}, \text{ B can claim } \left\{ \begin{array}{ccc} b'' & c'' & a'' \\ 300 & 100 & 0 \end{array} \right\}, \text{ C can claim } \left\{ \begin{array}{ccc} c''' & a''' & b''' \\ 200 & 200 & 0 \end{array} \right\};$$

∴ by (1) A's portion = $\frac{1}{3}(400+000+200) = 200$,

" (2) B's portion = $\frac{1}{3}(000+300+000) = 100$,

" (3) C's portion = $\frac{1}{3}(000+100+200) = 100$.

Net estate (d) = \$400.

PROBLEMS.

250. By L. REGAN, *Boonsboro, Iowa*.—Divide the line AB , geometrically, into three parts that shall be in harmonic proportion.

251. By DR. H. EGGERS, *Milwaukee, Wisconsin*.—Show that when the two lines, which bisect two angles of a triangle, are equal the triangle is isosceles.

252. By W. E. HEAL, *Wheeling, Ind.*—If the roots of a given cubic equation be not *real* and *positive* show that the equation can be transformed into another, of the same degree, in which *all* the roots are real and positive.

253. By PROF. E. J. EDMUNDS, *New Orleans, La.*—If $f(x)$ be a function whose roots are all real, show that the differential of the second order of that function has all its roots imaginary.

254. By ARTEMAS MARTIN, M. A., *Erie, Pa.*—Integrate

$$dI = xE(e, x)dx,$$

where $E(e, x)$ denotes an elliptic arc, eccentricity e and abscissa x .

255. By PROF. JOHNSON.—1. In a triangle ABC the angle $A = \varphi + \alpha$, $B = 2\varphi$; supposing AB to remain fixed, while φ varies, it is required to find the rectangular equation to the locus of C , and the equations to the asymptotes.

2. With the same data as above, taking A as the origin, and AB as the axis of x ; let $\alpha = 45^\circ$, and find the envelop of a straight line which passes through C and makes an angle $4\varphi + 90^\circ$ with Ax .

256. By PROF. H. T. EDDY, *Cincinnati, Ohio*.—If the given quantities x_1, x_2, x_3, x_4 have the probable errors r_1, r_2, r_3, r_4 , respectively, find the probable error r of the quantity x when $x_1 : x_2 :: x_3 + x : x_4 + x$.

257. By E. B. SEITZ, *Greenville, Ohio*.—Within a triangle ABC , determine a point P , such that $m.PA + n.PB + r.PC$ shall be a minimum, m, n, r being constants.

258. By J. M. ARNOLD, *Boston, Mass.*—The Balance of a watch is slightly heavier on one side than on the other. When the watch is placed vertically with the pendant up, the heavy point is at the lowest side when the balance is at rest.

If the watch be started with a small arc of vibration the rate will be much faster, in the vertical position, with the pendant up than with the pendant down. If the motive power be increased so that the arc of vibration becomes large, say one and a half revolutions or 270° each side the point of rest, it will be found to go faster, in the vertical position, pendant down than it does with the pendant up.

What arc of vibration must be given the balance, so that its rate shall be the same in any vertical position?

QUERY 1. By PROF. HALL.—When we descend below the surface of the Earth, does the Earth's attractive force increase or diminish?

QUERY 2. By MR. KUMMELL.—Let $\cos^2\varphi$ be the probability of a single event, hence $\sin^2\varphi$ that of its non-occurrence. Supposing m simultaneous events of either kind, then we have for the mean square of the difference in number, after an infinite number of trials, the expression:

$$\begin{aligned} \epsilon_m^2 &= \cos^{2m}\varphi(m)^2 + \frac{m}{1} \cos^{2m-2}\varphi \sin^2\varphi(m-2)^2 + \frac{m}{1} \frac{m-1}{2} \cos^{2m-4}\varphi \sin^4\varphi(m-4)^2 + \dots \\ &+ \frac{m}{1} \frac{m-1}{2} \dots \frac{m-n+1}{n} \cos^{2m-2n}\varphi \sin^{2n}\varphi(m-2n)^2 + \dots \sin^{2m}\varphi(-m)^2 = m(\sin^2 2\varphi + m \cos^2 2\varphi) \end{aligned}$$

Since $m(\cos^2\varphi - \sin^2\varphi) = m \cos 2\varphi$ is the most probable difference in the number of the two kinds of events, we have $\sin 2\varphi/\sqrt{m}$ for the mean error with respect to the most probable difference. How is this proved otherwise than by induction?

PUBLICATIONS RECEIVED.

Physical and Mathematical Principles of the Nebular Theory; 10 pages, 8vo:

The Origin of the Power which Causes the Stellar Radiations; 10 pages, 8vo:

Elements of Sideral Astronomy; 7 pages, 8vo: By JACOB ENNIS, A. M., Professor of Natural Science in the State Normal School, Shippensburg, Pa.

ERRATA.

On page 47, line 5, insert $\frac{1}{2}$ before the sign of integration.

" " " " 7, multiply the term under the sign of integration by $d\varphi$.

" " " " 9, for $n-3$, read $n-2$.

THE ANALYST.

VOL. VI.

MAY, 1879.

No. 3.

ON THE DEVELOPMENT OF $[p + (1-p)]^n$.

BY E. L. DE FOREST, M. A., WATERTOWN, CONN.

It is well known that when p and q denote respectively the probabilities that an event will happen and that it will fail to happen, in a single trial, so that $p + q = 1$, then if m trials are to be made independently of each other, the $m+1$ terms of the development

$$(p+q)^m = p^m + \frac{m}{1} p^{m-1} q + \frac{m(m-1)}{1.2} p^{m-2} q^2 + \dots + q^m, \quad (1)$$

are the several probabilities that the event will happen any number of times from m down to 0, and that it will fail to happen a corresponding number of times from 0 up to m . (See for instance Merriman's *Least Squares*, p. 123.) The terms of the series increase in value from either extremity toward an intermediate point where they attain a maximum. If we regard them as equidistant ordinates to a curve, then as m is increased, the form of this curve approximates rapidly to the probability curve

$$y = ce^{-h^2 x^2},$$

and this is found to be the limiting form, when m becomes infinite. I propose here to demonstrate it in a manner rather more simple than any I have met with, and also to consider some points not usually noticed in connection with it.

Let the terms of the development be denoted by y_0, y_1, y_2 &c., so that

$$(p+q)^m = y_0 + y_1 + y_2 + \dots + y_m.$$

Comparing this with (1) we see that

$$\frac{y_1 - y_0}{y_0} = \frac{q}{p} - 1, \quad \frac{y_2 - y_1}{y_1} = \frac{q(m-1)}{2p} - 1, \quad \frac{y_3 - y_2}{y_2} = \frac{q(m-2)}{3p} - 1,$$

and in general
$$\frac{y_{i+1} - y_i}{y_i} = \frac{q(m-i)}{p(i+1)} - 1. \quad (2)$$

When y_i is denoted simply by y , and the constant interval between two consecutive ordinates is Δx , and the origin is taken at the first term, then

$$y_{i+1} - y_i = \Delta y, \quad x = i\Delta x,$$

and (2) reduces to

$$\frac{\Delta y}{y} = \frac{(qm-p)\Delta x - x}{p(x+\Delta x)}. \quad (3)$$

The largest term in the series (1) is that term whose rank is $i = qm$, as can be ascertained by trial in any particular case where qm is a whole number, and also by considering that the term in question is then

$$\frac{\Gamma(m+1)}{\Gamma(pm+1)\Gamma(qm+1)} p^{pm} q^{qm},$$

being the probability that the event will happen pm times and fail qm times, while the terms next preceding and following are

$$\frac{\Gamma(m+1)}{\Gamma(pm+2)\Gamma(qm)} p^{pm+1} q^{qm-1}, \quad \frac{\Gamma(m+1)}{\Gamma(pm)\Gamma(qm+2)} p^{pm-1} q^{qm+1}.$$

Dividing each of these two by the former, we find the quotients

$$\frac{pm}{pm+1} \quad \text{and} \quad \frac{qm}{qm+1},$$

both of which are less than unity. We can always make qm a whole number by taking m sufficiently large.

In (3) let the origin be transferred from the first term in the series to the greatest term, by putting $x+qm\Delta x$ in place of x ; then

$$\frac{\Delta y}{y} = \frac{-(x+p\Delta x)}{p(qm\Delta x+x+\Delta x)}. \quad (4)$$

If m becomes infinite, and the successive ordinates are set close together so as to be consecutive, then at the limit Δx and Δy become dx and dy , and dx and pdx vanish in comparison with x , so that (4) reduces to

$$\frac{dy}{y} = \frac{-x}{p(qmdx+x)}. \quad (5)$$

To find the equation of the curve from this by integration, we must observe that the rules of the integral calculus presuppose that a differential infinitely smaller than the function from which it is derived. If the numerator dy in the first member is to be infinitely smaller than the denominator y , it will follow that in the second member, the numerator x must be infinitely less than $qmdx+x$, and consequently the x in the denominator is infinitely less than $qmdx$, and may be neglected, so that we may write

$$\frac{dy}{y} = \frac{-x}{pqmdx}. \quad (6)$$

But since x is finite, it cannot be infinitely less than $pqmdx$ unless mdx is infinite, and this cannot be unless m is an infinity of the second order, that is, of a magnitude comparable to $1 \div (dx)^2$. To make it such, we must suppose that the series extends throughout the whole infinite length of the axis of X . For dx is the interval between any two consecutive terms or ordinates, and it is contained ∞ times in a finite distance, say the unit of x , and this unit is contained ∞ times in the whole axis of X , so that dx is contained ∞^2 times in the whole axis, and thus the whole number of terms in the series becomes $m+1 = \infty^2+1$. Let (6) then be written

$$\frac{dy}{y} = \frac{-xdx}{pqm(dx)^2} \quad (7)$$

The denominator of the second member does not vary with x , and $m(dx)^2$ is a constant and finite area. We will express the denominator by means of a new constant h , so that

$$h = 1 \div [dx \sqrt{2pqm}], \quad (8)$$

and (7) becomes

$$\frac{dy}{y} = -2h^2 x dx,$$

and integration gives $\log' y = -h^2 x^2 + \log' c$; $\therefore y = ce^{-h^2 x^2}$. (9)

Since $p+q=1$, we have $(p+q)^m = 1$, and the sum of all the terms in the series (1) is unity, wherefore the sum of all the consecutive values of y in the curve (9) is also unity, if that curve fully represents the limit of the series. Multiplying the sum of the ordinates, and unity, each by dx , we have the condition

$$\int_{-\infty}^{\infty} y dx = dx; \therefore c \int_{-\infty}^{\infty} e^{-h^2 x^2} dx = dx. \quad (10)$$

The known value of the definite integral here is $\sqrt{\pi} \div h$. Thus (10) determines the constant

$$c = h dx \div \sqrt{\pi}, \quad (11)$$

and substituting it in (9), we get

$$y = \frac{h dx}{\sqrt{\pi}} e^{-h^2 x^2}, \quad (12)$$

the final equation of the limiting curve. We infer from (8) that h is the reciprocal of a finite line, so that $h^2 x^2$ is an abstract number, and finite as long as x is finite. It follows that y is an abstract number, as a probability should be, and is an infinitesimal of the first order.

Let i now be reckoned as x is, from the place of the maximum probability. As before, $x = i dx$ and at the limit $x = i dx$. Substituting in (12) this value of x and also the value of h from (8), we find

$$y = (2\pi pqm)^{-\frac{1}{2}} e^{-\frac{1}{2} \frac{i^2 + 2pqm}{m}} \quad (13)$$

When an infinite number m of trials are made, the most probable result, as we have seen, is that the event will happen pm times and fail qm times, and (13) gives the probability that it will happen exactly $pm+i$ times and fail $qm-i$ times. This formula holds good approximately also when m is a large finite number, provided that i is considerably smaller than pm or qm , a proviso which results from the fact that m in the formula is an infinity of the second order, while i is an infinity of the first order, since $i = x \div dx$, and x is finite.

Since y is the probability that the number of times the event happens will deviate from the most probable number pm by a certain number i , we may regard $x = idx$ as a deviation of position, or error of position, and regard y in general as the probability that any observed quantity, subject to accidental causes of error, will deviate from its most probable value by a certain quantity or error x . Equation (12) then is the equation of the probability curve, expressing the law of facility of error.

Let both its members be divided by dx , which is equivalent to multiplying them by a constant and infinitely large quantity. Denoting $y \div dx$ by Y , we have

$$Y = \frac{h}{\sqrt{\pi}} e^{-h^2 x^2}, \quad (14)$$

and since y is an infinitesimal of the first order, Y is finite. This is what is commonly known as the equation of the probability curve, but it is evident that here the probability of the error x is not the ordinate Y , but the elementary area $Ydx = y$. The whole area of the curve is

$$\int_{-\infty}^{\infty} Ydx = 1.$$

The probability that any error which occurs, when taken without regard to sign, will not exceed a certain limit x , is the sum of the probabilities of all the errors from $-x$ up to $+x$, and equals the area of the curve between those limits, or

$$P = \frac{h}{\sqrt{\pi}} \int_{-x}^x e^{-h^2 x^2} dx. \quad (15)$$

If we put $hx = t$, then $dx = dt \div h$, the limits $x = \pm x$ become $t = \pm hx$, and Y being a function of x^2 the curve is symmetrical on each side of the origin, so that

$$P = \frac{2}{\sqrt{\pi}} \int_0^t e^{-t^2} dt. \quad (16)$$

All works on probability contain the values of this integral arranged in a table, enabling us to find P for any given value of the argument $t = hx$.

It follows from (8) also that

$$t = i \div \sqrt{(2pqm)}, \quad (17)$$

so that such a table will give the sum of $2i$ values of y in (13), being the approximate probability that when a large finite number m of trials are made, and i is an integer or half-integer much smaller than pm or qm , the number of times the event will happen will be one of the group of $2i$ consecutive integers the middle of which falls nearest to the number pm .

We have hitherto supposed the constant h to be determined from (8) in terms of m , but in applying the theory of probability of errors it generally happens that there are given only a certain finite number of observed errors or deviations from the true value of a quantity or from some value which is regarded as the most probable one, while the whole number m of different possible errors is infinite or at least unknown. To find h from the observed errors, the best course generally is to compute the "mean error", or as it has been more properly called, the *quadratic error*, which is defined to be the square root of the mean of the squares of all the observed errors. In the most typical case, when the number of observed errors is very large or infinite, the square of the quadratic error is the quotient arising from dividing the sum of the squares of all the errors by the whole number of errors, that is

$$\epsilon^2 = \int_{-\infty}^{\infty} x^2 Y dx \div \int_{-\infty}^{\infty} Y dx, \quad (18)$$

each error x being supposed to occur a number of times proportional to its probability Ydx . The value of the divisor here is unity, as we have seen. The dividend is

$$\frac{h}{\sqrt{\pi}} \int_{-\infty}^{\infty} x^2 e^{-h^2 x^2} dx = \frac{1}{h^2 \sqrt{\pi}} \int_{-\infty}^{\infty} t^2 e^{-t^2} dt.$$

This last definite integral is known to have the value $\frac{1}{2}\sqrt{\pi}$. The quadratic error then is

$$\epsilon = 1 \div h\sqrt{2}; \quad \therefore h = 1 \div \epsilon\sqrt{2}. \quad (19)$$

Since ϵ^2 is the mean of all the observed values of x^2 , and $x = idx$, it follows that $(\epsilon \div dx)^2$ is the mean of all the observed values of i^2 . Substituting in (19) the value of h from (8), we get

$$\epsilon = dx \sqrt{(pqm)}; \quad \therefore \epsilon \div dx = \sqrt{(pqm)}. \quad (20)$$

This is the square root of the mean of the values of i^2 in (13), bearing the same relation to deviations i from the most probable number pm , as ϵ bears to deviations x from the most probable value of any observed quantity. It is of use in statistical investigations. (*Smithsonian Report* of 1873, p. 334.)

The constant h has been called the "measure of precision", because it varies inversely as the quadratic error; but in point of fact the accuracy of a

set of observations is usually estimated by the *probable error*, defined to be that error r , which occupies an intermediate position among all the observed errors taken without regard to sign, so that for an infinite number of errors the number greater than r is equal to the number less than r , and the probability that any error which occurs will be, for example, less than r , is $\frac{1}{2}$. By interpolation in a table of values of P , the value of $t = hx$ corresponding to $P = \frac{1}{2}$ is found to be

$$hr = 0.4769,$$

and substituting the value of h from (19) we get

$$r = .6745 \epsilon, \quad (21)$$

by which the probable error is computed from the quadratic error. We also have by (20)

$$r \div dx = .6745 \sqrt{pqm}, \quad (22)$$

which is the probable or intermediate value of i , so that it is an even chance that the event will happen a number of times not deviating from the most probable number pm by more than this probable value, when a large number m of trials are made.

It should be noticed, however, that the series is in general symmetrical on both sides of the maximum probability or axis of Y , only when m is infinite. If m is finite, the terms of the series are not identical on opposite sides of that axis, and the probable error or deviation is somewhat greater on one side, and less on the other, than the value which (22) gives. It is greatest on that side which is toward the more remote end of the series. But in the special case where $p = q = \frac{1}{2}$, the maximum term is in the middle of the series, and the probable errors are alike on both sides, whether m be infinite or finite. One of the usual elementary proofs of the law of probability demonstrates it as the limit of the development of $(\frac{1}{2} + \frac{1}{2})^{2n}$. (See for instance Price's *Calculus*, Vol. II., p. 376.)

Let us now return and consider the validity of our treatment of formula (5), from which (6) was obtained by neglecting the finite x in comparison with the infinite $qmdx$, where mdx represents the whole length of the axis of X , or the whole distance occupied by the series (1) when m is an infinity of the second order. It is evident, in the first place, that dy is not infinitely smaller than y in every part of the series. On the contrary, it is sometimes of a magnitude equal or comparable to that of y , and sometimes even infinitely greater than y , as appears from (5).

When we put successively $x = -qmdx + a$, $x = pmdx - a$, where a is finite and represents distance from either end of the series, (5) becomes

$$\frac{dy}{y} = \frac{qmdx - a}{ap} \quad \text{and} \quad \frac{dy}{y} = -\frac{(pmdx - a)}{(pmdx - ap)}.$$

As mdx is infinite, $dy \div y$ is infinite in the one case and -1 in the other, showing that, of two consecutive ordinates at any finite distance from either end of the series, that one which is nearest the end is infinitely smaller than the other one. Therefore, in going from either end toward the middle, the ratio of dy to y passes from infinity at any finite distance from the end, through intermediate and finite values, down to infinitesimal values for any finite distance from the point where y is a maximum; and it is only to this latter region that formula (6) applies. Have we a right, then, to say that the terms outside of this region are so small that their sum is infinitesimal, and that the sum of the terms within it is therefore equal to unity? For we assumed this, when we determined the constant c in (9) by the condition (10). It would hardly seem to be self-evident, but there are facts which confirm its truth, and one of them is reached by computing the quadratic error for a finite series, as follows.

In (1) let p , q and m have any consistent values, for instance

$$p = \frac{1}{8}, \quad q = \frac{2}{3}, \quad m = 4,$$

so that

$$(p+q)^m = \frac{1}{81}(1+8+24+32+16).$$

The second member is a series of five fractions which are the probabilities of the five different possible results, and may be represented by equidistant ordinates, the constant interval between them being Δx . Let the origin be taken at the point whose distance from the first ordinate is $qmdx = \frac{8}{9}\Delta x$. This point falls two thirds of the way from the term 24 to the next term 32. The negative errors or deviations then are

$$-\frac{2}{3}\Delta x, \quad -\frac{5}{9}\Delta x, \quad -\frac{8}{9}\Delta x,$$

and the positive ones are

$$\frac{1}{9}\Delta x, \quad \frac{4}{9}\Delta x.$$

The square of the quadratic error is the mean of the squares of all the errors, each one being supposed to occur a number of times proportional to y , so that

$$\epsilon^2 = \frac{\frac{1}{81}[1(\frac{8}{9})^2 + 8(\frac{5}{9})^2 + 24(\frac{2}{3})^2 + 32(\frac{1}{9})^2 + 16(\frac{4}{9})^2](\Delta x)^2}{\frac{1}{81}(1+8+24+32+16)},$$

from which we find

$$\epsilon^2 = \frac{8}{9}(\Delta x)^2 = pqm(\Delta x)^2.$$

In like manner, for all other consistent values of p , q and m which have been tried, it has been found that the quadratic error is invariably $\Delta x \sqrt{(pqm)}$. Hence we infer that at the limit, when m is an infinity of the second order, and Δx is reduced to dx

$$\epsilon = dx \sqrt{(pqm)}.$$

But this is the value which has already been found in (20) by integrating from the probability curve. In other words, the summation of x^2y for the whole of the infinite series gives the same result as its summation from the

curve (12) which can represent truly only a portion of the series; while the sum of y is unity in both cases. We conclude then that in all those portions of the infinite series to which (6) and (12) do not apply, the terms y are so small that the total of x^2y , and consequently the total of y , is infinitesimal.

This method of finding the expression (20) for the quadratic error inductively from the finite series is one which I have not seen in any of the published works on probability.

I take this opportunity to make a correction in my article on the Limit of Repeated Adjustments (*ANALYST*, Sept. 1878), where the curve, whose ordinates y are the coefficients of the limiting resultant formula of adjustment, was shown to be a probability curve, or a higher curve of analogous nature, and the condition that the sum of all the consecutive ordinates must be unity was employed in determining the constant parameters of the curves.

This condition was regarded as being expressed by

$$\int_{-\infty}^{\infty} y dx = 1,$$

whereas it should have been

$$\frac{1}{dx} \int_{-\infty}^{\infty} y dx = 1,$$

as we have seen in connection with formula (10) of the present paper. The consequence is, that the expressions for y obtained in formulas (10), (14) and (19) of the article referred to, are really expressions for $Y = y \div dx$, and the numerical values given for y in Tables I. and III. are really values of Y . The correction is one that affects the theory of the matter rather than the practical results obtained. It has been a fault of long standing with writers on the probability curve, to slur over the distinction between the infinitesimal probability here denoted by $y = Ydx$, and the finite ordinate Y which is proportional to y . When y is regarded as an ordinate, it belongs to a curve whose total area is dx ; but Y belongs to one whose area is unity.

It may be well to add a remark upon the three formulas (6), (7) and (8) of the article referred to, where k represents the number of applications of the adjusting process, and is supposed to become infinite at the limit. The infinity in these three cases must be of the second, fourth and sixth orders respectively, as appears from considerations similar to those arising in formula (6) of the present paper, where m becomes an infinity of the second order. There cannot well be any limit to the increase of an abstract number. If the number ∞ is a supposable quantity, then ∞^2 , ∞^4 and ∞^6 are supposable also.

For example, if dx is contained ∞ times in the unit of x , and this unit of x is contained ∞ times in the whole axis of X , then dx is contained ∞^2 times in the whole axis of X , and $(dx)^2$ is contained ∞^4 times in the infinite plane XY , and $(dx)^3$ is contained ∞^6 times in the infinite space XYZ .

The concrete result in the three cases of adjustment appears to be this, that for a finite number of applications, the limiting form is approached more rapidly in the first case than in the second, and more rapidly in the second than in the third. Thus under the first case, p. 133, the coefficients for only 16 applications approach the limiting form more nearly than the coefficients for 32 applications do under the second case at p. 138, the limit being apparently farther off in the latter case.

NOTE.—When a series is adjusted by two formulas successively, as at p. 66 of *ANALYST* for May 1878, where the two are denoted by

$$u'_0 = l_0 u_0 + l_1(u_1 + u_{-1}) + l_2(u_2 + u_{-2}) + \dots + l_m(u_m + u_{-m}),$$

$$u''_0 = L_0 u'_0 + L_1(u'_1 + u'_{-1}) + L_2(u'_2 + u'_{-2}) + \dots + L_n(u'_n + u'_{-n}),$$

the coefficients in the resultant formula are the same as those which belong to the several powers of x in the product of the two polynomials

$$l_m + l_{m-1}x + l_{m-2}x^2 + \dots + l_0x^m + l_1x^{m+1} + \dots + l_mx^{2m},$$

$$L_n + L_{n-1}x + L_{n-2}x^2 + \dots + L_0x^n + L_1x^{n+1} + \dots + L_nx^{2n}.$$

This property, which I did not perceive at first, systematizes the matter and simplifies it to the mind, although it makes no difference in the actual amount of computation to be done, in forming the resultant coefficients.

Since the resultant for k successive adjustments by one formula has the same coefficients as the corresponding polynomial raised to the k power, we are enabled to demonstrate the proposition (3) at p. 129 of the *ANALYST* for Sept., 1878, by means of the Multinomial Theorem, as I will take occasion to show hereafter. The demonstration includes the general case where the adjustment formula is not symmetrical on each side of the middle, so that λ_r and λ_{-r} are not necessarily equal.

GENERAL RULES FOR THE FORMATION OF MAGIC SQUARES OF ALL ORDERS.

BY PROF. A. B. NELSON, CENTRE COLLEGE, DANVILLE, KY.

THE following general rules for the formation of magic squares, whether of odd or even degree, may be new to the younger generation of mathematicians, and possibly to some others. The squares constructed by these

methods by no means exhaust all the possible arrangements, but they furnish squares in great number and variety when the degree is even. For the sake of completeness, we will begin by stating the rule (which is probably familiar to all) for producing squares when the number of cells on a side is odd.

CASE 1.

17	24	1	8	15
23	5	7	14	16
4	6	13	20	22
10	12	19	21	3
11	18	25	2	9

We begin by placing 1 in the middle cell of the upper row and, using the natural series of numbers,* proceed always diagonally upward to the right, except when we come to the edge of the square, or to a cell already filled. When the number would fall outside of the square, carry it to the extreme cell in that row or column in which the outside cell occurs. When a cell is already filled, or when the right hand upper corner cell is reached, place the number in the cell just below.

CASE 2. When the number on a side is even and of the form $4n$.

Imagine the square to be subdivided into squarelets of 4 cells each, the four central cells comprising one, and conceive these squarelets as of *two* kinds alternating with each other. Place 1 in the left upper corner cell and proceed horizontally to the right, filling the squarelets of *one* kind successively. When the end of one row is reached go to the extreme left cell of the next row and again advance to the right as before. For the 4-square the result of this operation is as follows:

1			4
	6	7	
	10	11	
13			16

Next begin with the right lower corner cell, considering 1 as falling on this, and proceed regularly to the left, filling the empty cells with the numbers belonging to them. The result of this second operation is as follows:

1	15	14	4
12	6	7	9
8	10	11	5
13	3	2	16

*It is hardly necessary to state that in forming squares of this kind, any arithmetical series may be used.

CASE 3. When the number of cells on a side is even and of the form $4n+2$.

Taking the 10-square for a particular example, first arrange the numbers in it as follows:

I.

1	1	2 ₁	3 ₂	4	5 ₃	6 ₃	7	8 ₂	9 ₁	10
2	11 ₃	12	13 ₁	14 ₂	15	16	17 ₂	18 ₁	19	20 ₃
3	21	22 ₃	23	24 ₁	25 ₂	26 ₂	27 ₁	28	29 ₃	30
4	31 ₂	32	33 ₃	34	35 ₁	36 ₁	37	38 ₃	39	40 ₂
5	41 ₁	42 ₂	43	44 ₃	45	46	47 ₃	48	49 ₂	50 ₁
6	51 ₁	52 ₂	53	54 ₃	55	56	57 ₃	58	59 ₂	60 ₁
7	61 ₂	62	63 ₃	64	65 ₁	66 ₁	67	68 ₃	69	70 ₂
8	71	72 ₃	73	74 ₁	75 ₂	76 ₂	77 ₁	78	79 ₃	80
9	81 ₃	82	83 ₁	84 ₂	85	86	87 ₂	88 ₁	89	90 ₃
10	91	92 ₁	93 ₂	94	95 ₃	96 ₃	97	98 ₂	99 ₁	100

Beginning at the left upper corner, designate the cells in the first row and first column respectively by the numbers from 1 to 10. Then we can denote any cell by the numbers standing at the beginning of the column and row in which it occurs. Thus, for instance, the cell containing 44 is denoted by (4, 5); that is, it occupies the 4th column and 5th row.

Next form the following auxiliary square with $2n+1$ rows and $2n+1$ columns:

II.

1	10	2	9	3	8	4	7	5	6
2	9	3	8	4	7	5	6	1	10
3	8	4	7	5	6	1	10	2	9
4	7	5	6	1	10	2	9	3	8
5	6	1	10	2	9	3	8	4	7

In the first column place the numbers from 1 to 5, and to their right in the same column the complements of these numbers to $4n+3$. In the top cell of the next column place 2, and descending write the numbers 1 to 5 with their respective complements, preserving the cyclic order of the first column. At the top of the next column place 3, and so on.

The numbers in the first column of II are identified with those of the first row of I; those of the remaining columns of II are identified with those of the first column of I. The numbers of I which are designated by the combination of one of the numbers in the 1st column of II with one of the two numbers of the same row in the *last* column, are indicated by

the suffix 1; those designated by a number of the 1st column of II combined with one in the column *next* the last, are indicated by the suffix 2; those designated by a number of the 1st column of II with a number of any *one* of the other columns except the last two are indicated by the suffix 3.

For example, in I, 51 has suffix 1 because it corresponds to the combination (1, 6) from the first and last columns of II; 76 has the suffix 2 because the combination (5, 8) is taken from the first and next to the last columns of II; and 68 has the suffix 3 because the combination (8, 7) is from the first and second columns of II. Four kinds of numbers are thus distinguished: (1), those without any suffix, which keep their original places in the magic square III; (2), those with suffix 1, any corresponding four of which—that is any two pair equidistant from the sides—are interchanged in the following manner: the arrangement

$a \dots b$ $d \dots c$; (3), those with $a \dots b$ $d \dots a$
 $:$ $:$ becomes $:$ $:$ suffix 2 are in- $:$ $:$ becomes $:$ $:$
 $c \dots d$ $a \dots b$ $c \dots d$ $b \dots c$
 terch'd thus:

(Instead of the $c \dots d$ $b \dots c$
 change in (2) we $:$ $:$, and for the one in (3), $:$ $:$). For exam.,
 may substitute $a \dots b$ $a \dots d$

35..36 66..65 3..8 98..3
 $:$ $:$ becomes $:$ $:$; and $:$ $:$ becomes $:$ $:$ (4). The num-
 65..66 35..36 93..98 8..93

bers with suffix 3 are arranged as the second class in the previous case (4n).

III.

1	99	98	4	96	95	7	3	92	10
90	12	88	87	15	16	14	83	19	81
21	79	23	77	76	25	74	28	72	30
70	32	68	34	66	65	37	63	39	41
60	59	43	59	45	46	54	48	42	61
41	49	53	47	55	56	44	58	52	50
40	62	38	64	35	36	67	33	69	61
71	29	73	24	26	75	27	78	22	80
20	82	13	17	85	86	84	18	89	11
91	2	2	94	6	5	97	93	9	100

The foregoing rules are condensed from an article in Hoffman and Nattani's *Mathematisches Wörterbuch*, Vol. V.

The following is another set of rules for even squares, abridged from a pamphlet by S. M. Drach, F. R. A. S.

1°. When the side cells = $4n$. Put $c = 16n^2 - 8i$, where i is 0, 1, 2, 3, ... $n^2 - 1$. These substituted successively in this squarelet

$$\begin{aligned} &8i+1, c-1, c-2, 8i+4, \\ &c-4, 8i+6, 8i+7, c-7, \\ &8i+8, c-6, c-5, 8i+5, \\ &c-3, 8i+3, 8i+2, c, \end{aligned}$$

give n^2 squarelets, all *equal* magic squares; so that they may be arranged any how n in a row, or turned each on its centre, and still give the same linear sum $= 32n^2 + 2n$.

2°. When the side cells $= 4n + 2$. Fill up the $4n$ -square by the previous rule and then add $8n + 2$ to each of its numbers. Surround this last square by a mono-cellular border of which the upper and lower rows are respectively

$$\begin{array}{c} 1 \\ s-4n-2 \end{array} \left| \begin{array}{c} s-2 \\ cp \end{array} \right| \left| \begin{array}{c} 8n+2 \\ cp \end{array} \right| \left| \begin{array}{c} cp \\ 8n+1 \end{array} \right| \left| \begin{array}{c} \dots \\ \dots \end{array} \right| \left| \begin{array}{c} cp \\ 6n+5 \end{array} \right| \left| \begin{array}{c} 6n+4 \\ cp \end{array} \right| \left| \begin{array}{c} \text{omit } 6n+3 \end{array} \right| \\ \left| \begin{array}{c} cp \\ 6n+2 \end{array} \right| \left| \begin{array}{c} cp \\ 6n+1 \end{array} \right| \left| \begin{array}{c} 6n \\ cp \end{array} \right| \left| \begin{array}{c} \dots \\ \dots \end{array} \right| \left| \begin{array}{c} 4n+4 \\ cp \end{array} \right| \left| \begin{array}{c} cp \\ 4n+3 \end{array} \right| \left| \begin{array}{c} 4n+2 \\ s-1 \end{array} \right|, \end{array}$$

where $s = [(4n + 2)^2 + 1]$, and cp = the complement to s of the opposite number. The vertical rows are,

$$\begin{array}{ll} \text{left,} & 1, s-3, 4, s-5, \dots, s-4n-1, 6n+3, s-4n-2, \\ \text{right} & 4n+2, 3, s-4, 5, \dots, 4n+1, s-6n-3, s-1. \end{array}$$

Hence to form the border, place 1 in N. W. corner, $s-1$ at S. E. corner. $4n+2$ at N. E. corner, and $s-4n-2$ at S. W. corner, next to which put 2; and place $4n+3, 4n+4$, up to $6n+1$, alternately from bottom, leftward; place $6n+2$ next $6n+1$ (omit $6n+3$), $6n+4$ at top, $6n+5$ at bottom, and so on; complete s in opposite cells.

For verticals, under $4n+2$ put 3, 4 opposite, 5 E., 6 W., 7 E., &c., to $(4n+1)$ E., $(6n+3)$ W.; their complements to s in the opposite cells, and it is done.

Any one wishing to go deeper into this subject will find it treated in a highly general and rigorous manner in a work entitled *Die Magischen Quadrate*, by Theodore Hugel.

According to this author the 3-square admits of 1 arrangement, the 4-square of 24, the 5-square of 6624, the 8-square of 834075520, the 11-sq, of 11736148838400000, &c.

According to an approximate calculation, the paper that would be required to contain all the squares with 13 cells on a side would cover the whole surface of the earth about 348 times.

ON THE REMOVAL OF TERMS FROM AN EQUATION OF THE FIFTH DEGREE.

BY W. E. HEAL, WHEELING, IND.

LET us take the equation $x^5 + p_1x^4 + p_2x^3 + p_3x^2 + p_4x + p_5 = 0$. We propose to remove any *three* of the mean terms and then reduce the equation to any of the forms $x^5 + q_4x + q_5 = 0$, $x^5 + q_3x^2 + q_5 = 0$, $x^5 + q_2x^2 + q_5 = 0$, or $x^5 + q_1x^4 + q_5 = 0$.

Let $y = a + bx + cx^2 + dx^3 + ex^4$. We wish to eliminate x and thus form an equation in y . Since for every value of x there is a corresponding value for y , the equation in y will be of the fifth degree.

Denote the roots of the equation $x^5 + p_1x^4 + p_2x^3 + p_3x^2 + p_4x + p_5 = 0$ by x_1, x_2, x_3, x_4, x_5 and the corresponding values of y by y_1, y_2, y_3, y_4, y_5 .

Then we have

$$\left. \begin{aligned} y_1 &= a + bx_1 + cx_1^2 + dx_1^3 + ex_1^4 \\ y_2 &= a + bx_2 + cx_2^2 + dx_2^3 + ex_2^4 \\ y_3 &= a + bx_3 + cx_3^2 + dx_3^3 + ex_3^4 \\ y_4 &= a + bx_4 + cx_4^2 + dx_4^3 + ex_4^4 \\ y_5 &= a + bx_5 + cx_5^2 + dx_5^3 + ex_5^4 \end{aligned} \right\} \quad (1)$$

Let $\Sigma(x^n) = x_1^n + x_2^n + x_3^n + x_4^n + x_5^n$; $\Sigma(y^n) = y_1^n + y_2^n + y_3^n + y_4^n + y_5^n$.

Adding the five equations of (1) we obtain

$$\Sigma(y) = 5a + b\Sigma(x) + c\Sigma(x^2) + d\Sigma(x^3) + e\Sigma(x^4). \quad (2)$$

Also squaring the equations of (1) and adding,

$$\Sigma(y^2) = 5a^2 + 2ab\Sigma(x) + (2ac + b^2)\Sigma(x^2) + 2(ad + bc)\Sigma(x^3) + (2ae + 2bd + c^2)\Sigma(x^4) + 2(be + cd)\Sigma(x^5) + (2ce + d^2)\Sigma(x^6) + 2de\Sigma(x^7) + e^2\Sigma(x^8). \quad (3)$$

In a similar manner we can express $\Sigma(y^3)$, $\Sigma(y^4)$ and $\Sigma(y^5)$ by a homogeneous equation in a, b, c, d, e , since $\Sigma(x^n)$ can be expressed in terms of p_1, p_2, p_3, p_4, p_5 .

Then, if we denote the coefficients of the equation in y , by P_1, P_2, P_3, P_4, P_5 , we have, by Newton's Theorem,

$$\Sigma(y^m) + P_1\Sigma(y^{m-1}) + P_2\Sigma(y^{m-2}) + \dots + P_{m-1}\Sigma(y) + mP_m = 0,$$

where m may be any integer less than 5.

From this we easily see that if $\Sigma(y)$, $\Sigma(y^2)$ and $\Sigma(y^3)$ be put equal to zero the equation will be reduced to the form $y^5 + q_4y + q_5 = 0$, and if $\Sigma(y)$, $\Sigma(y^2)$ and $\Sigma(y^4)$ all equal zero the equation will be of the form $y^5 + q_3y^2 + q_5 = 0$. But if $\Sigma(y) = 0$ we obtain from (2) $5a = -[b\Sigma(x) + c\Sigma(x^2) + d\Sigma(x^3) + e\Sigma(x^4)]$. Substituting this in (3) we will have a homogeneous equation in b, c, d and e ; denote it by

$$A_1b^2 + 2A_2bc + 2A_3bd + 2A_4be + A_5c^2 + 2A_6cd + 2A_7ce + A_8d^2 + 2A_9de + A_{10}e^2 = \text{function}(b, c, d, e). \quad (4)$$

Now assume

function $(b, c, d, e) = lb + mc + nd + pe)^2 + (qc + rd + se)^2 + (td + ne)^2 + (ve)^2$.

Expanding and equating to the corresponding coefficients of (4),

$l^2 = A_1$ (6); $lm = A_2$ (7); $ln = A_3$ (8); $lp = A_4$ (9); $m^2 + q^2 = A_5$ (10);
 $mn + qr = A_6$ (11); $mp + qs = A_7$ (12); $n^2 + r^2 + t^2 = A_8$ (13); $np + rs +$
 $nt = A_9$ (14); $p^2 + s^2 + u^2 + v^2 = A_{10}$. (15)

From (6) $l = A_1^{1/2}$; hence from (7), (8) and (9) we find m, n and p ; then from (10) we find q ; and from (11) and (12) we can then find r and s ; then from (13) and (14) we know t and u , and finally from (15) we find v . Thus equation (5) is always possible.

From $\Sigma(y^2) = (lb + mc + nd + pe)^2 + (qc + rd + se)^2 + (td + ue)^2 + (ve)^2 = 0$ we can assume $v^2 e^2 = -(lb + mc + nd + pe)^2$ and $(td + ue)^2 = -(qc + rd + se)^2$, as the equation will then be satisfied. Therefore

$$ve = \sqrt{-1}(lb + mc + nd + pe), \quad (16)$$

$$td + ue = \sqrt{-1}(qc + rd + se). \quad (17)$$

From (16) and (17) we can express d and e by two equations of the first degree in b and c . Then, substituting in $\Sigma(y^2) = 0$, we have a homogeneous equation of the third degree in b and c . One of these may now be arbitrarily assumed and the other will then be found by the solution of an equation of the third degree.

If we make the above substitution in $\Sigma(y^4) = 0$ the final equation will be of the fourth degree. Therefore the general equation of the fifth degree can be reduced to either of the forms $x^5 + q_4 x + q_5 = 0$ or $x^5 + q_3 x^2 + q_5 = 0$.

In these two equations let $x = 1 \div z$. Substituting and reducing we get the two forms

$$z^5 + \frac{q_4}{q_5} z^4 + \frac{1}{q_5} = 0, \quad \text{and} \quad z^5 + \frac{q_3}{q_5} z^3 + \frac{1}{q_5} = 0.$$

This result was first obt'd by E. S. Bring and afterward by Mr. Jerrard.

Their method of making the substitutions may be seen in Todhunter's Theory of Equations or Serret's Cours D'Algebre' Superieure.

The above method is much easier than theirs, especially in finding the equation in y , which is there obtained by Tschirnhausen's substitution.

NOTE, BY MARCUS BAKER.—The Note on p. 53 of the last No. is an important one and I would very much like to see it made fuller.

Prob. 5, p. 66, Vol. I, has not been solved in general. The proposed solution applying only to a special case of the problem.

Prob. 49, p. 61, Vol. II, is erroneous and incomplete. The proposed consideration.

Prob. 132, which you mention in your list, was considered by Hall at p. 55, Vol. IV. [This fact had escaped

But $t_1 = 2\sqrt{ac}$, $t_2 = 2\sqrt{cb}$, $t_3 = 2\sqrt{bd}$, $t_4 = 2\sqrt{da}$, and $T' = 2\sqrt{cd}$; $\therefore T \times 2\sqrt{cd} = 2\sqrt{ac} \times 2\sqrt{bd} + 2\sqrt{cb} \times 2\sqrt{da}$, whence $T^2 = 16ab = 4(2a)(2b)$; $\therefore AB^2 = (a-b)^2 + T^2 = a^2 + 14ab + b^2$; that is, the radii a, b being given, the distance between the centers of the circles A, B must be $\sqrt{a^2 + 14ab + b^2}$. We have then the following

Construction.—Take any radius r , not less than the half of HK . With A, B as centers and radii $r - a$ and $r - b$ describe arcs intersecting at E ; then with center E and radius r describe a circle; it will touch A and B .

The construction is completed by describing the circles C, D , each tang't to the circles A, B, E .

This problem, as stated on page 167, Vol. V, has been solved in a number of ways, and it is not necessary to give a construction here.

The method given in Chauvenet's *Geometry*, p. 366, can be readily applied.

Proof.—We have $t_1 = 2\sqrt{ac}$, $t_2 = 2\sqrt{cb}$, $t_3 = 2\sqrt{bd}$, $t_4 = 2\sqrt{da}$, and $T = 4\sqrt{ab}$. Substituting in (1), and reducing, we find $T' = 2\sqrt{cd}$; therefore the circles C, D touch each other.

It appears then that the relative position of the two given circles must be as stated above, and that any number of solutions of the problem with this condition can be made.

The circle E may be described so as to lie wholly on the space between the two common tangents of A and B , these circles touching it externally. Then the circles C, D will touch each other beyond B from A .

2. Let the circle B be wholly within the circle A , and let us use the same notation as in the first case. [The figure, to illustrate this case, must be drawn so that the circles C, D, E touch A internally and B externally.]

Now eq. (1) still holds, but t_1, t_4, T represent the common internal tangents of A and C, D and A, A and B , and are, therefore imaginary. We have $t_1 = 2\sqrt{-ac}$, $t_4 = 2\sqrt{-da}$, and t_2, t_3, T' as above. Hence (1) gives $T^2 = -16ab$. Therefore $AB^2 = (a+b)^2 + T^2 = a^2 - 14ab + b^2$; that is, the distance between the centers of the circles A, B must be $\sqrt{a^2 - 14ab + b^2}$. Moreover, b can not be greater than $(7 - 4\sqrt{3})a$.

The construction and proof are similar to those in the first case.

Problem 247. "Three points are taken at random on the surface of a circle; find the chance that the triangle formed by joining them is acute."

[By special request, we insert, below, the solution of this problem as submitted by the proposer, Mr. E. B. SERTZ.]

Let M, N, P be the three random points, O the center of the circle, AB , the chord through M, N , and CD the chord through M perpendicular to AB . Draw OH and OK perpendicular to AB and CD .

Now N being between B and M , the angle M of the triangle MNP is obtuse if P lies in the segment CAD .

Let $OA = r$, $MN = x$, $\angle AOH = \theta$, $\angle COK = \phi$, and ψ = the angle AB makes with some fixed line. Then $BH = r \sin \theta$, $OK = MH = r \sin \phi$, area seg. $CAD = r^2(\phi - \sin \phi \cos \phi)$; an element of the circle at M is $r \sin \theta d\theta r \sin \phi d\phi$, and at N it is $x dx d\psi$. The limits of θ are 0 and $\frac{1}{2}\pi$; of ϕ , $\frac{1}{2}\pi - \theta$ and $\frac{1}{2}\pi + \theta$; of x , 0 and $r(\sin \theta + \cos \phi) = x'$, and doubled; and of ψ , 0 and 2π .

Hence, since the whole number of ways the three points can be taken is $\pi^2 r^6$, the chance that the angle M is obtuse is

$$\begin{aligned} p_1 &= \frac{2}{\pi^2 r^6} \int_0^{\frac{1}{2}\pi} \int_{\frac{1}{2}\pi - \theta}^{\frac{1}{2}\pi + \theta} \int_0^{x'} r^2(\phi - \sin \phi \cos \phi) r \sin \theta d\theta r \sin \phi d\phi x dx d\psi \\ &= \frac{4}{\pi^2 r^3} \int_0^{\frac{1}{2}\pi} \int_{\frac{1}{2}\pi - \theta}^{\frac{1}{2}\pi + \theta} \int_0^{x'} (\phi - \sin \phi \cos \phi) \sin \theta \sin \phi d\theta d\phi x dx \\ &= \frac{2}{\pi^2} \int_0^{\frac{1}{2}\pi} \int_{\frac{1}{2}\pi - \theta}^{\frac{1}{2}\pi + \theta} (\phi - \sin \phi \cos \phi) (\sin \theta + \cos \phi)^2 \sin \theta \cos \phi d\theta d\phi \\ &= \frac{1}{3\pi^2} \int_0^{\frac{1}{2}\pi} (8\pi \sin^2 \theta + 3\theta - 12\theta \sin^2 \theta - 3 \sin \theta \cos \theta - 6 \sin^3 \theta \cos \theta) \sin^2 \theta d\theta \\ &= \frac{3}{8} - \frac{4}{3\pi^2}. \text{ Hence the chance that the triangle is obtuse is } 3p_1 \\ &= \frac{9}{8} - \frac{4}{\pi^2}, \text{ and the chance that it is acute is } p = 1 - 3p_1 = \frac{4}{\pi^2} - \frac{1}{8}. \end{aligned}$$



DEMONSTRATION OF A PROPOSITION.

BY THE EDITOR.

THE following *Proposition* has been sent to us by a subscriber, and we request that a demonstration be given.

As the proposition, in the form here given is new to us, and we have obtained a very elementary demonstration, assuming that it is new to our subscribers, also, we submit it to our readers.

Proposition.—If any parallelogram be divided into two trapezoids by a straight line, and a straight line be drawn through the points of intersection of the diagonals of the two trapezoids thus formed, it will bisect the parallelogram.

The truth of the above proposition may be inferred as a corollary depending upon the demonstration of the theorem of APOLLONIUS, viz.:

“If any quadrilateral be divided into two other quadrilaterals by a straight line, the intersections of the diagonals of the three quadrilaterals are on the same straight line.”

For, when the quadrilateral is a parallelogram, the intersection of its diagonals is at its center; therefore the straight line joining the other two intersections passes through the center of the parallelogram, and therefore bisects it.

But, as the demonstration of the theorem of APOLLONIUS is somewhat complicated (see *Am. Jour. of Science and Arts* for 1857, Vol. 23, p. 224), we sought and obtained the following, more elementary, demonstration of the foregoing proposition.

Let $ABCD$ represent any parallelogram; let it be divided into two trapezoids by the line EF , and draw the diagonals intersecting in G, H . Draw GH and prolong it in both directions to meet the sides produced in I and K .



Because of the parallel sides, and of the equal opposite angles at G , the triangles AGE and FGD are similar, as are, also, the triangles EGI and DGK ; and because the sides DG and GE are common to the two pairs of similar triangles, we have

$$AE : EI :: FD : DK;$$

and, similarly,

$$EB : BI :: CF : FK.$$

Or, putting $AE = a$, $EB = b$, $CF = c$, $FD = d$, $BI = h$ and $DK = k$,

$$a : b + h :: d : k, \text{ whence } ak = bd + dh, \quad (1)$$

$$c : d + k :: b : h, \quad \text{“} \quad ch = bd + bk. \quad (2)$$

Subtracting (2) from (1) we have

$$ak - ch = dh - bk,$$

or, transposing ch and bk ,

$$(a + b)k = (c + d)h.$$

But $a + b = c + d$; $\therefore h = k$; and because the angles at I and K are alternate angles they are equal, and the triangles BIh and DKm are identical; therefore $Am = Cn$, and consequently IK bisects the parallelogram.

ANSWER TO QUERY 1, PAGE 64.

BY PROF. D. J. MC ADAM, WASHINGTON, PENNSYLVANIA.

"When we descend below the surface of the Earth, does the Earth's attractive force increase or diminish?"

If the Earth were homogeneous, attraction for all points beneath the surface would vary directly as the distance from the center, hence would be a maximum at the surface.

The results of all the pendulum experiments made at the top and bottom of mines prove that the force of attraction is greater at the bottom of the mine than at the top. For instance, it was found in the experiment made in a mine near New Castle, England, that a pendulum which vibrated seconds at the top of the mine, gained $2\frac{1}{4}$ seconds per day when carried down 1256 ft. That is, the force of attraction at the bottom of the mine is about 0.00005 greater than at the surface. Concerning the force of attraction at other less distances from the center, we can only conjecture with more or less probability of being right.

We know from these and other experiments, that the mean density of the portion left after a shell about $\frac{1}{4}$ mile thick is taken off, is about 2 or $2\frac{1}{2}$ as great as the density of that shell.

Whether the remaining nucleus is homogeneous and the attractive force continually decreases as we approach its center, or whether the density varies according to any assumed law we cannot certainly say. We would be slow to accept the hypothesis that the bottom of the deepest mine happens to be in the dividing surface between the less dense shell and the homogeneous more dense nucleus. Indeed, that there is such dividing surface anywhere clearly marked is contradictory of all that we know of increase of density under increase of pressure.

It has been proved that if the Earth is homogeneous and its shape the same as if it were entirely fluid, the ellipticity must be $\frac{1}{230}$. Clairaut proved that Newton was in error when he supposed that if instead of being of uniform density the Earth were denser toward the center, the ellipticity ought to be greater, but, contrarywise, the ellipticity ought to be less than for a homogeneous Earth.

Measurements of arcs of meridians demonstrate that the ellipticity is less than $\frac{1}{230}$. So far there is agreement between the theory of heterogeneity and observation.

The hypothesis concerning the density of the interior strata which seems as probable as any that has been made is that assumed by Laplace. It had been given before by Legendre as an Exam. to which to apply his formula.

Laplace assumed such law of increase of density from surface to center as would make the differential equation for the ellipticities of the strata

$$\frac{d^2}{da^2} [\varphi(a)e] = \frac{6}{a^2} \varphi(a)e + 3a^2 e \frac{d\rho}{da} \quad (1)$$

integrable.

By assuming that the last term of the second member of equation (1) is a function of a , Pratt, in his "Figure of the Earth," deduced the result

$$\rho = Q \frac{\sin qa}{a},$$

in which ρ = the density, Q and q are constants, and a = radius of shell considered as spherical.

The constants q and Q are found from the conditions that the density at the surface is 2.75, and the mean density of the Earth is twice that of the surface.

By means of the densities thus found, Pratt deduces the masses of the nucleus, and four concentric shells, the radius of the nucleus and the thickness of each shell being equal to one fifth of the Earth's radius.

The masses in order from the surface are, 0.3372, 0.3257, 0.2237, 0.0981, 0.0153, the mass of the Earth being unity.

From these values the masses of the parts whose radii are R , $\frac{4}{5}R$, $\frac{3}{5}R$, $\frac{2}{5}R$ and $\frac{1}{5}R$ are found to be 1.0000, 0.6628, 0.3371, 0.1134, 0.0153.

Taking the formula for attraction at the surface, each mass considered a sphere, $A = m \div R^2$, we get in order from the surface, $\frac{1}{25}(1.0000)$, $\frac{1}{16}(0.6628)$, $\frac{1}{9}(0.3371)$, $\frac{1}{4}(0.1134)$, 0.0153, or, 0.0400, 0.0414, 0.03746, 0.0284, 0.0153.

If this is the true law, the force of attraction increases from the surface till we reach a point perhaps one-third distant from surface to center, from that point the force diminishes and becomes zero at the center.

SOLUTION OF THE PROBLEM PROPOSED BY MR. HARVILL AT THE CLOSE OF HIS SOLUTION ON PAGE 56.

BY THE EDITOR.

LET A represent the center of the sphere, radius r , BCD a tri-rectangular spherical triangle, and $AEFGHPCI$, the solid removed from this portion of the sphere in making the first hole.

Draw BPh , DPi and CP , arcs of great circles respectively through BP , DP and CP , and draw HPj , IPk , small circles thro' P with poles at B , D .

The solid $AEFGHPIC$ consists of a square pyramid, $AEFGP$ plus two similar and equal triangular spherical pyramids, $CBPA$ and $CDPA$, minus two equal sectors $BPIE$ and $DPHG$, of two equal spherical segments whose height is $BE = DG$.

Putting $2x$ for the width of the chisel, we have $AE = EF = x$, and consequently FP , the altitude of the square pyramid $= \sqrt{(r^2 - 2x^2)}$; hence the solidity of the pyramid $AEFGP = \frac{1}{3}x^2\sqrt{(r^2 - 2x^2)}$.

The base of the spherical pyramid $BCPA$, is the area of the spherical triangle $BCP = S$.

To find S , we have $\angle C = 45^\circ$, $\angle B = \sin^{-1}[x \div \sqrt{(r^2 - x^2)}]$, whence we find $\angle P = \sin^{-1}[\sqrt{(r^2 - 2x^2)} \div \sqrt{(2r^2 - 2x^2)}] + 90^\circ$; therefore the sum of the angles of the spherical triangle BCP is

$$\sin^{-1} \left[\frac{r+2x}{(r+x)\sqrt{2}} \right] + 135^\circ;$$

$$\therefore S = \left\{ \sin^{-1} \left[\frac{r+2x}{(r+x)\sqrt{2}} \right] - 45^\circ \right\} \frac{S'}{90^\circ},$$

where S' is the area of the spherical trian. $ABC = \frac{1}{8}$ the surface of the sphere $= \frac{1}{2}\pi r^2$. Hence the solidity of the two equal spherical pyramids is

$$\frac{1}{3}\pi r^3 \left\{ \sin^{-1} \left[\frac{r+2x}{(r+x)\sqrt{2}} \right] - 45^\circ \right\}.$$

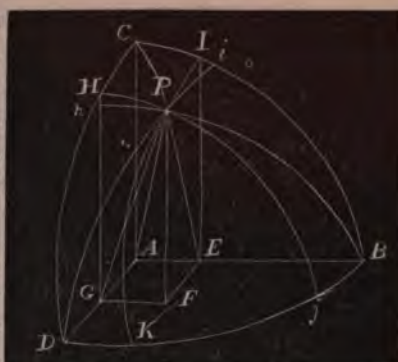
The solidity of the sectors to be subtr. is $\frac{\frac{1}{8}(4r^3 - 6r^2x + 2x^3)}{360^\circ} \cdot \sin^{-1} \frac{x}{\sqrt{(r^2 - x^2)}}$; hence $\frac{1}{8}$ the solidity removed in cutting the first hole through the sphere is

$$\frac{x^2\sqrt{(r^2 - 2x^2)}}{3} + \frac{1}{3}\pi r^3 \left\{ \sin^{-1} \left[\frac{r+2x}{(r+x)\sqrt{2}} \right] - 45^\circ \right\} \\ - \frac{\frac{1}{8}(4r^3 - 6r^2x + 2x^3)}{360^\circ} \cdot \sin^{-1} \frac{x}{\sqrt{(r^2 - x^2)}},$$

and consequently the part of the sphere removed in making the two holes will be 16 times the above $-8x^3$, the part common to both holes. Therefore

$$\frac{16x^2\sqrt{(r^2 - 2x^2)}}{3} + \frac{1}{3}\pi r^3 \left\{ \sin^{-1} \left[\frac{r+2x}{(r+x)\sqrt{2}} \right] - 45^\circ \right\} - 8x^3 \\ - \frac{\frac{1}{8}(4r^3 - 6r^2x + 2x^3)}{360^\circ} \cdot \sin^{-1} \frac{x}{\sqrt{(r^2 - x^2)}} = \frac{2\pi r^3}{3}.$$

By solving this transcendental equation the value of x will be found.



SOLUTIONS OF PROBLEMS IN NUMBER TWO.

SOLUTIONS of problems in number two have been received as follows:

From Prof. W. P. Casey, 250, 251, 257; George Eastwood, 255, 257; Prof. E. J. Edmunds, 251, 253, 255; Prof. A. B. Evans, 257; Prof. A. Hall, 256; A. S. Hathaway, 257; W. E. Heal, 250, 251, 252, 253, 254, 255; Prof. W. W. Johnson, 257; Chas. H. Kummell, 251, 254, 256, 257; Prof. J. H. Kershner, 250, 251, 252, 253, 254, 255, 256, 257; Artemas Martin, 254; Prof. M. C. Stevens, 250, 251; E. B. Seitz, 257; Prof. J. Scheffer, 250, 251, 253, 254, 255, 256, 257.

Besides the answer given on page 85, Query 1 was, also, answered by Prof. W. P. Casey and Prof. J. H. Kershner.

250. "Divide the line AB , geometrically, into three parts that shall be in harmonic proportion."

SOLUTION BY PROF. W. P. CASEY, SAN FRANCISCO, CAL.

As the ratio of the harmonic section of a line or angle may have any value, real or imaginary, a line or angle may be cut harmonically in an infinite number of ways; but the ratio of the harmonic section, or the position of one of the two conjugates is of course sufficient to determine the particular harmonic section in the case of a given line or angle.

Construction. — Assume any point C in the given line, and draw CD at right angles to it. Take any point D in this line, and join BD and produce it to meet the perpendicular AE in the point E . Make $AF = AE$; join FD , intersecting AC in the point O ; then is the line AB



divided in the required proportion, that is; $AB : BC :: AO : OC$. OB is evidently a harmonic mean between AB and BC ; and so is AC one between AB and AO .

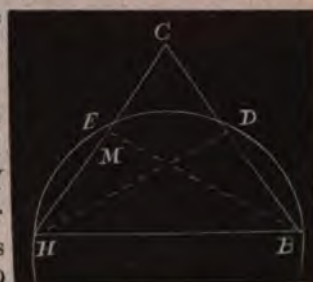
If it be required to divide AB into three parts that shall have this proportion, and that AB shall not form one of its terms, that is, having $AO : CB :: AO - OC : OC - CB$, it is also unlimited, but may be solved in a similar manner. It may be shown from the proportion $AB : BC :: AO : OC$ that $BO(AB + BC) = 2AB \cdot BC$; and also, that $OB(AO - OC) = 2AO \cdot OC$. If $AO \cdot CB$, $AO - CB$, $AC \cdot OB$, $AC^2 + OB^2$, or $AC^2 - OB^2$ were given the problem would be limited, and there would be only one solution.

251. "Show that when the two lines, which bisect two angles of a triangle, are equal the triangle is isosceles."

SOLUTION BY PROF. M. C. STEVENS, LAFAYETTE, IND.

Let ABC be a triangle such that when the angles A and B are bisected by AD and BE , $AD = BE$. Then the angle A must be equal to the angle B , and the triangle be isosceles.

Proof.—Through the three points A , B and D pass the circumference of a circle. It will also pass through E . For if it meets BE in M between B and E the arc DB must be greater than MD since DAB which is equal to EAD is greater than the angle MAD . Also the arc MD



is equal to the arc MA since the angle DBM is equal to the angle MBA . Whence the arc BDM is greater than DMA , and consequently the chord BM is greater than the chord AD . But $AD = BE$; $\therefore BM$ is greater than BE which is absurd. Hence the circle which passes through A , B and D cannot cut BE between B and E .

In like manner it may be shown that the circle cannot cut BE beyond E . It must therefore pass through E . Hence the angle DBE which is half of B is equal to EAD which is half of A . Therefore the angle B is equal to the angle A and the triangle is isosceles.

252. "If the roots of a given cubic equation be not *real* and *positive* show that the equation can be transformed into another, of the same degree, in which *all* the roots are real and positive."

SOLUTION BY W. E. HEAL, WHEELING, INDIANA.

Let x_1, x_2, x_3 and x_1^2, x_2^2, x_3^2 be the respective roots of the equations

$$x^3 + ax^2 + bx + c = 0, \quad (1)$$

$$x^3 + Ax^2 + Bx + C = 0. \quad (2)$$

We have $x_1 + x_2 + x_3 = -a$, $x_1x_2 + x_1x_3 + x_2x_3 = b$, $x_1x_2x_3 = -c$;
 $x_1^2 + x_2^2 + x_3^2 = -A$, $x_1^2x_2^2 + x_1^2x_3^2 + x_2^2x_3^2 = B$, $x_1^2x_2^2x_3^2 = -C$.

$$x_1^2 + x_2^2 + x_3^2 = a^2 - 2b = -A. \quad \therefore A = 2b - a^2. \quad (3)$$

$$x_1^4 + x_2^4 + x_3^4 = a^4 - 4a^2b + 4ac + 2b^2 = A^2 - 2B. \quad \therefore B = b^2 - 2ac. \quad (4)$$

$$x_1^2x_2^2x_3^2 = c^2 = -C. \quad \therefore C = -c^2. \quad (5)$$

If all the roots of an equation are real we can transform it by (3), (4) and (5) into another in which the roots shall be the squares of the roots of the given equation and therefore *all* the roots will be positive whatever the signs of the roots of the given equation.

If two roots are imaginary they must be of the form $p+q\sqrt{-1}$, $p-q\sqrt{-1}$. Let r denote the real root, which can be expressed by Cardan's formula.

We have $r+(p+q\sqrt{-1})+(p-q\sqrt{-1})=a$.

$$\therefore p = \frac{1}{2}(a-r).$$

Let $x = z - \frac{1}{2}(a-r)$. The equation found by substitution must have $\frac{1}{2}(3r-a)$, $q\sqrt{-1}$, $-q\sqrt{-1}$ for its roots. Transforming by (3), (4) and (5) we get an equation whose roots are $\frac{1}{4}(3r-a)^2$, $-q^2$, q^2 . Transforming again we have a cubic whose roots are $\frac{1}{16}(3r-a)^4$, q^4 , q^4 , all *real* and *positive*.

253. "If $f(x)$ be a function whose roots are all real, show that the differential of the second order of that function has all its roots imaginary."

ANSWER BY PROF. KERSHNER.

The second term of the "second differential" of the general equation

$$x^n + Ax^{n-1} + Bx^{n-2} + \&c. = 0,$$

after proper reduction, is $A(1 - \frac{1}{2}n)x^{n-3}$, from an inspection of which it is evident that no change such as predicated by the proposer is even possible. The problem, therefore, as stated, involves an error.

ANSWER BY PROF. SCHEFFER.

As far as I understand this problem—and I do not notice any ambiguity of expression which might admit of different interpretation—the assertion is incorrect, as can be easily demonstrated. If, for instance, the function is of the third degree, the second differential will be of the first, which certainly cannot admit of any imaginary roots, as such occur only by pairs.

254. "Integrate $dI = xE(e, x)dx$, where $E(e, x)$ denotes an elliptic arc, eccentricity e and abscissa x ."

SOLUTION BY CHAS. H. KUMMELL, U. S. LAKE SURVEY, DETROIT, MICH.

Integrating by parts we have

$$\begin{aligned} I &= \frac{1}{2}x^2 E(e, x) - \frac{1}{2} \int_0^x x^2 dx \sqrt{\frac{1-e^2x^2}{1-x^2}} \quad (\text{semi major axis} = 1) \\ &= \frac{1}{2}x^2 E(e, x) - \frac{1}{2}I'. \end{aligned} \quad (1)$$

Let

$$R = \sqrt{(1-x^2)(1-e^2x^2)}; \quad (2)$$

then we have

$$\begin{aligned} d(Rx) &= \left(R + \frac{-(1+e^2)x^2 + 2e^2x^4}{R} \right) dx \\ &= \frac{1 - 2(1+e^2)x^2 + 3e^2x^4}{R} dx \\ &= \frac{1 + (1-2e^2)x^2}{R} dx - 3dI' \\ &= \frac{1 + [(1-2e^2) \div e^2]}{R} dx - \frac{1-2e^2}{e^2} d[E(e, x)] - 3dI' \\ &= \frac{1-e^2}{e^2} d[F(e, x)] - \frac{1-2e^2}{e^2} d[E(e, x)] - 3dI'. \end{aligned} \quad (3)$$

Substituting this into (1) we obtain

$$I = \frac{1}{2}x^2 E(e, x) + \frac{1-2e^2}{6e^2} E(e, x) - \frac{1-e^2}{6e^2} F(e, x) + \frac{1}{6}x \sqrt{(1-x^2)(1-e^2x^2)}.$$

255. "In a triangle ABC the angle $A = \varphi + \alpha$, $B = 2\varphi$; supposing AB to remain fixed, while φ varies, it is required to find the rectangular equation to the locus of C , and the equations to the asymptotes.

2. With the same data as above, taking A as the origin, and AB as the axis of x ; let $\alpha = 45^\circ$, and find the envelop of a straight line which passes through C and makes an angle $4\varphi + 90^\circ$ with AX ."

SOLUTION BY GEORGE EASTWOOD, SAXONVILLE, MASS.

Let ABC be a triangle in which $\angle A = \varphi + \alpha$, $\angle B = 2\varphi$, base $AB = \beta$, and let CD be a line which makes with AB produced an angle $= 90^\circ + 4\varphi$.

Then, since $2A - B = 2\alpha$, we have

$$\tan(2A - B) = \frac{\tan 2A - \tan B}{1 + \tan 2A \tan B} = \tan 2\alpha,$$

a given quantity.

But if x, y be the coordinates of the vertex C ,

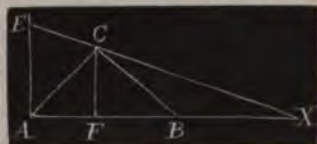
$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A} = \frac{2xy}{x^2 - y^2}, \text{ and } \tan B = \frac{y}{\beta - x}.$$

Therefore

$$\frac{\tan 2(\varphi + \alpha) - \tan 2\varphi}{1 + \tan 2(\varphi + \alpha) \tan 2\varphi} = \frac{2y(\beta - x)x - x^2y + y^3}{x^2(\beta - x) - (\beta - 3x)y^2} = \tan 2\alpha;$$

$$\therefore y^3 + \tan 2\alpha(\beta - 3x)y^2 - (3x^2 - 2\beta x)y - \tan \alpha(\beta - x)x^2 = 0,$$

which indicates a locus of the third degree.



2. The equation of the line DC is

$$y = x \tan(90^\circ + 4\varphi) + b, \quad (1)$$

in which b represents the intercept AE .

Now in the triangle ACB we have $BC = \beta \sin(\varphi + a) \div \sin(\varphi + a + 2\varphi)$, and in the triangle BCD , we find

$$BD = \frac{-\beta \sin(\varphi - a) \cos 6\varphi}{\sin[(\varphi + a) + 2\varphi] \cos 4\varphi}, \quad AD = \beta - \frac{\beta \sin(\varphi + a) \cos 6\varphi}{\sin[(\varphi + a) + 2\varphi] \sin 4\varphi},$$

$$AE = \beta \cot 4\varphi - \frac{\beta \sin(\varphi + a) \cos 6\varphi}{\sin[(\varphi + a) + 2\varphi] \sin 4\varphi} = b. \quad (2)$$

Put $\tan(90^\circ + 4\varphi) = p$; then $\tan 4\varphi = -1 \div p$, $\sin 4\varphi = -1 \div (1 + p^2)^{1/2}$, and equation (1) becomes

$$y = px - \beta p + \frac{\beta \sin(\varphi + a) \cos 6\varphi \sqrt{1 + p^2}}{\sin[(\varphi + a) + 2\varphi]}. \quad (3)$$

Differentiating equation (3) with respect to the parameter p , and equating dy to zero, we obtain, by developing $\sin(\varphi + a)$, $\sin[(\varphi + a) + 2\varphi]$ in functions of x and y , and putting $m = \sqrt{(\beta - x)^2 + y^2}$,

$$p = \frac{\beta - x}{\sqrt{m^2 \cos^2 6\varphi - (\beta - x)^2}}.$$

Substituting this value of p in equation (3), we obtain, after reduction,

$$[y^2 - 3(\beta - x)^2][y^2 - 3(\beta - x)^2] = 0,$$

which indicates, as the required envelop, two equal hyperbolas whose centers are at B and whose semi axes are $\sqrt{[3\beta(\beta - 1)]}$ and $\sqrt{[\beta(\beta - 1)]}$.

Remark.—Since this result is independent of a , it would appear that the datum, $a = 45^\circ$, is not necessary to the solution of the problem.

[Mr. Eastwood has not given the equation to the asymptotes. Professor Scheffer determines the equation of the asymptotes as follows: Putting $y = \mu x + \beta$ for the equation of the asymptote, he substitutes this value of y in his Eqn. of the locus, $(y^2 - x^2)[y \cos 2a + (a - x) \sin 2a] + 2xy[(a - x) \cos 2a - y \sin 2a] = 0$, and thence, by putting the coef. of x^3 and x^2 , respectively = 0, he obtains equations (4) and (5). From (4) he derives the cubic equation, $\mu^3 - 3\mu^2 \tan 2a - 3\mu + \tan 2a = 0$, from which he finds the three roots μ_1, μ_2, μ_3 . Substituting the first of these roots ($\tan \frac{2}{3}a$) in (5) he finds $\beta = -\frac{1}{3}a \tan \frac{2}{3}a$. ($a = AB$.) Substituting in the eqn. $y = \mu x + \beta$, he gets $y = (x - \frac{1}{3}) \tan \frac{2}{3}a$, for the eq. of one of the asymptotes.]

256. "If the given quantities x_1, x_2, x_3, x_4 have the probable errors r_1, r_2, r_3, r_4 , respectively, find the probable error r of the quantity x when $x_1 : x_2 :: x_3 + x : x_4 + x$."

SOLUTION BY PROF. HALL.

If we have given $x = \varphi(x_1, x_2, x_3, x_4)$ and r_1, r_2, r_3, r_4 are the probable errors of x_1, x_2, x_3, x_4 , we have, by the theory of least squares, the probable error of x

$$= \pm \left\{ \left(\frac{d\varphi}{dx_1} \right)^2 r_1^2 + \left(\frac{d\varphi}{dx_2} \right)^2 r_2^2 + \left(\frac{d\varphi}{dx_3} \right)^2 r_3^2 + \left(\frac{d\varphi}{dx_4} \right)^2 r_4^2 \right\}^{\frac{1}{2}}.$$

In the present case we have

$$x = \frac{x_1 x_4 - x_2 x_3}{x_2 - x_1}.$$

Hence we have the probable error of x

$$r = \pm \frac{1}{(x_2 - x_1)^2} \sqrt{[x_2^2(x_4 - x_1)^2 r_1^2 + x_1^2(x_3 - x_4)^2 r_2^2 + x_2^2(x_1 - x_2)^2 r_3^2 + x_1^2(x_2 - x_1)^2 r_4^2]}.$$

257. "Within a triangle ABC , determine a point P , such that $m.PA + n.PB + r.PC$ shall be a minimum, m, n, r being constants."

SOLUTION BY PROF. W. W. JOHNSON, ANNAPOLIS, MD.

Denoting PA by ρ_1 , PB by ρ_2 , PC by ρ_3 , we are to have

$$m\rho_1 + n\rho_2 + r\rho_3 = \text{a minimum.}$$

When P is in the proper position the locus of

$$m\rho_1 + n\rho_2 = \text{a constant}$$

must touch the circle whose radius is ρ_3 , otherwise by moving P along this locus we could diminish ρ_3 without increasing $m\rho_1 + n\rho_2$. This locus is a Cartesian Oval; s being an arc of this curve, the cosine of the angle between ρ_1 and the tangent is evidently $d\rho_1 \div ds$, and that of the angle between ρ_2 and the same end of the tangent is $-d\rho_2 \div ds$; and since from the equation of the locus,

$$md\rho_1 = -nd\rho_2,$$

these cosines are inversely in the ratio $m : n$. Now ρ_2 being perpendicular to the tangent, the sines of the angles at P subtended by b and a are inversely as $m : n$. A corresponding relation existing between the sines of each pair of angles, the sines of the angles subtended by a, b and c are directly as $m : n : r$.

To find P construct a triangle whose sides are m, n, r ; the required angles are the supplements of the angles of this triangle, and P may be determined by the intersection of circular segments constructed on two of the sides and containing the proper angles.

SOLUTION BY PROF. W. P. CASEY, SAN FRANCISCO, CAL.

Construction.— Determine a triangle QRS having its sides pass through the vertices of the given triangle ABC and similar to that determined by the three multiples m, n, r ; and having AP, BP, CP perpendicular to RS, SQ, QR respectively.

Upon AC and AB describe two segments of circles containing angles respectively $=$ to the supplements of the angles mr, mn ; the intersection of the circles will give the point P .

Join PA, PB, PC ; the sides RS, SQ and QR being respectively perpendicular to PA, BP and CP, P is the point required; for, take any other point, O , and draw the perpendiculars, Oy, Ox, Oz ; then, by a well known property in modern geometry, $m.PA + n.PB + r.PC = m.Oy + n.Ox + r.Oz$, which is less than $m.OA + n.OB + r.OC$.

When the three given multiples m, n, r are incapable of forming a triangle, this method of determining P fails, but it is easily seen, at once without any construction, that if any of the multiples m, n, r be $=$ or $>$ than the sum of the other two, the point itself corresponding to that multiple is that for which the sum $m.PA + n.PB + r.PC$ is the minimum.



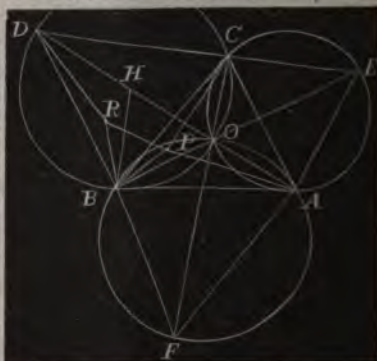
SOLUTION BY E. B. SEITZ, GREENVILLE, OHIO.

Dividing the expression by m , we have $OA + (n \div m) \cdot OB + (r \div m) \cdot OC$, which is to be a minimum.

Take any point P , within the triangle, and join PA, PB, PC . Construct the triangle BPR , such that $BP : PR : RB = m : n : r$, and construct the triangle BRD , making it similar to BPC .

Then we have $PR = (n \div m) \cdot PB$, and $RD = (r \div m) \cdot PC$; $\therefore AP + PR + RD = PA + (n \div m) \cdot PB + (r \div m) \cdot PC$.

The angle $CBD = CBR + RBD = CBR + CBP = PBR$, and $BC : BD :: BP : BR$; hence the triangle BCD is similar to the triangle BPR . The point D is, therefore known.



Now $AP + PR + RD$, or $PA + (n \div m).PB + (r \div m).PC$ will be least when $APRD$ becomes a straight line. This will be the case when P is taken so that the angle APB is the supplement of BPR , and BPC or BRD is the supplement of BRP . Hence we have the following

Construction.—On the sides BC, CA, AB construct the triangles BCD, CAE, ABF , making them similar to BPR , the angle $CBD = ABF = CEA = PBR$, $BCD = ACE = AFB = BPR$, and $CAE = BAF = BDC = BRP$. Circumscribe circles about the triangles BCD, CAE, ABF .

Now since $BDC + CEA + AFB = 180^\circ$, the sum of the angles inscribed in the arcs BC, CA, AB , is 360° ; hence the circumferences intersect in one point O , which is the required point. Join OD, OE, OF , and make the angle $OBH = CBD$.

The angle BOD is equal to BCD , which by construction is equal BFA , and BFA is the supplement of AOB ; hence BOD is the supplement of AOB , and AOD is, therefore, a straight line. Similarly it can be proved that BOE and COF are straight lines.

The triangle BOH being similar to BCD , and BHD to BOC , we have $OH = (n \div m).OB$, and $HD = (r \div m).OC$; $\therefore AD = OA + (n \div m).OB + (r \div m).OC$, and $m.AD = n.OA + m.OB + r.OC =$ the req'd minimum.

In like manner we prove that $n.BE$ and $r.CF$ each = the req'd min.

If Δ, Δ_1 are the areas of the $\triangle ABC$ and a $\triangle$ with sides m, n, r , and if α, β, γ represent the angles A, B, C , of the $\triangle ABF$, we find,

$$OA = \frac{2m\Delta(\cot A + \cot \alpha)}{\sqrt{[2\Delta_1(a^2 \cot \alpha + b^2 \cot \beta + c^2 \cot \gamma + 4\Delta)]}}$$

$$OB = \frac{2n\Delta(\cot B + \cot \beta)}{\sqrt{[2\Delta_1(a^2 \cot \alpha + b^2 \cot \beta + c^2 \cot \gamma + 4\Delta)]}}$$

$$OC = \frac{2r\Delta(\cot C + \cot \gamma)}{\sqrt{[2\Delta_1(a^2 \cot \alpha + b^2 \cot \beta + c^2 \cot \gamma + 4\Delta)]}}$$

PROBLEMS.

259. By PROF. J. SCHEFFER, *Mercersburg, Pa.*—In a triangle, one side = 400 ft., and the two adjacent angles, 70° and 80° , are given; to compute the other two sides without the aid of trigonometry.

260. By GEO. M. DAY, *Lockport, N. Y.*—A sphere, radius r , rolls down the concave arc of a circle, radius R . At the beginning of the motion, the center of the sphere is on the horizontal diameter of the circle. Find the time of descent of the sphere in terms of the coordinable of its center.

261. By ALEX. S. CHRISTIE, *U. S. Coast Survey, Washington, D. C.*—A curve of the n th degree rolls upon any curve whatever: to determine the degree and position of the locus of the centers of curvature of all the elements described, at any given instant, by the points of the rolling curve.

262. By PROF. A. B. EVANS, *Lockport, N. Y.*—Prove, x being an integer, that $x(x+1)(x+2)(x+3)(x+4)(x+5)(x+6)$ cannot be the square of a commensurate number.

263. By W. L. MARCY, *Colorado Springs, Col.*—On the 17th of Aug., 1878, at 5^h 20^m A. M., an observation of polaris was taken in Lat. 37°30', Long. 107° W. from Greenwich, with an engineer's transit; from the meridian thus obtained, the Sun's azimuth at 6^h23^m53^s, A. M., was 79°10'30". What was the error of the watch, and of the meridian?

264. By PROF. J. H. KERSHNER, *Mercersburg, Pa.*—It is required to divide a given straight line into three parts such that the triangle formed of them shall have its circumscribing circle a minimum and its inscribed circle a maximum.

265. By GEORGE EASTWOOD, *Saxonville, Mass.*—At age a , a person takes out a life policy of k dollars, for which he agrees to pay an annual premium of p dollars. At age $a+n$ he is not able to make his annual payment and wants to sell, or surrender, his life-policy for full-paid insurance policy. If the n premiums that have been paid be each viewed as the sum of a series of infinitely small annuities, payable at infinitely small intervals of time, within each year, what ought to be the present value of the new policy?

PUBLICATIONS RECEIVED.

Transactions of the Wisconsin Academy of Science, Arts and Letters. Madison, Wisconsin. 1879. 320 pages, 8vo.

The Mathematical Visitor, No. 3. Artemas Martin, A. M., Editor and Publisher. Erie, Pa. 36 pages, 4to. Price, 50 cents.

ERRATA.

On page 63, line 20, for Prof. Johnson, read, Prof. Hendrickson.

" " 67, " 9, insert a period after $\infty + 1$, and for "Le", read Let.

" " 49, " 4, Vol. III, for 4058, read 4059, and for 4059, read, 4060.

" " 23, Vol. IV, insert the factor, $\cos \phi$, in the numerator of the value of Δ .

THE ANALYST.

VOL. VI.

JULY, 1879.

No. 4.

REDUCTION OF OBSERVATION EQUATIONS WHICH CONTAIN MORE THAN ONE OBSERVED QUANTITY.

BY CHAS. H. KUMMELL, ASST. U. S. LAKE SURVEY, DETROIT, MICH.

THE ordinary case, in the application of the method of Least Squares, where a quantity is observed which is a known function of certain unknown quantities, and the case, occurring principally in higher geodesy, where the observed quantities have to fulfill rigorous conditions, have received by Gauss, Bessel, Hansen, Encke and others such an exhaustive treatment, that it would be difficult to add anything of importance to their labors. The case mentioned in the heading of this article, which is nothing but an extension of the ordinary case, whenever it occurs, is generally forced into the treatment of the ordinary case, either by considering one class of observed quantities without error (comparing standards of length at different temperatures, we really have two observed quantities, viz., difference of length and temperature, but in this case the temperature is considered in the reduction without error), or some function of the observed quantities is treated as an observed quantity.—In Mr. Hill's reduction of Regnault's experiments on the volume of atmospheric air, ANALYST, Vol. IV, No. 4, there are three classes of observed quantities, viz., Vol., V , Pressure, P and temperature, T . In the first class of experiments, where the temperature is nearly the same there are then only two variable observed quantities, V and P , and, in applying Least Squares Mr. Hill makes the sum of the squares of the corrections to $\log(PV)$ a minimum.—Upon close examination it will be found that a great many observations are of a more or less complex nature so that the observation equations contain more than one observed quantity, and it may be of interest to show how such observation equations should be treated.

Suppose we have the following m observation equations:

$$f_1(a, b, \dots x_1 + \delta x_1, y_1 + \delta y_1, \dots) = 0, \quad (1_1)$$

$$f_2(a, b, \dots x_2 + \delta x_2, y_2 + \delta y_2, \dots) = 0, \quad (1_2)$$

$$\dots \dots \dots$$

$$f_m(a, b, \dots x_m + \delta x_m, y_m + \delta y_m, \dots) = 0, \quad (1_m)$$

where the form of these functions may be considered different, provided the same unknown quantities, $a, b, \dots$ either all or in part, occur in them. Also the observed quantities need not be of the same class in all observation equations but I suppose here each observed value to enter only one of them.

Since the corrections $\delta x_1, \delta y_1, \dots \delta x_2, \delta y_2, \dots$ must be small quantities, we have, if $f_1, f_2, \dots f_m$ denote the values of $(1_1), (1_2), \dots (1_m)$, if the uncorrected values, $x_1, y_1, \dots x_2, y_2, \dots x_m, y_m, \dots$ are substituted,

$$f_1 + \frac{df_1}{dx_1} \delta x_1 + \frac{df_1}{dy_1} \delta y_1 + \dots = 0, \quad (1'_1)$$

$$f_2 + \frac{df_2}{dx_2} \delta x_2 + \frac{df_2}{dy_2} \delta y_2 + \dots = 0, \quad (1'_2)$$

$$\dots \dots \dots$$

$$f_m + \frac{df_m}{dx_m} \delta x_m + \frac{df_m}{dy_m} \delta y_m + \dots = 0. \quad (1'_m)$$

Let $p_1 = \text{weight of } x_1; q_1 = \text{weight of } y_1; \dots \quad (2_1)$

$p_2 = \text{ " " } x_2; q_2 = \text{ " " } y_2; \dots \quad (2_2)$

$\dots \dots \dots$

$p_m = \text{ " " } x_m; q_m = \text{ " " } y_m; \dots \quad (2_m)$

The quantities $f_1, f_2, \dots f_m$ are independent with respect to the observations as we here suppose. Let $P_1, P_2, P_3, \dots P_m$ be their weights respectively then we have, evidently,

$$\frac{1}{P_1} = \frac{1}{p_1} \left(\frac{df_1}{dx_1} \right)^2 + \frac{1}{q_1} \left(\frac{df_1}{dy_1} \right)^2 + \dots \quad (3_1)$$

$$\frac{1}{P_2} = \frac{1}{p_2} \left(\frac{df_2}{dx_2} \right)^2 + \frac{1}{q_2} \left(\frac{df_2}{dy_2} \right)^2 + \dots \quad (3_2)$$

$$\dots \dots \dots$$

$$\frac{1}{P_m} = \frac{1}{p_m} \left(\frac{df_m}{dx_m} \right)^2 + \frac{1}{q_m} \left(\frac{df_m}{dy_m} \right)^2 + \dots \quad (3_m)$$

which expressions are in general functions of $a, b, \dots$ and of the observed quantities.

According to the principle of least squares we must make

$$[p\delta x^2] + [q\delta y^2] + \dots = \text{a minimum}, \quad (4)$$

or, by (1') $[p\delta x^2] + \left[q \left(\frac{f + (df \div dx)\delta x + \dots}{df \div dy} \right)^2 \right] + \dots = \text{a min.}$ (4')

Equating to zero the differential coefficients with respect to $\delta x_1, \delta x_2, \dots, \delta x_m$, successively, solving for these corrections, and doing the same for $\delta y_1, \delta y_2, \dots, \delta y_m$, etc., we obtain regarding (3)

$$\delta x_1 = -\frac{P_1 f_1}{p_1} \cdot \frac{df_1}{dx_1}; \quad \delta y_1 = -\frac{P_1 f_1}{q_1} \cdot \frac{df_1}{dy_1} \dots \dots \quad (5_1)$$

$$\delta x_2 = -\frac{P_2 f_2}{p_2} \cdot \frac{df_2}{dx_2}; \quad \delta y_2 = -\frac{P_2 f_2}{q_2} \cdot \frac{df_2}{dy_2} \dots \dots \quad (5_2)$$

$$\dots \dots \dots \delta x_m = -\frac{P_m f_m}{p_m} \cdot \frac{df_m}{dx_m}; \quad \delta y_m = -\frac{P_m f_m}{q_m} \cdot \frac{df_m}{dy_m} \dots \dots \quad (5_m)$$

and substituting these expressions into (4) we obtain

$$[Pf^2] = \text{a minimum}; \quad (4'')$$

$$\therefore \frac{d}{2da} [Pf^2] = 0, \quad (6_a)$$

$$\frac{d}{2db} [Pf^2] = 0, \quad (6_b)$$

.....

Only in the simplest case these conditions may be employed for the direct solution of our problem. Before attempting the general problem, I take such a simple case. Let

$$f_1 = ax_1 - y_1 + b = -a\delta x_1 + \delta y_1, \quad (a_1)$$

$$f_2 = ax_2 - y_2 + b = -a\delta x_2 + \delta y_2, \quad (a_2)$$

$$\dots \dots \dots$$

$$f_m = ax_m - y_m + b = -a\delta x_m + \delta y_m, \quad (a_m)$$

be the observation equations, then we have by (3)

$$\frac{1}{P_1} = \frac{a^2}{p_1} + \frac{1}{q_1}, \quad (b_1)$$

$$\frac{1}{P_2} = \frac{a^2}{p_2} + \frac{1}{q_2}, \quad (b_2)$$

$$\dots \dots \dots$$

$$\frac{1}{P_m} = \frac{a^2}{p_m} + \frac{1}{q_m}. \quad (b_m)$$

We have then by (4'') $\left[\frac{(ax - y + b)^2}{\frac{a^2}{p} + \frac{1}{q}} \right] = \text{a minimum},^*$

*If $p = q = 1$ we have $[(ax - y + b)^2 \div (a^2 + 1)] = \text{a minimum}$. This is Mr. Adcock's principle of making the normals as short as possible. See ANALYST, Vol. V, p. 54.

$$\text{and by (6)} \quad \frac{d}{2da} \left[\frac{(ax-y+b)^2}{\frac{a^2}{p} + \frac{1}{q}} \right] = 0, \quad (c_1)$$

$$\frac{d}{2db} \left[\frac{(ax-y+b)^2}{\frac{a^2}{p} + \frac{1}{q}} \right] = 0, \quad (c_2)$$

$$\text{or} \quad \left[\frac{x}{\frac{a^2}{p} + \frac{1}{q}} (ax-y+b) \right] - \left[\frac{a+p}{\left(\frac{a^2}{p} + \frac{1}{q}\right)^2} (ax-y+b)^2 \right] = 0, \quad (d_1)$$

$$\left[\frac{ax-y+b}{\frac{a^2}{p} + \frac{1}{q}} \right] = 0. \quad (d_2)$$

Unless we make a certain assumption with regard to the weights of the observed quantities, we cannot solve, by a direct process, even these eq'ns. Such an assumption is: that the precisions of x and y which enter the same observation equation, are considered to have a constant ratio. The x and y observations having been made as nearly as possible simultaneously, they are made under the same conditions and are therefore as good as those conditions permit. From this however it does not follow that they have equal weight, more particularly if they are heterogeneous quantities, as for example, temperature readings and microscope readings on the comparator in comparing standards of length, or signalling the transit of a star over a thread in a telescope on the chronograph and reading the level, in time observations, etc. If weight is defined as the reciprocal square of the probable error, it is clear that the unit in which a quantity is expressed makes a great difference in the weight to be assigned. All we can safely assume therefore with regard to the weights of simultaneous observations that enter one observation equation is this: that there is, throughout, a constant ratio in their precisions, the square of which is the constant ratio of their weights. To obtain this we may take the reciprocal ratio of the practical limits of error in each class of observations for the ratio of the precisions. Such limits are known sufficiently near by any experienced observer. Even a computer can draw some inference from the results given to him, which will enable him to get some rough idea of this ratio; for a good observer will not give his results with any more accuracy than he thinks he can estimate. Now suppose x to be given to μ , and y to ν decimal places, whatever be the units in which they are expressed; then it is the computer's duty not to alter the respective *figures* (not the absolute values) any more than necessary to establish consistency. This he will effect if he takes y $10^{\nu-\mu}$ times as precise as x . If however, for example, y is the mean of k

directly observed values, he should take $y 10^{\nu-\mu} \sqrt{k}$ times as precise as x , and similarly he might reason in other more complex cases. Suppose then

$$q_1 = kp_1, \quad (e_1)$$

$$q_2 = kp_2, \quad (e_2)$$

$$\dots$$

$$q_m = kp_m, \quad (e_m)$$

then we have, substituting in (d_a) and (d_b) ,

$$a[px^2] - [pxy] + b[px] - \frac{a}{a^2 + (1 \div k)} \left\{ [py^2] - 2a[pxy] + a^2[px^2] - 2b[py] + 2ab[px] + b^2[p] \right\} = 0, \quad (d'_a)$$

$$a[px] - [py] + b[p] = 0. \quad (d'_b)$$

$$\text{From } (d'_b) \text{ we have } b = (1 \div [p]) \{ [py] - a[px] \} \quad (f)$$

Substituting this into (d'_a) , we obtain, after reduction,

$$a^2 + a \left\{ \frac{(1 \div k)([px^2] - (1 \div [p])[px]^2) - ([py^2] - (1 \div [p])[py]^2)}{[pxy] - (1 \div [p])[px][py]} \right\} - \frac{1}{k} = 0, \quad (g)$$

and solving we obtain

$$a = -\frac{(1 \div k)[px^2 \cdot 1] - [py^2 \cdot 1]}{2[pxy \cdot 1]} \pm \sqrt{\left[\frac{1}{k} + \frac{\{ (1 \div k)[px^2 \cdot 1] - [py^2 \cdot 1] \}^2}{4[pxy \cdot 1]^2} \right]}, \quad (h)$$

where use has been made of Gauss' notation of auxiliaries as used in his method of substitution.

To decide which of the signs must be taken for the minimum minimorum, I take two extreme cases:

1. Let the weight of the x be infinite, or let us suppose the x without error, then $k=0$ in order that $q=kp$ = a finite quantity. Then (h) becomes

$$a' = \frac{[qxy \cdot 1]}{[qx^2 \cdot 1]} \text{ or } = \infty. \quad (h')$$

2. Let the y be without error, or $k = \infty$, then (h) becomes

$$a'' = \frac{[py^2 \cdot 1]}{[pxy \cdot 1]} \text{ or } = 0. \quad (h'')$$

Only the results given first, which correspond to the upper sign, can be used; for they agree with the results from the ordinary treatment if 1, the (y) 's and 2, the (x) 's are corrected. We have then to adopt the upper sign in (h) .

If $p_1 = p_2 = \dots p_m = 1$ and also $k = 1$ we have Mr. Adcock's problem, Vol. V, page 54, and (h) becomes

$$a = -\frac{[x^2] - (1 \div m)[x]^2 - [y^2] + (1 \div m)[y]^2}{2[xy] - (2 \div m)[x][y]}$$

$$+\sqrt{1+\frac{1}{4}\left\{\frac{[x^2]-(1\div m)[x]^2-[y^2]+(1\div m)[y]^2}{[xy]-(1\div m)[x][y]}\right\}^2} \cdot (h_0)$$

This however is not identical with Mr. Adcock's formula in consequence of an error which he committed in eliminating b .

Having found a and b we compute the f and P by formulæ (b) and (c), and the minimum sum of squares by (4'').

We have then the probable error of an observation of weight 1, since there are two unknown quantities,

$$r = \rho \sqrt{\left(\frac{2[pf^2]}{m-2}\right)}. \quad (i)$$

Although the probable errors of a and b could be deduced in terms of the probable error of observation from their expressions just given, yet since this requires tedious operations and since the approximate solution which I am going to give of the general problem also gives the probable errors of the unknown quantities, I pursue this special case no farther.

Assume

$$P_1 = \frac{1^*}{N_1} \dots (7_1); P_2 = \frac{1}{N_2} \dots (7_2); \dots P_m = \frac{1}{N_m}; \quad (7_m)$$

then (4'') becomes $[f^2 \div N] = \text{a minimum.} \quad (4''')$

Let $a_0, b_0, \dots$ be some approximate values of the unknown quantities, and $\Delta a_0, \Delta b_0, \dots$ their corrections, then we have, in (4''') exact to quantities of the second order,

$$\begin{aligned} \left[\frac{f^2}{N}\right] &= \left[\frac{f^2}{N}\right]_0 + \frac{d}{da} \left[\frac{f^2}{N}\right]_0 \Delta a_0 + \frac{d}{db} \left[\frac{f^2}{N}\right]_0 \Delta b_0 + \dots \\ &\quad + \frac{d^2}{2da^2} \left[\frac{f^2}{N}\right]_0 \Delta a_0^2 + \frac{d^2}{2dad b} \left[\frac{f^2}{N}\right]_0 \Delta a_0 \Delta b_0 + \dots \\ &\quad + \frac{d^2}{2dad b} \left[\frac{f^2}{N}\right]_0 \Delta a_0 \Delta b_0 + \frac{d^2}{2db^2} \left[\frac{f^2}{N}\right]_0 \Delta b_0^2 + \dots \\ &\quad + \dots \dots \dots \quad (4''') \\ &= \text{a minimum;} \end{aligned}$$

and the conditions for a minimum, or the normal equations, are

$$\begin{aligned} \frac{d}{2\Delta a_0} \left[\frac{f^2}{N}\right] &= \frac{d}{2da} \left[\frac{f^2}{N}\right]_0 + \frac{d^2}{2da^2} \left[\frac{f^2}{N}\right]_0 \Delta a_0 + \frac{d^2}{2dad b} \left[\frac{f^2}{N}\right]_0 \Delta b_0 \\ &\quad + \dots = 0, \quad (6'_a) \\ \frac{d}{2\Delta b_0} \left[\frac{f^2}{N}\right] &= \frac{d}{2db} \left[\frac{f^2}{N}\right]_0 + \frac{d^2}{2dad b} \left[\frac{f^2}{N}\right]_0 \Delta a_0 + \frac{d^2}{2db^2} \left[\frac{f^2}{N}\right]_0 \Delta b_0 \\ &\quad + \dots = 0, \quad (6'_b) \\ &\dots \dots \dots \end{aligned}$$

*This is done because the N 's are easier to differentiate than the P 's.

We have

$$\frac{d}{2da} \left[\frac{f^2}{N} \right]_0 = \left[\frac{f}{N} \left(\frac{df}{da} - \frac{f}{2N} \frac{dN}{da} \right) \right]_0, \quad (8_a)$$

$$\frac{d}{2db} \left[\frac{f^2}{N} \right]_0 = \left[\frac{f}{N} \left(\frac{df}{db} - \frac{f}{2N} \frac{dN}{db} \right) \right]_0, \quad (8_b)$$

$$\frac{d^2}{2da^2} \left[\frac{f^2}{N} \right]_0 = \left[\frac{1}{N} \left(\frac{df}{da} - \frac{f}{2N} \frac{dN}{da} \right)^2 \right]_0 + \left[\frac{f}{N} \left(\frac{d^2f}{da^2} - \frac{f}{2N} \frac{d^2N}{da^2} \right) \right]_0, \quad (8_{aa})$$

$$\begin{aligned} \frac{d^2}{2dad b} \left[\frac{f^2}{N} \right]_0 &= \left[\frac{1}{N} \left(\frac{df}{da} - \frac{f}{2N} \frac{dN}{da} \right) \left(\frac{df}{db} - \frac{f}{2N} \frac{dN}{db} \right) \right]_0 \\ &\quad + \left[\frac{f}{N} \left(\frac{d^2f}{dad b} - \frac{f}{2N} \frac{d^2N}{dad b} \right) \right]_0, \quad (8_{ab}) \end{aligned}$$

$$\frac{d^2}{2db^2} \left[\frac{f^2}{N} \right]_0 = \left[\frac{1}{N} \left(\frac{df}{db} - \frac{f}{2N} \frac{dN}{db} \right)^2 \right]_0 + \left[\frac{f}{N} \left(\frac{d^2f}{db^2} - \frac{f}{2N} \frac{d^2N}{db^2} \right) \right]_0, \quad (8_{bb})$$

Having computed the first and second differential coefficient of f and N , these coefficients can be found and used for the normal equations (6').

If we place

$$\frac{df_1}{da} - \frac{f_1}{2N_1} \frac{dN_1}{da} = X_1; \quad \frac{df_1}{db} - \frac{f_1}{2N_1} \frac{dN_1}{db} = Y_1, \dots \quad (9_1)$$

$$\frac{df_2}{da} - \frac{f_2}{2N_2} \frac{dN_2}{da} = X_2; \quad \frac{df_2}{db} - \frac{f_2}{2N_2} \frac{dN_2}{db} = Y_2, \dots \quad (9_2)$$

$$\frac{df_m}{da} - \frac{f_m}{2N_m} \frac{dN_m}{da} = X_m; \quad \frac{df_m}{db} - \frac{f_m}{2N_m} \frac{dN_m}{db} = Y_m, \dots \quad (9_m)$$

we have sufficiently near (replacing now $1/N$ by P)

$$[PXf]_0 + [PX^2]_0 da_0 + [PXY]_0 db_0 + \dots = 0, \quad (6'')$$

$$[PYf]_0 + [PXY]_0 da_0 + [PY^2]_0 db_0 + \dots = 0, \quad (6''')$$

These normal equations we might consider to have been formed from the following fictitious observation equations:

$$A_1 = f_1 + X_1 da_0 + Y_1 db_0 + \dots \quad (\text{weight} = P_1) \quad (10_1)$$

$$A_2 = f_2 + X_2 da_0 + Y_2 db_0 + \dots \quad (\text{weight} = P_2) \quad (10_2)$$

$$A_m = f_m + X_m da_0 + Y_m db_0 + \dots \quad (\text{weight} = P_m) \quad (10_m)$$

if $[PA^2]$ is made a minimum.

The equations $(6'')$, $(6''')$. . . not only give the corrections Δa_0 , Δb_0 , . . . to the assumed values of the unknown quantities, but also their weights in the usual manner.

If n = number of unknown quantities there are $m-n$ conditions between the observed quantities alone. We have then for the probable error of an observation of weight 1

$$r = \rho \sqrt{\left(\frac{2[Pf^2]}{m-n} \right)}; \quad (11)$$

where $[Pf^2]$ is to be computed by means of the final values of a , b , . . . and if p_a , p_b , . . . are the weights of a , b , . . . as deduced from $(6'')$, $(6''')$. . . we have their probable errors,

$$r_a = \frac{r}{\sqrt{p_a}}, \quad (12_a)$$

$$r_b = \frac{r}{\sqrt{p_b}}, \quad (12_b)$$

.

Applying this process to the above simple case we have

$$\frac{df_1}{da} = x_1; \quad \frac{df_1}{db} = 1; \quad \frac{d^2 f_1}{da^2} = \frac{d^2 f_1}{dad b} = \frac{d^2 f_1}{db^2} = 0, \quad (j_1)$$

$$\frac{df_2}{da} = x_2; \quad \frac{df_2}{db} = 1; \quad \frac{d^2 f_2}{da^2} = \frac{d^2 f_2}{dad b} = \frac{d^2 f_2}{db^2} = 0, \quad (j_2)$$

$$\frac{dN_1}{da} = \frac{2a}{p_1}; \quad \frac{dN_1}{db} = 0; \quad \frac{d^2 N_1}{da^2} = \frac{2}{p_1}; \quad \frac{d^2 N_1}{dad b} = \frac{d^2 N_1}{db^2} = 0, \quad (k_1)$$

$$\frac{dN_2}{da} = \frac{2a}{p_2}; \quad \frac{dN_2}{db} = 0; \quad \frac{d^2 N_2}{da^2} = \frac{2}{p_2}; \quad \frac{d^2 N_2}{dad b} = \frac{d^2 N_2}{db^2} = 0, \quad (k_2)$$

.

We have then

$$\frac{d}{2da} \left[\frac{f^2}{N} \right]_0 = \left[\frac{f}{N} \left(x - \frac{f}{N} \cdot \frac{a}{p_1} \right) \right]_0 = \left[Pf \left(x - \frac{Pf}{p} a \right) \right]_0 \quad (l_a)$$

$$\frac{d}{2db} \left[\frac{f^2}{N} \right]_0 = \left[\frac{f}{N} \right]_0 = [Pf]_0, \quad (l_b)$$

$$\frac{d^2}{2da^2} \left[\frac{f^2}{N} \right]_0 = \left[P \left(x - \frac{2Pf}{p} a \right)^2 \right]_0 - \left[\frac{P^2}{p} f^2 \right]_0, \quad (l_{aa})$$

$$\frac{d^2}{2dad b} \left[\frac{f^2}{N} \right]_0 = \left[P \left(x - \frac{2Pf}{p} a \right) \right]_0, \quad (l_{ab})$$

$$\frac{d^2}{2db^2} \left[\frac{f^2}{N} \right]_0 = [P]_0. \quad (l_{bb})$$

Substituting into (6') we have

$$\left[Pf \left(x - \frac{Pf a}{p} \right) \right]_0 + \left[P \left(x - \frac{2Pf a}{p} \right)^2 - \frac{P^2 f^2}{p} \right]_0 \Delta a_0 + \left[P \left(x - \frac{2Pf a}{p} \right) \right]_0 \Delta b_0 = 0, \quad (m_a)$$

$$\left[Pf \right]_0 + \left[P \left(x - \frac{2Pf a}{p} \right) \right]_0 \Delta a_0 + \left[P \right]_0 \Delta b_0 = 0. \quad (m_b)$$

Employing (9) (6'') and (10) we have also

$$X_1 = x_1 - \frac{P_1 f_1 a}{p_1}; \quad Y_1 = 1, \quad (n_1)$$

$$X_2 = x_2 - \frac{P_2 f_2 a}{p_2}; \quad Y_2 = 1, \quad (n_2)$$

.

Should the results of these two methods differ materially, the indication is that there is considerable uncertainty about the determination of the unknown quantities or that their probable errors are large. It would not be judicious in that case to attempt a more exact solution.

SYMMETRICAL FUNCTIONS OF THE SINES OF THE ANGLES INCLUDED IN THE EXPRESSION

$$\alpha_0 + \frac{2k\pi}{n}.$$

BY PROF. W. W. JOHNSON, ANNAPOLIS, MARYLAND.

DENOTING these angles by $\alpha_0, \alpha_1, \alpha_2, \dots, \alpha_{n-1}$, the values of $\sin n\alpha$ and $\cos n\alpha$ will be the same for all; hence, putting x for $\sin \theta$ and $n\alpha$ for $n\theta$ in the developments of $\cos n\theta$ and $\sin n\theta$ in terms of $\sin \theta$ (quoted on page 17) we have the eq'ns whose roots are $\sin \alpha_0, \sin \alpha_1$, etc.; viz., when n is even,

$$\cos n\alpha = 1 - \frac{n^2}{2} x^2 + \frac{n^2(n^2-2^2)}{4!} x^4 \dots + (-1)^{\frac{1}{2}n} \frac{n^2(n^2-2^2)(n^2-4^2) \dots [n^2-(n-2)^2]}{n!} x^n, \quad (1)$$

and when n is odd,

$$\sin n\alpha = nx - \frac{n(n^2-1^2)}{3!} x^3 \dots + (-1)^{\frac{n-1}{2}} \frac{n(n^2-1^2)(n^2-3^2) \dots [n^2-(n-2)^2]}{n!} x^n. \quad (2)$$

On dividing by the coefficient of x^n each of these eq'ns takes the form

$$0 = x^n - \frac{n(n-1)}{n^2-(n-2)^2} x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{[n^2-(n-2)^2][n^2-(n-4)^2]} x^{n-4} - \dots,$$

the equations differing only in the last terms. Reducing the coefficients,

the equation becomes

$$0 = x^n - \frac{n}{4}x^{n-2} + \frac{n(n-3)}{4.8}x^{n-4} - \frac{n(n-4)(n-5)}{4.8.12}x^{n-6}, \dots \quad (3)$$

the general term being

$$(-1)^r \frac{n(n-r-1) \dots (n-2r+1)}{4^r \cdot r!} x^{n-2r}.$$

But when n is even the last term, [as found from (1)] is obtained by putting $1 - \cos na$ in place of x^0 when $r = \frac{1}{2}n$; it is therefore

$$(-1)^{\frac{1}{2}n} \frac{n(\frac{1}{2}n-1)!}{2^n \cdot (\frac{1}{2}n)!} (1 - \cos na) = (-1)^{\frac{1}{2}n} \frac{\sin^2 \frac{1}{2}na}{2^{\frac{n-2}{2}}}: \quad (4)$$

and when n is odd the last term [as found from (2)] is obtained by putting $-\sin na$ in place of nx in the term corresponding to $r = \frac{1}{2}(n-1)$; the last two terms are therefore

$$(-1)^{\frac{n-1}{2}} \frac{n \cdot (\frac{n-1}{2})!}{2^{n-1} (\frac{n-1}{2})!} x - (-1)^{\frac{n-1}{2}} \frac{\sin na}{2^{\frac{n-1}{2}}}. \quad (5)$$

Since the last terms only involve a , all the symmetrical rational integral functions of $\sin a_0, \sin a_1$, etc., of a degree less than n have values independent of a_0 , and the form of equation (3) shows that all those of an odd order vanish. With this restriction we have

$\Sigma \sin a_0 \sin a_1 = \frac{1}{2}n$, $\Sigma \sin a_0 \sin a_1 \sin a_2 \sin a_3 = \frac{1}{8}n(n-3)$,
and in general

$$\Sigma \sin a_0 \dots \sin a_{2r-1} = (-1)^r \frac{n \cdot (n-r-1)!}{4^r \cdot r! \cdot (n-2r)!}$$

By employing Hirsch's tables of symmetrical functions (given in an Appendix to Salmon's Higher Algebra) we may calculate any function of degree not higher than the tenth, and any function involving only even powers, up to the twentieth degree. I had by this means obtained

$$\Sigma \sin^2 a = \frac{1}{2}n, \quad \Sigma \sin^4 a = \frac{3}{8}n, \quad \Sigma \sin^6 a = \frac{5}{16}n, \quad \Sigma \sin^8 a = \frac{35}{128}n,$$

which are remarkable as containing the first power only of n , before I noticed that it is easily shown *a priori* that this is universally true. For compare the values of $\Sigma \sin^{2r} a$ for two values of n, n' and n'' , (both greater than $2r$) with its value when $n = n'n''$. It is evident that the angles $a_0 + (2k\pi \div n'n'')$ are equivalent to n'' sets of angles of the form $a'_0 + (2k\pi \div n')$, hence the $n'n''$ terms of $\Sigma^{n'n''} \sin^{2r} a$ are equivalent to n'' sets of terms each of which has for its sum $\Sigma^{n'} \sin^{2r} a$; that is, $\Sigma^{n'n''} \sin^{2r} a = n'' \Sigma^{n'} \sin^{2r} a = n' \Sigma^{n''} \sin^{2r} a$. Therefore $\Sigma^{n'} \sin^{2r} a = kn'$, where k is a numerical constant.

The value of $\Sigma \sin^{2r} a$ is now easily calculated since it is necessary to retain only the first power of n in the result. Since each of the coefficients in eq. (3) contains n as a factor, it is unnecessary to retain any terms in the expression for $\Sigma \sin^{2r} a$ in terms of the coefficients which involve more than one coefficient. Now the only term of Σa^{2r} which is of the first degree in the coefficients is $-2rp_{2r}$ (p_{2r} being the coefficient of x^{n-2r} in the equation whose roots are a, β , etc). Moreover, in the value of p_2 (see general term above) it is only necessary to retain the numerical part of each of the binomial factors. Therefore

$$\Sigma \sin^{2r} a = -2rp_{2r} + \&c. = \frac{2rn.(r+1) \dots (2r-1)}{4^r.r!} = \frac{n}{2^{2r-1}} \frac{(2r-1)!}{r!(r-1)!}$$

the coefficient of $n \div 2^{2r-1}$ being the middle coefficient of the expansion of $(a+b)^{2r-1}$.

We may now readily calculate any symmetrical function by means of the formulæ

$$\Sigma a^m \beta^p = \Sigma a^m . \Sigma a^p - \Sigma a^{m+p}, \&c.;$$

for example

$$\Sigma \sin^2 a_0 . \sin^6 a_1 = \frac{5}{32} n^2 - \frac{35}{128} n,$$

and

$$\Sigma \sin^3 a_0 . \sin^6 a_1 = -\frac{35}{128} n.$$

The values (4) and (5) of the last term of equation (3) show that, when n is even, the product

$$\sin a_0 . \sin a_1 \dots \sin a_{n-1} = (-1)^{\frac{1}{2}n} \sin^2 \frac{1}{2} n a \div 2^{n-2},$$

and when n is odd

$$\sin a_0 . \sin a_1 \dots \sin a_{n-1} = (-1)^{\frac{1}{2}(n-1)} \sin n a \div 2^{n-1}.$$

In the first of these expressions one half of the factors are negative, and numerically equal to the positive factors; hence putting $2n'$ in place of n , the product of the sines of the angles included in the expres'n $a_0 + (k\pi \div n')$ [$a_0 < (\pi \div n)$], is $\sin a_0 . \sin a_1 \dots \sin a_{n'-1} = \sin n' a \div 2^{n'-1}$.

These expressions of course vanish when $a_0 = 0$, but in that case if we divide by $\sin a_0$ and evaluate, we have, when n is odd,

$$\sin \frac{2\pi}{n} \sin \frac{4\pi}{n} \dots \sin \frac{2(n-1)\pi}{n} = (-1)^{\frac{n-1}{2}} \frac{n}{2^{n-1}},$$

and for all values of n'

$$\sin \frac{\pi}{n'} \sin \frac{2\pi}{n'} \dots \sin \frac{(n'-1)\pi}{n'} = \frac{n'}{2^{n'-1}}.$$

Equation (1) shows that the sum of the reciprocals of the roots, $\Sigma \operatorname{cosec} a_0$ vanishes when n is even, but

$$\Sigma \operatorname{cosec} a_0 \operatorname{cosec} a_1 = -n^2 \div (4 \sin^2 \frac{1}{2} n a), \text{ whence } \Sigma \operatorname{cosec}^2 a = \frac{1}{2} n^2 \operatorname{cosec}^2 \frac{1}{2} n a:$$

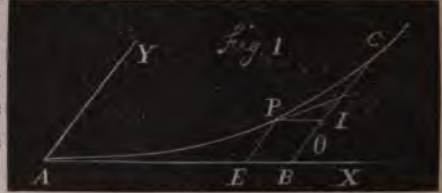
and equation (2) shows that when n is odd,

$$\Sigma \operatorname{cosec} a_0 = n \operatorname{cosec} n a, \quad \Sigma \operatorname{cosec} a_0 \operatorname{cosec} a_1 = 0, \text{ etc.}$$

CURVE OF PURSUIT GENERALIZED.

BY PROF. L. G. BARBOUR, RICHMOND, KY.

LET a particle start from B and describe a straight line BC with any given rate of motion, either uniform, or varying according to a given law. At the same instant let another particle start from A and maintain a velocity every where m times the velocity of the first particle, and a course always directed toward it; so that if the first particle should stop at any point as H , a tangent to the curve described by the other particle at the point P , which it has reached at the same instant, will pass through H .



Let AX be the axis of abscissas; A , the origin; AY parallel to BC , the axis of ordinates; AB , the unit of distance at the start; CBX , or θ , any angle from 0 to 180° .

The co-ordinates of P are $AE = x$ and $PE = y$. Draw PI parallel to AX . Let $AP = u$. By the conditions, $u = m.BH = m.BI + m.IH$. But

$$PI : IH :: 1 - x : IH :: dx : dy; \therefore IH = (1 - x) \frac{dy}{dx}.$$

It will be noticed that the obliqueness of the co-ordinates is recognized in the differentials. Therefore

$$u = m \left\{ y + (1 - x) \frac{dy}{dx} \right\}; \therefore du = m dy + m(1 - x) \frac{d^2y}{dx^2} - m dy = m(1 - x) \frac{d^2y}{dx^2}$$

Also $du^2 = dx^2 + dy^2 + 2dx dy \cos \theta$. Therefore

$$m(1 - x) \frac{d^2y}{dx^2} = dx \sqrt{\left(1 + \frac{dy^2}{dx^2} + 2 \frac{dy}{dx} \cos \theta\right)}.$$

Let $v = dy \div dx$, then $m(1 - x)dv = dx \sqrt{(1 + v^2 + 2v \cos \theta)}$. It must be remembered that θ is an arbitrary constant.

Separating the variables, $\frac{dv}{\sqrt{(1 + v^2 + 2v \cos \theta)}} = \frac{dx}{m(1 - x)}$. Integrating,

$$\log[v + \cos \theta + \sqrt{(1 + v^2 + 2v \cos \theta)}] = -(1 \div m) \log(1 - x) + C.$$

To determine the constant, let $x = 0$. This is at A . The curve at this pt. is tangent to axis of abscissas, $\therefore (dy \div dx) = v = 0$; $\therefore C = \log(\cos \theta + 1)$.

$$\therefore \log[v + \cos \theta + \sqrt{(1 + v^2 + 2v \cos \theta)}] = -(1 \div m) \log(1 - x) + \log(\cos \theta + 1);$$

$$\therefore \frac{dy}{dx} + \cos \theta + \sqrt{(1 + v^2 + 2v \cos \theta)} = (1 - x)^{-\frac{1}{m}} (\cos \theta + 1);$$

$$\therefore \sqrt{\left(1 + \frac{dy^2}{dx^2} + 2\frac{dy}{dx} \cos \theta\right)} = (1-x)^{-\frac{1}{m}} (\cos \theta + 1) - \frac{dy}{dx} - \cos \theta;$$

$$du = dx \sqrt{\left(1 + \frac{dy^2}{dx^2} + 2\frac{dy}{dx} \cos \theta\right)} = (1-x)^{-\frac{1}{m}} (\cos \theta + 1) dx - dy - \cos \theta dx.$$

Integrating,

$$u = -\frac{(1-x)^{\frac{m-1}{m}}}{\frac{m-1}{m}} (\cos \theta + 1) - y - x \cos \theta + C'.$$

To find C' , let $x=0$, $\therefore y=0$, $u=0$; $\therefore C' = [m \div (m-1)](\cos \theta + 1)$.

Integrating between the limits $x=0$, $x=1$, and $y=0$, $y=BC$, we get

$$u = my = -y - \cos \theta + [m \div (m-1)](\cos \theta + 1);$$

$$\therefore y = \frac{m + \cos \theta}{m^2 - 1}; \therefore u = m \frac{m + \cos \theta}{m^2 - 1}.$$

DISCUSSION. (1). If $\theta = 0$, $y = 1 \div (m-1)$, $u = m \div (m-1)$. This is the well known algebraic problem of the couriers. BC becomes a prolongation of AB , and the curve APC degenerates into a straight line. It is readily seen that the hindmost particle travels over $AB = 1$, and in addition as far as the foremost particle moves, i. e.,

$$u = \frac{m}{m-1} = 1 + \frac{1}{m-1}.$$

If $\theta = 180^\circ$, the two particles approach each other;

$$y = \frac{1}{m+1}; u = \frac{m}{m+1}; u+y = 1.$$

If $\theta = 90^\circ$, we have the ordinary form of the curve of pursuit;

$$y = \frac{m}{m^2-1}; u = \frac{m^2}{m^2-1}.$$

(2). If $m = 1$, $\theta = 180^\circ$, then $y = u = \frac{1}{2}$. But if $m = 1$, $\theta = 0$, then $y = \infty$, as in the problem of the couriers.

If $m = 1$ and $\theta = 90^\circ$, $y = \infty$, and the straight line BC is asymptote to the curve APC . The curve is then the *Tractrix*, whose evolute is the Catenary. When $m = \infty$, then $y = 0$, $u = 1 = AB$.

Let $m = 2$, $\theta = 0$, then $y = \frac{2}{3}$; $\theta = 60^\circ$, $y = \frac{5}{6}$; $\theta = 90^\circ$, $y = \frac{4}{3}$; $\theta = 120^\circ$, $y = \frac{3}{2}$; $\theta = 180^\circ$, $y = \frac{2}{3}$.

Ordinarily the particle starting from B is supposed to move at a constant rate. This, however has not entered into our demonstration at all. As stated in the outset, any rate may be assumed, provided both particles are subjected to the same law. We thus find easy solutions to problems like the following:

I. A falcon is trained to fly after a tennis ball that has been projected vertically, the bird heading always toward the ball. — Given the length of a straight line from the ball to the bird when he started, the angle that line makes with the vertical line described by the ball, and the velocity of the ball = $1 \div m$ that of the bird. — Required the distance the falcon flies before catching the ball.

II. A hawk, carrying a fish in its mouth, is pursued by an eagle. At length both pause mid air for a moment. The hawk drops the fish, and the eagle, being on the alert, darts after it at the same instant. Same data and requirement as in No. I.

To find the value of u , x and y at any point of the curve, we can assume a value for x in the general equation which is

$$u = -\frac{m(1-x)^{\frac{m-1}{m}}}{m-1}(\cos \theta + 1) - y - x \cos \theta + \frac{m}{m-1}(\cos \theta + 1).$$

The value of y and of $u = my$ can be readily obtained.

But if we assume a value for y and then attempt to deduce that of x , it will make, in some cases, a difficult though not insoluble problem. It is best to remove the origin from A to B . Designate BE by x' ; $\therefore 1 - x = x'$ and $-x = -1 + x'$. Substituting in the general equation, we have

$$u = my = -\frac{m}{m-1}x'^{\frac{m-1}{m}}(\cos \theta + 1) - y - \cos \theta + x' \cos \theta + \frac{m}{m-1}(\cos \theta + 1).$$

For any given values of m , θ and y , the value of x' can be found. The general form to which it can be reduced is

$$x'^{\frac{m-1}{m}} - ax = b.$$

Let $x' = z^m$; then $x'^{\frac{m-1}{m}} = z^{m-1}$; $\therefore z^{m-1} - az^m = b$; a , b and m being numbers.

In the demonstration the origin might have been put at B ; then dx would be a decreasing function of x , which is apt to perplex the signs.

CURVE OF CAPTURE.—If a hare starts from B and runs toward A , and a hound at the same moment starts from A and runs toward B , they will meet at some intermediate point on the line AB . The particular point will be determined by the value of m in the expression $(m + \cos \theta) \div (m^2 - 1)$. Now let us make m the constant and θ the variable. For the purpose in hand, m must never be less than 1; if m be greater than 1, the hound will catch the hare, in whatever direction the hare may run from B , that is, whatever may be the value of θ . In fig. 1, θ is an acute angle, and the point of overtaking is at C . If θ were zero, the point would be on the axis of abscissas to the right of B . If in succession a hundred hares, running in a

hundred different directions, should be pursued by as many hounds, the hares would be caught each at a different point; but the hundred points would all be situated on one and the same curve. It is of course understood that each hound runs m times as fast as the hare he pursues; m being an arbitrary constant, each value chosen would give a different curve but all the curves would belong to the same family.

In fig. 2, let B be the starting point of the hares. The locus of the p'ts where they are caught will be the *Curve of Capture*.

From the previous formula, generalizing and calling the distance from B to the point of capture ρ , we get the polar equation

$$\rho = \frac{m + \cos \theta}{m^2 - 1}. \quad \therefore IB = \frac{1}{m+1}; BH = \frac{1}{m-1}.$$

Also $IB + BH = 2m \div (m^2 - 1)$. Make $IC = CH$ and we find

$$IC = \frac{m}{m^2 - 1}; \quad \therefore BC = \frac{1}{m^2 - 1}.$$

Since $BE = m \div (m^2 - 1)$, EBH being a right angle, $BE = IC$. Since $BC : IC :: 1 : m$, we may take any line as IH , bisect it at C , divide IC into m parts of which BC is one, and at B erect a perpendicular equal to IC ; we shall have three points, I , E and H , of the curve. In this case, m is given. But we may suppose m not given. Then at any point of IC erect a perpendicular equal to IC , and we have three points of the Curve of Capture: Also $BC \div IC = m$.

We here suppose IC constant, and B a movable point. Place B at C , and we have $BC = 0$, $m = \infty$, and the curve becomes a circle. Remove C to an infinite distance, then $BC = IC$; $\therefore m = 1$, and the curve becomes a straight line.

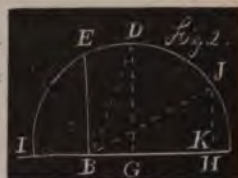
To find the points where the curve cuts the axis of abscissas; any abscissa, as $BK = \rho \cos \theta = m \cos \theta + \cos^2 \theta$, $\therefore -m \sin \theta - 2 \cos \theta \sin \theta = 0$, $\therefore \sin \theta = 0$, $\therefore \theta = 0$ or 180° , for the maximum value of abscissas. The curve is perpendicular to the axis at I and H . Or the equation may be satisfied by $-m - 2 \cos \theta = 0$; $\therefore \cos \theta = -\frac{1}{2}m$, which gives negative values for cosine, and $\theta > 90^\circ$; our limits forbid the consideration of these.

The maximum and minimum values of ρ are easily derived from

$$\rho = \frac{m + \cos \theta}{m^2 - 1};$$

$\therefore -\sin \theta = 0$; $\therefore \theta = 0$, or 180° , as we found in the case of abscissas.

From the genesis of the curve it is evidently symmetrical to the axis of abscissas. The figure is drawn as if CD were the maximum ordinate. Let us see if this is true. $BC = 1 \div (m^2 - 1) = BD \cos \theta$; therefore



$$\frac{m \cos \theta + \cos^2 \theta}{m^2 - 1} = \frac{1}{m^2 - 1}; \therefore \cos \theta = \frac{-m \pm \sqrt{m^2 + 4}}{2}.$$

This is the value of $\cos \theta$ when $\theta = DBC$. To see whether CD is a maximum ordinate, take any ordinate, as $JK = JB \sin \theta = \rho \sin \theta = (m \sin \theta + \cos \theta \sin \theta) \div (m^2 - 1)$; for a maximum, $m \cos \theta + \cos^2 \theta = \sin^2 \theta = 1 - \cos^2 \theta$. $\therefore 2 \cos^2 \theta - \frac{1}{2} m \cos \theta = \frac{1}{2}$; $\therefore \cos \theta = \frac{1}{4} [-m \pm \sqrt{m^2 + 8}]$, which differs from the value for $\theta = DBC$. It will be found moreover that no value for m will satisfy both equations. Therefore the curve is not symmetrical to the axis CD . Thus for $m=2$, we get $DBC = 65^\circ 32'$; whereas, for the max. ordinate, $\theta = 68^\circ 31'$; showing that the highest point is to the left of D .

SOLUTION OF A PROBLEM IN MENSURATION, BY REQUEST. *Problem.*—

Four cylinders of equal radii, r , intersect each other at right angles, two and two, in the same horizontal plane, each pair being tangent to each other and having their axes parallel. Required the contents of the solid bounded by the surfaces of the four cylinders and their common tangent plane.

Solution.—Let $ADEF$ represent one-fourth of the surface bounded by the common tangent plane, and $ADEFB$, one fourth of the required solid; then will ABC and BEG represent one-fourth the sectional areas of two of the intersecting cylinders.

Let $adef$ be a section of the required solid parallel to the common tangent plane and cutting the radius CA in H , and put $AH = x$. Then from property of the circle, we have

$$aH = \sqrt{(2rx - x^2)},$$

and consequently

$$fa = r - \sqrt{(2rx - x^2)},$$

= one of the equal sides of the variable plane $adef$. Hence the area of the variable plane $adef = [r - \sqrt{(2rx - x^2)}]^2$

$$= r^2 + 2rx - x^2 - 2r\sqrt{(2rx - x^2)}.$$

Because the required solid is the sum of all its parallel planes, putting dx for the thickness of each plane and S for one-fourth the required solid, we have

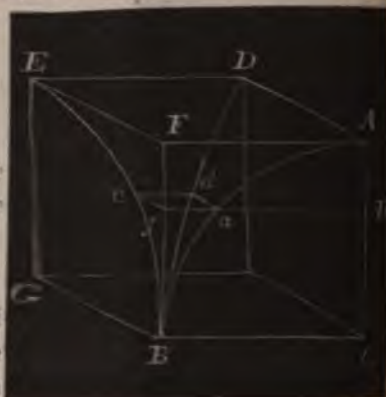
$$dS = r^2 dx + 2rx dx - x^2 dx - 2r\sqrt{(2rx - x^2)} dx;$$

$$\therefore S = r^2 x + rx^2 - \frac{1}{3} x^3 - \int_0^x \sqrt{(2rx - x^2)} dx$$

$$= \frac{2}{3} r^3 - \frac{1}{2} r \times \text{area of circle whose radius is } r.$$

$$\therefore 4S = \frac{2}{3} r^3 - 2r \times \text{area of circle whose radius is } r.$$

If $r = 7\text{ft } 3\text{in}$, this becomes $2540.52 - 2394.39 = 146.13$ cubic feet.



SOLUTION OF A PROBLEM.

BY R. J. ADCOCK, MONMOUTH, ILLINOIS.

Problem.—The velocity of the wind being v , and its force being proportional to the square of the velocity; and turning a wind wheel with a horizontal axis, the driving surface of the wings being described by a straight line perpendicular to, and moving uniformly along, the axis, while it rotates uniformly about the axis; what then must be the inclination to the plane, perpendicular to the axis, of the part of the wing farthest from the axis, in order that the wheel when moving with the proper angular velocity, may be doing a maximum amount of work in a given time?

Solution.—Let f = the force of the wind on a unit of surface when moving with a unit of velocity, then fv^2 = its force for velocity v .

Let b = dist. moved along axis by gen. line of wing surf. in one revol'n,

x = distance of any point of generatrix from axis,

z = angle made with plane perp. to axis by helix at distance x ,

z_1 = same at distance r , the length of generatrix,

α = angular velocity of rotation of wheel. Then

$$b = 2\pi x \tan z = 2\pi r \tan z_1,$$

$$\sqrt{(b^2 + 4\pi^2 x^2)} dx = -\frac{b^2 dz}{2\pi \sin^3 z} = \text{elementary area of wings,}$$

$v - \alpha x \tan z$ = velocity with which the wind strikes this elementary surface,

$-f \sin z \frac{\cos^2 z}{\sin z} (v - \alpha x \tan z)^2 \frac{b^2 dz}{4\pi^2 \sin^3 z} = -f(v - \alpha x \tan z)^2 \frac{b^2 \cos^2 z dz}{4\pi^2 \sin^3 z}$ = elementary moment; and

$-f \alpha (v - \alpha x \tan z)^2 \frac{b^2 \cos^2 z dz}{4\pi^2 \sin^3 z}$ = elementary work done in a unit of time;

which in case of a maximum for α , gives

$$(v - \alpha x \tan z)^2 - 2\alpha(v - \alpha x \tan z)x \tan z = 0,$$

$$v - \alpha x \tan z = 0, v - \alpha x \tan z - 2\alpha x \tan z = 0, \alpha x \tan z = \frac{1}{3}v.$$

Since, from (1), $x \tan z = r \tan z_1$, $\alpha r \tan z_1 = \frac{1}{3}v$. Integrating elementary area,

$$-\int_0^{z_1} \frac{b^2 dz}{2\pi \sin^3 z} = \pi r^2 \tan z_1 \left(\frac{\cos z_1}{\sin^2 z_1} - \log \tan \frac{1}{2} z_1 \right) = \text{entire area.}$$

By reduction the elementary moment is

$$-f(v - \alpha x \tan z)^2 \frac{b^2 \cos^2 z dz}{4\pi^2 \sin^3 z} = -\frac{1}{9} f v^2 \frac{b^2 \cos^2 z dz}{\pi^2 \sin^3 z} = -\frac{8}{9} f v^2 \pi r^3 \tan^3 z_1 \frac{\cos^2 z dz}{\sin^3 z}.$$

And the entire moment when moving with proper angular velocity is

$$\int_{\frac{\pi}{2}}^{z_1} -\frac{8}{9} f v^2 \pi r^3 \tan^3 z_1 \frac{\cos^2 z dz}{\sin^3 z} = \frac{8}{9} \pi f v^2 r^3 \tan^3 z_1 \left(\frac{\cos z_1}{\sin^2 z_1} + \log \tan \frac{1}{2} z_1 \right).$$

Hence, $\frac{\frac{8}{9} f v^2 r \tan z_1 [(\cos z_1 \div \sin^2 z_1) + \log \tan \frac{1}{2} z_1]}{(\cos z_1 \div \sin^2 z_1) - \log \tan \frac{1}{2} z_1}$ is a maximum for z_1 , (2)

will give the position of wings when the work being done is a maximum.

Therefore, placing the first differential coefficient = 0, and solving for $\log \tan \frac{1}{2} z_1$, there results

$$\log \tan \frac{1}{2} z_1 = (\cos z_1 \div \sin^2 z_1) [1 + \cos^2 z_1 \pm \sqrt{(4 + \cos^4 z_1)}].$$

Substituting in (2) this second value, the only one applicable,

$$\frac{\tan z_1 [1 + 1 + \cos^2 z_1 - \sqrt{(4 + \cos^4 z_1)}]}{1 - 1 - \cos^2 z_1 + \sqrt{(4 + \cos^4 z_1)}} = \frac{\tan z_1 [2 + \cos^2 z_1 - \sqrt{(4 + \cos^4 z_1)}]}{\sqrt{(4 + \cos^4 z_1)} - \cos^2 z_1}$$

$$= -\tan z_1 + \frac{1}{2} \tan z_1 [\cos^2 z_1 + \sqrt{(4 + \cos^4 z_1)}] = \text{maximum for } z_1.$$

Differentiating a second time and reducing,

$$\cos^6 z_1 - \frac{7}{2} \cos^4 z_1 - 2 \cos^2 z_1 + 2 = 0, \text{ from which}$$

$$\cos^2 z_1 = 0.55154; \therefore z_1 = 42^\circ 21'.$$

SOLUTION OF A PROBLEM.

BY PROF. C. M. WOODWARD, WASHINGTON UNIV., ST. LOUIS, MO.

Problem.—Given a frustrum of an oblique cone with a circular base; the frustrum is cut in two by a plane perpendicular to the principal plane of the cone, and tangent to the two bases. Find the ratio of the volumes of the two parts of the frustrum.

Solution.—Let $vABCD$ be the section of the cone made by the principal plane, and let DBN be the trace of the intersecting plane perpendicular to it. The section, projected in DB , is obviously an ellipse of which DB is the major axis. Let $ov = h$, $As = r$, $Dk = R$ and $on = x$. Then

$$DB = (R + r) \frac{\sin \varphi}{\sin \theta}.$$

The minor axis is the chord of a circle whose radius is $\frac{1}{2}(R + r)$, and whose dist. from the center is $\frac{1}{2}(R - r)$; its length is therefore $2\sqrt{(Rr)}$.



The volume of the cone $v-DB$ is

$$V_0 = \pi \frac{R+r}{2} \frac{\sin \varphi}{\sin \theta} (Rr)^{\frac{1}{2}} \frac{h \sin \theta}{3} = \frac{1}{3} \pi h \sin \varphi \frac{R+r}{2} (Rr)^{\frac{1}{2}},$$

but $\frac{R+r}{2} = \frac{Rr}{x}$; hence $V_0 = \frac{1}{3x} \pi h \sin \varphi R^{\frac{3}{2}} r^{\frac{3}{2}}.$

Now $vs = \frac{hr}{x}$ and $vk = \frac{hR}{x}$, hence

cone $v-AB = V_1 = \frac{1}{3x} \pi h \sin \varphi r^3$, cone $v-CD = V_2 = \frac{1}{3x} \pi h \sin \varphi R^3.$

$$\text{Vol. } ABD = V_0 - V_1 = \frac{1}{3x} \pi h \sin \varphi \left(R^{\frac{3}{2}} - r^{\frac{3}{2}} \right) r^{\frac{3}{2}},$$

$$\text{Vol. } BDC = V_2 - V_0 = \frac{1}{3x} \pi h \sin \varphi \left(R^{\frac{3}{2}} - r^{\frac{3}{2}} \right) R^{\frac{3}{2}},$$

hence $\frac{\text{Vol. } ABD}{\text{Vol. } BDC} = \frac{\sqrt[3]{r^3}}{\sqrt[3]{R^3}}.$

From the volumes of the three cones we see that the elliptical cone is a mean proportional between the other two. (This striking analogy between the cones $v-AB$, $v-BD$, $v-CD$, and the triangles AvB , DvB , DvC , I had never before noticed, though it must have been well known. Neither had I ever observed that the semi-conjugate axis of *such* a conic section is a mean proportional between the radii of the bases of the frustum.)

It is obvious that the plane AC divides the frustum in the same ratio as the plane BD .

If the altitude of the frustum is p we have $h \sin \varphi \div x = p \div (R-r)$ and

$$V_0 = \frac{1}{3} \pi p [\sqrt[3]{R^3 r^3} \div (R-r)],$$

$$V_1 = \frac{1}{3} \pi p [r^3 \div (R-r)], \quad V_2 = \frac{1}{3} \pi p [R^3 \div (R-r)],$$

$$V_0 - V_1 = \frac{1}{3} \pi p \frac{\sqrt[3]{R^3} - \sqrt[3]{r^3}}{R-r} r^{\frac{3}{2}},$$

$$V_2 - V_0 = \frac{1}{3} \pi p \frac{\sqrt[3]{R^3} - \sqrt[3]{r^3}}{R-r} R^{\frac{3}{2}}.$$

SOLUTION OF PROBLEM 255.

BY PROF. W. W. HENDRICKSON, NAVAL ACADEMY, ANNAPOLIS, MD.

Denoting the distance AB by a , and taking the axes as represented in the figure, the equations to the lines AC and BC are (1.) $y = x \tan (\varphi + a)$,

COMPUTATION OF THE CUBE ROOT OF 2.

BY ARTEMAS MARTIN, M. A., EDITOR OF THE MATH'L VISITOR, ERIE, PA.

HAVING recently computed the cube root of 2 to 52 places of decimals, by the method of approximation found in Simpson's Algebra, I submit it, with the work, for publication.

Let R = the true n th root of a number N , and r = a near approximate root, and put $q = nr^n \div (N - r^n)$; then (Simpson's Algebra, p. 169)

$$R = r + \frac{r(2q+n)}{q(2q+2n-1) + \frac{1}{2}(n-1)(2n-1)}, \text{ very nearly,}$$

which he says (p. 165) "quintuples the number of figures at every operation."

Taking $n = 3$ we have for the cube root of N ,

$$R = r + \frac{r(2q+3)}{q(2q+5) + \frac{5}{8}}, \text{ very nearly.}$$

To compute the cube root of 2, take $r = 1.25 = \frac{5}{4}$; then

$$\sqrt[3]{2} = 1.25 + \frac{\frac{5}{4}(253)}{125(255) + \frac{5}{8}} = \frac{5}{4} + \frac{759}{76504} = \frac{96389}{76504} = 1.2599210498 +,$$

which is true to the last figure.

$$\text{Now take } r = \frac{96389}{76504}, \text{ then } r^3 = \frac{895534711311869}{447767355672064},$$

and after some reductions we get

$$\begin{aligned} \sqrt[3]{2} &= \frac{96389}{76504} + \frac{5569174100732765358417747}{368129177985585128169959391884736464} \\ &= \frac{463813700424535044109807007546772121}{368129177985585128169959391884736464} \\ &= 1.2599210498948731647672106072782283505702514647015079+. \end{aligned}$$

ON THE TRISECTION OF AN ANGLE.

BY PROF. J. SCHEFFER, MERCERSBURG COLLEGE, PENN'A.

I SHALL here give some of the different methods which have been devised for the solution of this celebrated problem of the Platonic school.

1. Let $AB = a$ represent an arc, whose radius is r , and let F represent the point in which AB is trisected.

Denoting CD and BD respectively by x' and y' , and the coordinates of F by x and y , we have $x' = r \cos \alpha$, $y' = r \sin \alpha$; $x = r \cos \frac{1}{3}\alpha$, $y = r \sin \frac{1}{3}\alpha$.

Since $r^2 \sin \frac{2}{3}\alpha = 2r \sin \frac{1}{3}\alpha \times r \cos \frac{1}{3}\alpha$, and also $= r \sin \alpha \times r \cos \frac{1}{3}\alpha - r \cos \alpha r \sin \frac{1}{3}\alpha$, we have $2xy = xy' - x'y$, or

$$y = \frac{xy'}{2x + x'} \quad (1)$$

Putting $x = \frac{1}{2}x'$ in place of x , and $-y + \frac{1}{2}y'$ in place of y we obtain the simple equation

$$xy = \frac{1}{2}x'y', \quad (2)$$

which represents an equilateral hyperbola referred to its asymptotes.

From equation (2) we can easily construct the curve: Make $CE = \frac{1}{2}CD$, $EG = \frac{1}{2}BD$; through G draw the two axes respectively parallel to AA' and BD , and construct the equilateral hyperbola, cutting the arc AB at F , then will the arc AB be trisected at F .

Combining eqn. (1) with the equation $x^2 + y^2 = r^2$ of the circle we obtain the biquadratic equation, $x^4 + x'^2x^2 - \frac{3}{4}r^2x^2 - r^2x'x - \frac{1}{4}r^2x'^2 = 0$; whose four roots are

$$x_1 = -x' = -r \cos \alpha,$$

$$x_2 = r \cos \frac{1}{3}\alpha,$$

$$x_3 = -r \cos (60^\circ + \frac{1}{3}\alpha),$$

$$x_4 = -r \cos (60^\circ - \frac{1}{3}\alpha),$$

which shows that there are four points of intersection; x_1 corresponds to the point F' ; x_2 , to the point F ; x_3 to the point F''' , consequently $A'F''' = \frac{1}{3}A'F''B$; and x_4 corresponds to F'' , consequently $AF'' = \frac{1}{3}A'F''B$. For $\alpha = 45^\circ$, F' and F'' coincide. It follows from this that the hyperbola always trisects both the acute and its supplementary obtuse angle. If therefore an obtuse angle is to be trisected, it is only necessary to trisect its adjacent acute angle.

Pappus was the first who devised and employed the hyperbola for the solution of this problem.

2. Let ABC be the angle to be trisected. [The reader will readily construct this, and the subsequent figures in this Art., from their description.]

Let fall the perpendicular AD ; make $CD = 2AB$, and describe, with B as pole and AD as base, the upper Conchoid, CE ; draw AE parallel to BC and join BE , intersecting AD in F , then is $\angle CBE = \frac{1}{3}\angle ABC$.



For, bisecting FE in G and drawing AG , $CD = FE = 2AB$ according to the nature of the conchoid. Since $FG = GE = AG = AB$, we easily find $\angle CBE = \frac{1}{3} \angle ABC$. By the lower conchoid $C'E'$, the obtuse adjacent angle is trisected. For, bisecting $E'F'$ ($= C'D = 2AB$) in G' and drawing AG' , we have $E'G' = G'A = G'F' = AB$, from which at once follows $\angle E'BC' = \frac{1}{3} \angle ABC'$.

Nicomedes devised the conchoid for the trisection of an angle and the duplication of a cube.

3. Let BEC represent an Archimedean Spiral. Divide the radius BC of the circular arc AC , into three equal parts so that $BD = \frac{1}{3}BC$, then, describing from B , with radius BD , an arc which intersects the spiral at E , the angle $ABE = \frac{1}{3}$ angle ABC . For, according to the definition of the spiral, $AB : BE (= BD) = \angle ABC : \angle ABE$.

4. *Montucla* ascribes the following two solutions to the Platonic school.

1. Let ACB be the angle to be trisected. From C , with any radius, describe a semi-circle, and through B draw BE , intersecting the circle in D , so as to make DE = the radius of the circle, then angle at $E = \frac{1}{3}ACB$.

2. Let ABC be the angle to be trisected. Complete the rectangle above BC . Produce the upper side, and through B draw BE meeting the upper side produced in E and intersecting the perpendicular CA in D , so as to make $DE = 2AB$, then angle $DBC = \frac{1}{3}ABC$, as can be easily proved by drawing AG to the middle point of DE .

5. The jesuit *Thomas Ceva* devised an instrument for the trisection of an angle. It consists of four rulers, AE, AF, DB, DC , which form a rhombus, $ABDC$, and are movable around A, B, C, D . (The points B, G and C, H being, respectively, on AE and AF .) If the angle GDH is to be trisected, we take $GD = DH = BD$, fasten the instrument at D , and move the rulers so as to make AE and AF pass through G and H , then angle $EAF = \frac{1}{3}GDH$.

6. By approximation we can trisect the angle $BCA = a$, in the following manner:

Make $AD = \frac{1}{4}a$, $DE = \frac{1}{4}AD = \frac{1}{16}a$, $EF = \frac{1}{4}DE = \frac{1}{64}a$, &c.; then we obtain for the sum of all these arcs, by summing the infinite geometric series $(\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \dots)a = \frac{1}{3}a$.

NOTE ON THE CATENARY, BY PROF. W. W. JOHNSON.—The following formulæ arise in the consideration of the measurement of a base line by means of a steel tape which is allowed to assume the form of a catenary.

The equation of the curve being

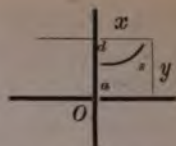
$$y = \frac{a}{2} \left(e^{\frac{x}{a}} + e^{-\frac{x}{a}} \right), \quad (1)$$

we have

$$s = \frac{a}{2} \left(e^{\frac{x}{a}} - e^{-\frac{x}{a}} \right), \quad (2)$$

whence

$$s^2 = y^2 - a^2 = (y-a)(y+a). \quad (3)$$



Now suppose the ends of the tape to be on a level, and that d , the deflection of the curve, is measured; the length of the tape is $2s$, and s and d are the known quantities. But $y-a=d$, hence from (3), $y+a=s^2 \div d$; whence

$$y = \frac{s^2 + d^2}{2d}, \quad \text{and} \quad a = \frac{s^2 - d^2}{2d}. \quad (4)$$

From (1) and (2),
$$e^{\frac{x}{a}} = \frac{y+s}{a},$$

and substituting from (4),

$$e^{\frac{x}{a}} = \frac{s^2 + d^2 + 2ds}{s^2 - d^2} = \frac{s+d}{s-d};$$

whence

$$x = a \log \frac{s+d}{s-d} = \frac{s^2 - d^2}{2d} \log \frac{s+d}{s-d}, \quad (5)$$

or, putting L for $2x$, the length to be measured, and S for $2s$, the whole length of the tape,

$$L = \frac{S^2 - 4d^2}{4d} \log \frac{S+2d}{S-2d}, \quad (6)$$

the logarithm being Napierian.

If we develop the logarithm by the formula

$$\log \frac{1+x}{1-x} = 2x \left[1 + \frac{x^2}{3} + \frac{x^4}{5} + \dots \right],$$

we have

$$\begin{aligned} L &= S \left[1 - \frac{(2d)^2}{S^2} \right] \left[1 + \frac{(2d)^2}{3S^2} + \frac{(2d)^4}{5S^4} + \dots \right] \\ &= S \left[1 - \frac{2(2d)^2}{3S^2} - \frac{2(2d)^4}{3.5S^4} - \frac{2(2d)^6}{5.7S^6} - \frac{2(2d)^8}{7.9S^8} - \dots \right]. \quad (7) \end{aligned}$$

The second term is of course the same that occurs in the case of a circular arc.

SOLUTIONS OF PROBLEMS IN NUMBER THREE.

SOLUTIONS of problems in No. 3 have been received, as follows:

From Prof W. P. Casey, 259, 264; Alexander S. Christie, 261; Geo. Eastwood, 265; W. V. McKnight, 259; W. L. Marcy, 263; E. B. Seitz, 264, 265; Prof. J. Scheffer, 259 and answer to query at page 55. Also, Prof. J. H. Kershner and R. J. Adcock each sent ans. to query at p. 55.

259. "In a triangle, one side, = 400 ft., and the two adjacent angles, 70° and 80° , are given; to compute the other two sides without the aid of trigonometry."

SOLUTION BY PROF. W. P. CASEY, SAN FRANCISCO, CAL.

About the triangle describe a circle, and bisect the angle ABC by the line BD , join DC and draw Bn perpendicular to AC .

Angle $ABC = 80^\circ$, $\angle BAC = 70^\circ$; $\therefore \angle ACB = 30^\circ$; $\therefore BO = AB = a$. Let $BC (= BD) = x$ and $OC (= DC) = y$; then $x : a :: y : OA = \frac{ay}{x}$; $\therefore On = \frac{ay}{2x}$ and $a^2 - \frac{a^2 y^2}{4x^2} = Bn^2 = \frac{1}{4}x^2$.

$$\therefore 4a^2 - \frac{a^2 y^2}{x^2} = x^2; \quad (1)$$

but (Eucl., B. VI.) $AB \times BC = AO \times OC + BO^2$;

$$\therefore y^2 = x^2 - ax. \quad (2)$$

By substituting this value for y^2 in (1) we get

$$x^3 - a^2 x = a^3;$$

from which we find $x = 751.752$ ft. The other values of x are imaginary.

The value of $y (= CO) = 514.227$ ft. and $AO = 273.615$ ft., $\therefore AC = 787.842$ ft.

Cor. AB is the side of a hexagon inscribed in the circle $ABCD$, and if we allow ph to be its opposite side, then will arc $pC = 2Ch$, and the chord pC is the side of a figure of nine sides, inscribed in the circle, and that of Ch the side of a figure of eighteen sides.

$Ah = \text{diam.} = 2AB = 800$, $Ch = \sqrt{(Ah^2 - Ch^2)} = \sqrt{[(800)^2 - (787.842)^2]}$; and $Cp = \sqrt{(Bp^2 - BC^2)} = \sqrt{[(800)^2 - (751.752)^2]}$; $\therefore$ an arc of 60° may be trisected without the aid of trigonometry.



260. "A sphere, radius r , rolls down the concave arc of a circle, radius R . At the beginning of the motion, the center of the sphere is on the horizontal diameter of the circle. Find the time of descent of the sphere in terms of the coordinates of its center."

No solution of this problem has been received.

[Putting φ for the arc of the circle over which the sphere has rolled in any given time t , and θ for the angular rotation of the sphere in the same time, we shall have

$$r(\theta - \varphi) = R\varphi. \quad (1)$$

The equations of motion are (see Routh's Rigid Dynamics, page 103)

$$(R-r) \frac{d^2\varphi}{dt^2} = +g \sin \varphi - \frac{F}{m}, \quad (2)$$

$$(R-r) \left(\frac{d\varphi}{dt} \right)^2 = -g \cos \varphi + \frac{R'}{m}, \quad (3)$$

$$\frac{d^2\theta}{dt^2} = \frac{rF}{mk^2}. \quad (4)$$

Substituting for $\frac{d^2\theta}{dt^2}$ in (4), its equivalent, $\frac{R+r}{r} \cdot \frac{d^2\varphi}{dt^2}$, as found from (1), we have $\frac{R+r}{r} \cdot \frac{d^2\varphi}{dt^2} = \frac{rF}{mk^2}$, $\therefore \frac{d^2\varphi}{dt^2} = \frac{r}{R+r} \cdot \frac{rF}{mk^2} = \frac{r^2F}{(R+r)mk^2}$. (5)

Combining (5) and (2) we get

$$\frac{r^2F}{(R+r)mk^2} = \frac{g \sin \varphi}{R-r} - \frac{F}{m(R-r)}; \therefore F = \frac{mk^2(R+r)g \sin \varphi}{(R-r)r^2 + (R+r)k^2}$$

Substituting for F in (2) its value as here found and reducing, we get

$$\frac{d^2\varphi}{dt^2} = \frac{g \sin \varphi}{R-r} \left(1 - \frac{(R+r)k^2}{(R-r)r^2 + (R+r)k^2} \right) = \frac{5g}{7R-3r} \sin \varphi. \quad (6)$$

Multiplying by $2d\varphi \div dt$ and integrating we get, after correcting,

$$\left(\frac{d\varphi}{dt} \right)^2 = \frac{10g}{7R-3r} (1 - \cos \varphi) = a(1 - \cos \varphi).$$

Therefore we have $dt = \frac{d\varphi}{\sqrt{a(1 - \cos \varphi)}}$, whence $t = \int_0^{\frac{\pi}{2}} \frac{d\varphi}{\sqrt{a(1 - \cos \varphi)}}$
 $= \frac{1}{\sqrt{a}} \int_0^{\frac{\pi}{2}} \frac{d\varphi}{\sqrt{1 - \cos \varphi}} = \frac{1}{\sqrt{a}} \left[\frac{1}{\sqrt{2}} \log \frac{\sqrt{1 + \cos \varphi} - \sqrt{2}}{\sqrt{1 - \cos \varphi}} \right]_0^{\frac{\pi}{2}},$
 $= (2 \div \sqrt{a}) \log (2 - \sqrt{2}). \text{—Ed.]}$

261. "A curve of the n th degree rolls upon any curve whatever: to determine the degree and position of the locus of the centers of curvature of all the elements described, at any given instant, by the points of the rolling curve."

SOLUTION BY ALEX. S. CHRISTIE, U. S. COAST SURVEY, WASHINGTON, D. C.

In the more general case of a curve of the n th degree,

$$0 = u_0 + u_1 + u_2 + \dots + u_p + \dots + u_r \quad (1)$$

(where u_p denotes the aggregate of terms of the p th degree), carried by any curve rolling on any other, if we assume, as x and y axes, the tangent and normal to the point of contact of the two curves, and reckon θ from the common normal, a property of the circle of inflexions gives

$$\rho = \frac{e \cos \theta \rho_1}{e \cos \theta + \rho_1} = \frac{ey}{k^2} \rho_1,$$

where ρ is the vector of a point of the carried curve, ρ_1 the vector of the corresponding point of the sought locus, e the diameter of the circle of inflexion, and $k^2 = ey + x^2 + y^2$.

Hence the required locus is

$$0 = u_0 k^{2n} + u_1 ey k^{2(n-1)} + \dots + u_n (ey)^n, \quad (2)$$

which reduces, when (1) rolls on the fixed curve, to

$$0 = y^2 \left\{ -\frac{2c_0 r}{r-e} k^{2(n-1)} + u_2 k^{2(n-2)} + u_3 ey k^{2(n-3)} + \dots + u_n (ey)^{n-2} \right\}, \quad (3)$$

where c_0 is the coefficient of x^2 in (1), and r is the radius of curvature of the fixed curve at the point of contact.

Equation (2) holds in form in whatever manner (1) moves. When (1) is a right line, (2) is a parabola, an hyperbola, or an ellipse, according as (1) touches, cuts, or does not meet the circle of inflexions, and (3) gives the parabola $y^2 = 0$, a special case of the parabola of (2).

When (1) is a conic, (2) is of the 4th degree, and (3) is a conic touching the rolling conic at the instantaneous center of rotation, with the parabola $y^2 = 0$ ($x = dx^2$).

In general (2) and (3) are each of the $(2n)$ th deg. with n and $n+1$ branches, respectively, passing through the origin tangent to the rolling curve.

262. No solution received.

263. "On the 17th of Aug., 1878, at 5^h 20^m A. M., an observation of polaris was taken in Lat. 37° 30', Long. 107° W. from Greenwich, with an engineer's transit; from the meridian thus obtained, the Sun's azimuth at 6^h 23^m 53^s, A. M., was 79° 10' 30". What was the error of the watch, and of the meridian?

SOLUTION BY W. L. MARCY, C. E., LEADVILLE, COL.

The right ascension and declination of polaris, for Aug. 17, 1878, Greenwich mean noon, were, respectively, 1^h 14^m 35^s, and 88° 39' 34". The diff. of time for 107° of longitude is 7^h 8^m; hence the observation of polaris was taken at 20 minutes after Greenwich mean noon. Correcting the R. S. of polaris to correspond with the time of observation, I find the azimuth of solaris for lat. 37° 30' to be 46° 57'', nearly.

With the bearing of polaris at the time of observation, corrected for his zimuth, I lay off the approximate meridian, LP' .

Reducing the time of the Sun-observation to Greenwich time, I find it to be 1^h 31^m 53^s after Greenwich mean noon. The Sun's right ascension and declination for Aug. 17, 1878, were, respectively, 9^h 46' 48.6'', and 13° 23'

56.9'', from which, correcting for the time past noon, the Sun's declination at the time of the observation is found to be $13^{\circ} 22' 43.4''$. Hence, if S represents the apparent place of the Sun, we have the two sides LP' and SP in the spherical triangle $LP'S$ and the angle SLP' , opposite SP' to find angle $SP'L = 90^{\circ} 14' 13\frac{1}{2}''$. Converting this angle into time, and correcting for equation of time, I get $6^h 2^m 55.5^s$ for mean time by first calculation. The difference between this and the assumed time is $20^m 57\frac{1}{2}^s$.

Repeating the calculation, with corrected time, I get for azim. of polaris, at second trial, $38' 23''$, at third trial, $38' 48''$, at fourth trial, $38' 46.4''$; and for the time of Sun-observation, I find, from second cal. $6^h 3^m 55.3^s$, and from third calculation, $6^h 3^m 52.4^s$. Hence I find the error of the watch to be $20^m 1.6^s$, and that of the meridian, $8' 10.4''$.

264. "It is required to divide a given straight line into three parts such that the triangle formed of them shall have its circumscribing circle a minimum and its inscribed circle a maximum."

SOLUTION BY E. B. SEITZ, GREENVILLE, OHIO.

Let $2a$, x , y and $2a-x-y$ = the length of the line and the three pieces, respectively, u = the area of the triangle formed, v and w = the radii of the circumscribed and inscribed circles. Then we have

$$u = \sqrt{[a(a-x)(a-y)(x+y-a)]}, \quad (1)$$

$$u = aw, \quad (2)$$

$$4vu = xy(2a-x-y). \quad (3)$$

Differentiating (1), (2) and (3), regarding y a constant, we find

$$\frac{du}{dx} = \frac{a(a-y)(2a-2x-y)}{u}, \quad (4)$$

$$\frac{dw}{dx} = \frac{1}{a} \cdot \frac{du}{dx} = 0, \quad (5)$$

$$\frac{dv}{dx} = \frac{y(2a-2x-y)}{4u} - \frac{v}{u} \cdot \frac{du}{dx} = 0. \quad (6)$$

From (4) and (5), or from (4) and (6) we find

$$2a - 2x - y = 0. \quad (7)$$

Differentiating (1), (2) and (3), regarding x a constant, we find

$$\frac{du}{dy} = \frac{a(a-x)(2a-x-2y)}{u}, \quad (8)$$

$$\frac{dw}{dy} = \frac{1}{a} \cdot \frac{du}{dy} = 0, \quad (9)$$

$$\frac{dv}{dy} = \frac{x(2a-x-2y)}{4u} - \frac{v}{u} \cdot \frac{du}{dy} = 0. \quad (10)$$

From (8) and (9), or from (8) and (10) we find

$$2a - x - 2y = 0. \quad (11)$$

From (7) and (11) we find $x = \frac{2}{3}a$, $y = \frac{1}{3}a$, $2a - x - y = \frac{2}{3}a$.

[Prof. E. J. Edmunds writes, in relation to this problem, "It seems to me that one of the conditions is unnecessary. For the second condition alone determines the three divisions of the line, each of which must be equal to one-third of the line.

"It is well known that $r = s \div p$; and r will be a max. when s is a max. If x, y, z are the three parts, $s = \sqrt{[p(p-x)(p-y)(p-z)]}$, where $x+y+z = 2p$. S will be a max. with $(p-x)(p-y)(p-z)$, and this will be when $p-x = p-y = p-z$; $\therefore x = y = z = \frac{2}{3}p$."]]

265. "At age a , a person takes out a life policy of k dollars, for which he agrees to pay an annual premium of p dollars. At age $a+n$ he is not able to make his annual payment and wants to sell, or surrender, his life-policy for full-paid insurance policy. If the n premiums that have been paid be each viewed as the sum of a series of infinitely small annuities, payable at infinitely small intervals of time, within each year, what ought to be the present value of the new policy?"

SOLUTION BY GEORGE EASTWOOD, SAXONVILLE, MASS.

Let i = rate of interest allowed by the insurance company, $f(a)$ = number of persons alive at age a , as per mortality tables, ${}^tC_a$ = present worth of a life series of \$1 per annum, payable at equal intervals t , and resting on a life of age a . Then the probability that the insured will be alive at age $a+t$ will be expressed by $f(a+t) \div f(a)$, and ${}^tC_a$ will be defined by

$${}^tC_a = t \left[\frac{f(a+t)}{f(a)} (1+i)^{-t} + \frac{f(a+2t)}{f(a)} (1+i)^{-2t} + \dots \right]. \quad (1)$$

If in (1) we diminish t indefinitely, we shall have

$${}^0C_a = \int_0^\infty \frac{f(a+x)}{f(a)} (1+i)^{-x} dx. \quad (2)$$

Adapting this to Euler's formula, we deduce an approx. int. of the form

$$\int_0^\infty \frac{f(a+x)}{f(a)} (1+i)^{-x} dx = \sum_{x=0}^\infty \frac{f(a+x)}{f(a)} (1+i)^{-x} + \frac{t}{2} - \frac{t^2}{12} \frac{f'(a)}{f(a)} \dots \quad (3)$$

By virtue of eq's (1) and (2), the first two right-hand terms of (3) give

$${}^0C_a = {}^tC_a + \frac{1}{2}t, \quad (4)$$

$$\text{and when } t = 0, \quad {}^0C_a = C_a + \frac{1}{2}. \quad (5)$$

$$\text{Whence, from (4) and (5)} \quad {}^tC_a = C_a + \frac{1}{2} - \frac{1}{2}t. \quad (6)$$

2. Let P_a = single premium to be paid by the insured to the company to insure payment to him, by the latter, if \$1 at his decease. The probability that the insured will be alive at age $a+x$, and that he will *not* be alive at age $a+x+dx$, being, respectively, $f(a+x) \div f(a)$, and $1 - [f(a+x+dx) \div f(a+x)]$, the probability that he will decease, during the time dx , will be expressed by

$$\frac{f(a+x)}{f(a)} \left[1 - \frac{f(a+x+dx)}{f(a+x)} \right] = -\frac{df(a+x)}{f(a)}. \quad (7)$$

By multiplying eq. (7) by $(1+i)^{-x}$, we shall have what French Actuaries term the mathematical hope of the event whose probability we are in search of. If now we take the sum of all the analogous hopes, relatively to the values of x comprised between zero and the extreme limit of the mortality tables, we shall have

$$P_a = -\int_{x=0}^{x=\infty} \frac{df(a+x)}{f(a)} (1+i)^{-x}. \quad (8)$$

Integrating by parts, equation (8) gives

$$P_a = \frac{1}{f(a)} \left[f(a) - \int_0^{\infty} f(a+x) (1+i)^{-x} \log(1+i) dx \right]. \quad (9)$$

The integral included in eq. (9) is the continuous annuity ${}^0C_a$ of eq. (2); hence

$$P_a = 1 - {}^0C_a \log(1+i).$$

$$\text{Hence, also} \quad P_{a+n} = 1 - {}^0C_{a+n} \log(1+i). \quad (10)$$

3. In our problem, p being the annual premium, the value of the unpaid premiums at the time of default of payment will be represented by

$$p(1 + C_{a+n});$$

while the value of the whole insurance, considered as a commercial commodity, will be expressed by

$$KP_{a+n}.$$

The insured then may be considered to have a claim upon the company at age $a+n$ worth, at his decease,

$$KP_{a+n} - p(1 + C_{a+n}).$$

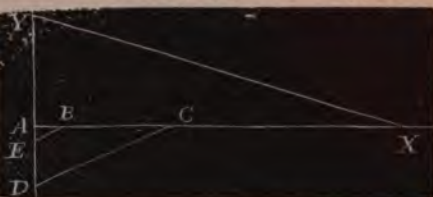
If X denote its present value, then obviously,

$$1 : P_{a+n} :: X : KP_{a+n} - p(1 + C_{a+n}).$$

Whence

$$\begin{aligned} X &= K - [p \div (P_{a+n})] (1 + C_{a+n}) \\ &= K - [p(1 + C_{a+n}) \div \{ 1 - {}^0C_{a+n} \log(1+i) \}] \\ &= K - [p(1 + C_{a+n}) \div \{ 1 - {}^tC_{a+n} \log(1+i) \} + \frac{1}{2}t]. \end{aligned}$$

NOTE BY PROF. P. E. CHASE.—To construct π , approximately, draw the rectangular axes AX, AY ; make $AX = 60$, and take, on the line $AX, AB = 3, AC = 20$. On the axis AY , lay off, negatively, $AD = 9$. Join CD and draw BE parallel to CD ; make $EY = AC = 20$ and join XY . Then is $XY \div AC = 3.14158499 = \pi$, very nearly.



[The demonstration, received in answer to query at p. 55, is deferred, for want of room, to a future number.]

PROBLEMS

266. By PROF. J. H. KERSHNER, *Mercersburg, Pa.*—Sum the series,

$$\cos 2\theta \cos \theta + \frac{\cos^2 2\theta \cos 3\theta}{2 \cdot 3} + \frac{1 \cdot 3}{2 \cdot 4} \cos^3 2\theta \frac{\cos 5\theta}{5} + \dots \text{to infinity.}$$

267. By WM. HOOVER, *Bellefontaine, Ohio.*—What was the duration of a building and loan association in which, for the first eight years, money was loaned at an average premium of \$45 per share (of \$200), interest paid being 6 per cent.

268. By E. B. OPDYCKE, *Pulaski, Ohio.*—A conical vessel whose upper diameter is a , lower diameter b ($a > b$) and height h , contains a quantity of water. Find the greatest perpendicular depth of the water when the vessel is so tipped on its bottom as to balance, the vessel being uniform in thickness, and of one material.

269. By GEO. H. HARVILL, *Bonner, La.*—Show that the equations

$$x = (r_1 - r_2) \cos \varphi + m r_2 \cos \left(\frac{r_1 - r_2}{r_2} \varphi \right),$$

$$y = (r_1 - r_2) \sin \varphi - m r_2 \sin \left(\frac{r_1 - r_2}{r_2} \varphi \right),$$

represent the prolate and curtate hypocycloids; and also that when $r_1 = 2r_2$ the curve becomes the ellipse

$$\frac{x^2}{(1+m)^2 r_2^2} + \frac{y^2}{(1-m)^2 r_2^2} = 1.$$

270. By PROF. E. J. EDMUNDS, *New Orleans, La.*—Find the locus of the centres of circles tangent to a parabola and to the tangent at its vertex.

271. By PROF. J. H. KERCHNER, *Mercersburg, Pa.*—If three points be taken at random in the circumference of a circle, required the probability that the triangle formed by joining them will be acute.

272. By. PROF. W. W. JOHNSON, *Annapolis Md.*—Two lines rotate, uniformly, in opposite directions about two fixed points, the velocity of one being n times that of the other; find the rectangular equation of their intersection.

273. By E. B. SEITZ, *Greenville, Ohio.*—If m and n be the masses of the earth and moon, a the distance between their centers, r the radius of the earth, and if a body fall toward the earth from the point of equal attraction in the line joining their centers, find the time of falling from the height h to the earth's surface.

274. By PROF. L. G. BARBOUR, *Richmond, Ky.*—Required the shortest distance between two curves whose equations are

$$\begin{aligned} 4x^2 + 9y^2 - 144 &= 0, \\ x'^2 + y'^2 - 26x' - 32y' + 25 &= 0. \end{aligned}$$

275. By Prof. A. HALL, *Washington, D. C.*—Find the moments of inertia of an elliptical disk: (1), about a right line in the plane of the disk and parallel to the axis of x : (2), about a right line parallel to the axis of y , the equation of the disk being

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0.$$

QUERY BY W. E. HEAL.—Can an elliptic function of the *third* species be expressed in terms of elliptics of the *first* and *second* species? If so, how?

PUBLICATIONS RECEIVED.

Radiant Points of Meteors. By EDWARD F. SAWYER. From the American Journal of Science and Arts. June, 1879.

The American Journal of Mathematics, Vol II, No. 1. Baltimore, Maryland. March, 1879. 100 pages, 4to.

ERRATA.

On page 83, line 11, for $r \sin \phi$ read, $r \cos \phi$.

" " 87, multiply the formula in lines 2, 6 and 9, from bottom, by the factor π .

" " 108, Fig. 1, insert the letter H between I and C .

" " 111, Fig. 2, for G , read C .

" " 118, F , in the Fig., should mark the intersection with the arc AB .

THE ANALYST.

VOL. VI.

SEPTEMBER, 1879.

No. 5.

THE MOTION OF A SATELLITE.

BY PROF. ASAPH HALL, NAVAL OBSERVATORY, WASHINGTON, D. C.

(1). THE apparent orbit of a satellite around its primary will be in general an ellipse, which will be the projection of the real orbit of the satellite on a plane perpendicular to the line of sight, or to a right line drawn from the observer to the centre of the planet. Assuming that the orbit of the satellite around its primary is known, and that its motion is purely elliptical, or free of perturbation, we have the problem of determining from the known orbital motion of the satellite its apparent motion. The solution of this problem has been given by Bessel in the *Astronomische Nachrichten*, Bd. 9, p. 7; and again in his *Astronomische Untersuchungen*, Bd. 2, p. 37; and also by Marth in the *Astronomische Nachrichten*, Bd. 44, p. 113.

Denote by s and p the apparent distance and angle of position of a satellite, or its polar coordinates referred to the centre of the planet. The zero of the angle p is the circle of declination passing through the centre of the planet, this angle being counted from the north toward the east, and from 0° to 360° .

Let α and δ be the right ascension and declination of the planet, α' and δ' the same quantities for the satellite. From the spherical triangle between the pole of the equator, the planet and the satellite, we have

$$\left. \begin{aligned} \cos s &= \sin \delta \sin \delta' + \cos \delta \cos \delta' \cos (\alpha' - \alpha) \\ \sin s \cos p &= \cos \delta \sin \delta' - \sin \delta \cos \delta' \cos (\alpha' - \alpha) \\ \sin s \sin p &= \cos \delta' \sin (\alpha' - \alpha) \end{aligned} \right\}. \quad (1)$$

If we denote by r the radius vector of the satellite, and by α and δ its right ascension and declination seen from the planet, and also by ρ and ρ' the distances of the planet and the satellite from the Earth, we shall have by equating the values of the rectangular coordinates,

$$\begin{aligned}\rho' \cos \delta' \cos a' &= \rho \cos \delta \cos a + r \cos d \cos a, \\ \rho' \cos \delta' \sin a' &= \rho \cos \delta \sin a + r \cos d \sin a, \\ \rho' \sin \delta' &= \rho \sin \delta + r \sin d,\end{aligned}$$

or by a simple transformation

$$\begin{aligned}\rho' \cos \delta' \cos (a' - a) &= \rho \cos \delta + r \cos d \cos (a - a), \\ \rho' \cos \delta' \sin (a' - a) &= r \cos d \sin (a - a), \\ \rho' \sin \delta' &= \rho \sin \delta + r \sin d.\end{aligned}$$

If we multiply equations (1) by ρ' we have by means of the last equations

$$\left. \begin{aligned}\rho' \cos s &= \rho + r \cdot [\sin d \sin \delta + \cos d \cos \delta \cos (a - a)] \\ \rho' \sin s \cos p &= r \cdot [\sin d \cos \delta - \cos d \sin \delta \cos (a - a)] \\ \rho' \sin s \sin p &= r \cdot \cos d \sin (a - a)\end{aligned} \right\}. \quad (2)$$

If we place now

$$\begin{aligned}\xi &= \frac{r}{\rho} \cdot \cos d \sin (a - a), \\ \eta &= \frac{r}{\rho} \cdot [\sin d \cos \delta - \cos d \sin \delta \cos (a - a)], \\ \zeta &= \frac{r}{\rho} \cdot [\sin d \sin \delta + \cos d \cos \delta \cos (a - a)],\end{aligned}$$

equations (2) become

$$\left. \begin{aligned}\rho' \cos s &= \rho \cdot (1 + \zeta) \\ \rho' \sin s \cos p &= \rho \cdot \eta \\ \rho' \sin s \sin p &= \rho \cdot \xi\end{aligned} \right\}. \quad (3)$$

We have therefore

$$\begin{aligned}\tan s \sin p &= \frac{\xi}{1 + \zeta}, \\ \tan s \cos p &= \frac{\eta}{1 + \zeta}.\end{aligned}$$

If we assume $\tan s = s$, which may generally be done without sensible error, and put $x = s \sin p$, $y = s \cos p$; we shall have in seconds of arc

$$x = \frac{\omega \cdot \xi}{1 + \zeta}; \quad y = \frac{\omega \cdot \eta}{1 + \zeta};$$

where $\omega = 206264.8$. The equations (3) give the values of ρ' , s and p when those of ξ , η and ζ are known. It is evident also that ζ has no influence on the value of p , and since it is generally very small with respect to unity its influence on the value of s is slight.

We have now to express the quantities ξ , η , ζ , in terms of the elements of the orbit of the satellite. Continuing the use of Bessel's notation, let N be the longitude of the ascending node of the orbit of the satellite on the Equator, and J its inclination to the Equator. Let u be the argument of the latitude, or the angle between the node and the satellite counted in the direction of the motion. From the right angled spherical triangle having u for a hypotenuse, and the sides d , and $a-N$, we have

$$\cos d \sin (a-N) = \sin u \cos J,$$

$$\cos d \cos (a-N) = \cos u,$$

$$\sin d = \sin u \sin J.$$

Eliminating N from the left hand side of these equations we have

$$\cos d \cos (a-a) = \sin u \sin (a-N) \cos J + \cos u \cos (a-N),$$

$$\cos d \sin (a-a) = \sin u \cos (a-N) \cos J - \cos u \sin (a-N),$$

$$\sin d = \sin u \sin J.$$

Denote the mean distance of the planet from the sun by ρ_0 , and the semi-major axis of the orbit of the satellite by $\rho_0 \sin \Delta$; if we express the radius vector r by means of the semi-major axis the value of r will be

$$\rho_0 \sin \Delta \cdot r,$$

or approximately $\rho_0 \cdot \frac{\Delta}{\omega} \cdot r.$

The values of ξ , η , ζ , are therefore

$$\left. \begin{aligned} \omega \xi &= r \cdot \frac{\rho_0}{\rho} \Delta \cdot \left\{ \sin u \cos (a-N) \cos J - \cos u \sin (a-N) \right\} \\ \omega \eta &= r \cdot \frac{\rho_0}{\rho} \Delta \cdot \left\{ \sin u [\cos \delta \sin J - \sin \delta \cos J \sin (a-N)] \right. \\ &\quad \left. - \cos u \sin \delta \cos (a-N) \right\} \\ \omega \zeta &= r \cdot \frac{\rho_0}{\rho} \Delta \cdot \left\{ \sin u [\sin \delta \sin J + \cos \delta \cos J \sin (a-N)] \right. \\ &\quad \left. + \cos u \cos \delta \cos (a-N) \right\} \end{aligned} \right\} \quad (4)$$

Introducing the auxiliary quantities f , F , g , G , h , H , by the formulæ

$$\sin f \cos F = \cos (a-N) \cos J,$$

$$\sin f \sin F = -\sin (a-N),$$

$$\cos f = -\cos (a-N) \sin J,$$

$$\sin g \cos G = \cos \delta \sin J - \sin \delta \cos J \sin (a-N),$$

$$\sin g \sin G = -\sin \delta \cos (a-N),$$

$$\cos g = \cos \delta \cos J + \sin \delta \sin J \sin (a-N),$$

$$\sin h \cos H = \sin \delta \sin J + \cos \delta \cos J \sin (a-N),$$

$$\sin h \sin H = \cos \delta \cos (a-N),$$

$$\cos h = \sin \delta \cos J - \cos \delta \sin J \sin (a-N),$$

we have

$$\left. \begin{aligned} \omega.\xi &= r \cdot \frac{\rho_0}{\rho} \Delta \sin f \sin (F + u) \\ \omega.\eta &= r \cdot \frac{\rho_0}{\rho} \Delta \sin g \sin (G + u) \\ \omega.\zeta &= r \cdot \frac{\rho_0}{\rho} \Delta \sin h \sin (H + u) \end{aligned} \right\}. \quad (5)$$

Denoting the mean longitude at the instant from which the time t is counted by E , the daily motion by λ , the longitude of the perihelion by P , the eccentricity of the orbit by e , the mean, the eccentric and the true anomalies by μ , ε and φ , we have

$$\left. \begin{aligned} \mu &= E + \lambda t - P = \varepsilon - e \sin \varepsilon \\ \tan \frac{1}{2} \varphi &= \sqrt{\frac{1+e}{1-e}} \cdot \tan \frac{1}{2} \varepsilon \\ r &= 1 - e \cos \varepsilon. \end{aligned} \right\}. \quad (6)$$

The preceding formulæ are those given by Bessel. The six auxiliary quantities f, F, g, G, h, H , are similar to those introduced by Gauss for computing the place of a planet (Zach's Monatliche Correspondenz, Mai, 1804, or Werke, Bd. 6, p. 94), and their geometrical meaning is readily seen. Thus, if we take the origin of rectangular coordinates at the centre of the planet, the line drawn from the planet to the Earth as the axis of x and take the axis of y in the plane of the declination circle, we shall find that f, g, h , are the arcs drawn from the north pole of the orbit of the satellite to the points where the axes of x, y and z cut the sphere; and that F, G, H , are the angles at this pole between the arcs drawn to the north pole of the Equator, and those drawn to the points where the sphere is cut by the axes of x, y and z .

By introducing a single additional auxiliary quantity Marth has given a very simple and elegant method of computing s and p . Thus if we compute the angle k by the formulæ

$$\left. \begin{aligned} \sin h \sin k &= \cos (\alpha - N) \sin J \\ \sin h \cos k &= -\sin (\alpha - N) \sin J \sin \delta - \cos J \cos \delta \end{aligned} \right\}, \quad (7)$$

and denote by σ the supplement of the angle between the line drawn from the Earth to the planet and the radius vector of the satellite, we have from the spherical triangle formed by the directions of the selines and the pole of the satellite's orbit,

$$\left. \begin{aligned} \sin \sigma \sin (p - k) &= \cos (H + u) \\ \sin \sigma \cos (p - k) &= \sin (H + u) \cos h \\ \cos \sigma &= \sin (H + u) \sin h \end{aligned} \right\}. \quad (8)$$

We evidently have

$$\begin{aligned}\rho' \sin s &= r \sin \sigma, \\ \rho' \cos s &= r \cos \sigma + \rho,\end{aligned}$$

and therefore we can find ρ' , s and p from the equations

$$\left. \begin{aligned}\rho' \sin s \sin (p-k) &= r \cos (H+u) \\ \rho' \sin s \cos (p-k) &= r \sin (H+u) \cos h \\ \rho' \cos s &= r \sin (H+u) \sin h + \rho\end{aligned} \right\}. \quad (9)$$

By putting

$$\sin (H+u) \sin h + \frac{\rho}{r} = \frac{1}{\tau}$$

and eliminating ρ' we have for computing s and p ,

$$\begin{aligned}\tan s \sin (p-k) &= \tau \cos (H+u) \\ \tan s \cos (p-k) &= \tau \sin (H+u) \cos h.\end{aligned}$$

If we wish to use the angle σ we may compute s by means of the equations

$$\sin s = \frac{r}{\rho} \cdot \sin (\sigma-s),$$

and for ρ' we have

$$\rho' = \rho \cdot \frac{\sin \sigma}{\sin (\sigma-s)}.$$

Both these equations come directly from the plane triangle between the Earth, the centre of the planet and the position of the satellite in its orbit. The preceding formulæ given by Marth are rigorous, but generally it will be sufficient to use the first two equations of (9). If we neglect the difference between ρ and ρ' , and express s and r in seconds of arc, we have

$$\left. \begin{aligned}s \sin (p-k) &= \frac{r}{\rho} \cdot \cos (H+u) \\ s \cos (p-k) &= \frac{r}{\rho} \cdot \sin (H+u) \cos h\end{aligned} \right\}. \quad (10)$$

Equations (10) are convenient for use if from the observed values of s and p we wish to compute r and u . For this purpose we may use also the auxiliary quantities of Bessel, which give

$$\begin{aligned}r \cdot \sin [\tfrac{1}{2}(F+G)+u] &= \frac{s}{2} \cdot \frac{\sin g \sin p + \sin f \cos p}{\sin f \sin g \cos \tfrac{1}{2}(F-G)}, \\ r \cdot \cos [\tfrac{1}{2}(F+G)+u] &= \frac{s}{2} \cdot \frac{\sin g \sin p - \sin f \cos p}{\sin f \sin g \sin \tfrac{1}{2}(F-G)},\end{aligned}$$

but it is more convenient to use the formulæ of Marth.

In computing the quantities f , F , g , G , h , H , we use the quantities J and N , which are themselves subject to slow changes on account of precession. But we may easily form a table of the values of J and N which will

enable us to take account of these changes in the values of the auxiliary quantities. If moreover we wish to take account also of the nutation, and denote by $\Delta\omega$ and Δl the nutation in the obliquity of the ecliptic and the longitude, we may employ the differential equations of a spherical triangle.

If ω be the obliquity of the ecliptic we have

$$dJ = \cos N \cdot \Delta\omega - \sin N \sin \omega \cdot \Delta l,$$

$$dw = \frac{\sin N \cdot \Delta\omega + \cos N \sin \omega \cdot \Delta l}{\sin J},$$

$$dN = -\cos J \cdot dw + \cos \omega \cdot \Delta l,$$

or
$$dN = -(\sin N \cdot \Delta\omega + \cos N \sin \omega \cdot \Delta l) \cotg J + \cos \omega \cdot \Delta l.$$

The first and last of these formulæ will give the corrections to the values of J and N .

(2). If we assume values of the elements of the orbit of a satellite, the preceding formulæ enable us to compare our observations with the elements. In order to correct the assumed elements we must form equations of condition by differentiating the first two of equations (5). Denoting the partial differential coefficients by a, b, c, d, e, f ; and by ν the residual = computed — observed value, the equation of condition will be of the form

$$a \cdot dE + b \cdot edP + c \cdot de + d \cdot dA + e \cdot \sin J dN + f \cdot dJ + \nu = 0.$$

Each observation gives two equations, one from the value of ξ , and one from the value of η . The changes in the value of ζ can be neglected.

If we denote by n the longitude of the ascending node of the orbit of the satellite on the Ecliptic, by i its inclination to the Ecliptic, by w the distance of the Ecliptic from the Equator, measured on the orbit of the satellite, by v the longitude of the satellite in its orbit, we have

$$v - n + w = u, \quad \text{and } \varphi = v - P.$$

If we put

$$f' = \frac{\rho_0}{\rho} \Delta \sin f; \quad g' = \frac{\rho_0}{\rho} \Delta \sin g; \quad h' = \frac{\rho_0}{\rho} \Delta \sin h,$$

$$F' = F + w - n; \quad G' = G + w - n; \quad H' = H + w - n,$$

the values of ξ, η and ζ become

$$\omega \cdot \xi = r \cdot f' \sin (F' + v),$$

$$\omega \cdot \eta = r \cdot g' \sin (G' + v),$$

$$\omega \cdot \zeta = r \cdot h' \sin (H' + v).$$

The value of $\omega \cdot \xi$ may be written in the form

$$\omega \cdot \xi = f' \cdot [\sin (F' + P) \cdot r \cos (v - P) + \cos (F' + P) \cdot r \sin (v - P)].$$

Differentiating with respect to the elements in the plane of the orbit we have

$$\omega \cdot d\xi = f' \sin (F' + P) \cdot d[r \cos (v - P)] + f' \cos (F' + P) \cdot d[r \sin (v - P)] \\ + r f' \cos (F' + v) \cdot dP.$$

We have also, since $v-P$ is the true anomaly,

$$\begin{aligned} r \cdot \cos(v-P) &= \cos \varepsilon - e, \\ r \cdot \sin(v-P) &= \sin \varepsilon \sqrt{1-e^2}, \\ E-P &= \varepsilon - e \sin \varepsilon, \end{aligned}$$

and therefore

$$\begin{aligned} d[r \cos(v-P)] &= -\sin \varepsilon \cdot d\varepsilon - de, \\ d[r \sin(v-P)] &= \cos \varepsilon \cdot \sqrt{1-e^2} \cdot d\varepsilon - \frac{e \sin \varepsilon de}{\sqrt{1-e^2}}, \\ d(E-P) &= (1 - e \cos \varepsilon) \cdot d\varepsilon - \sin \varepsilon \cdot de, \end{aligned}$$

or

$$d\varepsilon = \frac{d(E-P) + \sin \varepsilon \cdot de}{1 - e \cos \varepsilon}.$$

Eliminating $d\varepsilon$ we have

$$\begin{aligned} d[r \cos(v-P)] &= -\frac{\sin \varepsilon}{1 - e \cos \varepsilon} \cdot d(E-P) - \left(1 + \frac{\sin \varepsilon^2}{1 - e \cos \varepsilon}\right) \cdot de, \\ d[r \sin(v-P)] &= \frac{\cos \varepsilon \cdot \sqrt{1-e^2}}{1 - e \cos \varepsilon} \cdot d(E-P) \\ &\quad + \left(\frac{\cos \varepsilon \cdot \sqrt{1-e^2}}{1 - e \cos \varepsilon} - \frac{e}{\sqrt{1-e^2}}\right) \cdot \sin \varepsilon \cdot de. \end{aligned}$$

But since

$$\frac{\sin \varepsilon}{1 - e \cos \varepsilon} = \frac{\sin(v-P)}{\sqrt{1-e^2}}; \text{ and } \cos(v-P) = \frac{\cos \varepsilon - e}{1 - e \cos \varepsilon},$$

the values of these differentials become

$$\begin{aligned} d[r \cos(v-P)] &= -\frac{\sin(v-P)}{\sqrt{1-e^2}} \cdot d(E-P) - \left(1 + \frac{\sin \varepsilon \sin(v-P)}{\sqrt{1-e^2}}\right) \cdot de, \\ d[r \sin(v-P)] &= \frac{\cos(v-P) + e}{\sqrt{1-e^2}} \cdot d(E-P) + \frac{\sin \varepsilon \cos(v-P)}{\sqrt{1-e^2}} \cdot de. \end{aligned}$$

Multiplying the first differential by $f' \sin(F'+P)$, and the second by $f' \cos(F'+P)$, the value of $\omega \cdot d\xi$ becomes

$$\begin{aligned} \omega \cdot d\xi &= \frac{f'}{\sqrt{1-e^2}} \cdot \left\{ \cos(F'+v) + e \cos(F'+P) \right\} \cdot dE \\ &\quad + \frac{f'}{\sqrt{1-e^2}} \cdot \left\{ \cos(F'+v) [r\sqrt{1-e^2} - 1] - e \cos(F'+P) \right\} \cdot dP \\ &\quad + \frac{f'}{\sqrt{1-e^2}} \cdot \left\{ \cos(F'+v) \sin \varepsilon - \sin(F'+P) \cdot \sqrt{1-e^2} \right\} \cdot de. \end{aligned}$$

If we substitute the value of $r = 1 - e \cos \varepsilon$, in the coefficient of dP , and observe that the value of the partial differential coefficient d is evident, we shall have the values of the first four coefficients given by Bessel; *Astronomische Nachrichten*, Bd. 9, p. 10.

In order to find the two other coefficients, we take the value of $\omega \cdot \xi$

$$\omega \cdot \xi = r \cdot \frac{\rho_0}{\rho} \Delta \cdot [\sin u \cos (\alpha - N) \cos J - \cos u \sin (\alpha - N)].$$

Differentiating with respect to the elements that determine the position of the plane of the orbit we have

$$\begin{aligned} \omega \cdot d\xi = r \cdot \frac{\rho_0}{\rho} \Delta \cdot \left\{ [\cos u \cos (\alpha - N) \cos J + \sin u \sin (\alpha - N)] \cdot du \right. \\ \left. + [\sin u \sin (\alpha - N) \cos J + \cos u \cos (\alpha - N)] dN - \sin u \cos (\alpha - N) \sin J dJ \right\} \end{aligned}$$

The values of the auxiliary quantities f and F give

$$\begin{aligned} \omega \cdot d\xi = r \cdot \frac{\rho_0}{\rho} \Delta \cdot \left\{ \sin f \cos (F' + v) \cdot du + \sin f \cos (F' + v) \cos J \cdot dN \right. \\ \left. - \cos u \cos f \sin J \cdot dN + \sin u \cos f \cdot dJ \right\}. \end{aligned}$$

If we now consider the spherical triangle formed by the equator, the ecliptic and the orbit of the satellite, we shall have by the formulæ for the differentials of a spherical triangle,

$$dw = \cos i \cdot dn - \cos J \cdot dN,$$

$$dn = -\frac{\sin w}{\sin i} \cdot dJ + \frac{\cos w \sin J}{\sin i} \cdot dN,$$

$$\text{or } dw = -\frac{\sin w \cos i}{\sin i} \cdot dJ + \frac{\cos w \cos i \sin J}{\sin i} \cdot dN - \cos J \cdot dN.$$

We have therefore

$$d(w - n) = (-\cos J - \cos w \sin J \tan \frac{1}{2}i) \cdot dN + \sin w \tan \frac{1}{2}i \cdot dJ.$$

Since $du = d(w - n)$, by substituting this value of du we have

$$\begin{aligned} \omega \cdot d\xi = -\sin J \cdot \left\{ f' \cdot \tan \frac{1}{2}i \cos w \cdot r \cos (F' + v) + \frac{\rho_0}{\rho} \Delta \cos f \cdot r \cos u \right\} \cdot dN \\ + \left\{ f' \cdot \tan \frac{1}{2}i \sin w \cdot r \cos (F' + v) + \frac{\rho_0}{\rho} \Delta \cos f \cdot r \sin u \right\} \cdot dJ. \end{aligned}$$

The values therefore of the partial differential coefficients a, b, c, d, e, f , are as follows:

$$a = \omega \cdot \frac{d\xi}{dE} = \frac{f'}{\sqrt{1-e^2}} \cdot \left\{ \cos (F' + v) + e \cos (F' + P) \right\}$$

$$\begin{aligned} b = \omega \cdot \frac{d\xi}{edP} = \frac{-f'}{\sqrt{1-e^2}} \cdot \left\{ \cos (F' + v) \cdot \left(\cos \varepsilon \sqrt{1-e^2} \right) \right. \\ \left. + \frac{e}{1+\sqrt{1-e^2}} \right\} + \cos (F' + P) \} \end{aligned}$$

$$c = \omega \cdot \frac{d\xi}{de} = \frac{f'}{\sqrt{1-e^2}} \cdot \left\{ \cos (F' + v) \sin \varepsilon - \sin (F' + P) \cdot \sqrt{1-e^2} \right\}$$

$$\begin{aligned}d &= \omega. \frac{d\xi}{dJ} = \frac{\omega. \xi}{A}, \\e &= \omega. \frac{d\xi}{\sin J dN} = -f'. \text{tang } \frac{1}{2}i \cos w. r \cos (F' + v) - \frac{\rho_0}{\rho} \Delta. \cos f. r \cos u \\f &= \omega. \frac{d\xi}{dJ} = f'. \text{tang } \frac{1}{2}i \sin w. r \cos (F' + v) + \frac{\rho_0}{\rho} \Delta. \cos f. r \sin u.\end{aligned}\tag{11}$$

In order to obtain the coefficients in η we have only to put g' and G' in the place of f' and F' .

The preceding formulæ give the values of the differential coefficients adapted to the quantites

$$\xi = x = s \sin p; \quad \eta = y = s \cos p;$$

and we must therefore combine our observed quantities s and p to find the residuals for the equations of condition. Generally it is better to compare directly with each of the observed quantities, since one of them may be erroneous and the other not. If we wish to do this in the present case, we must use the formulæ

$$\begin{aligned}ds &= \sin p. dx + \cos p. dy, \\s. dp &= \cos p. dx - \sin p. dy.\end{aligned}$$

By means of equations (11) the value of dx is

$$dx = a. dE + b. edP + c. de + d. dA + e. \sin J dN + f. dJ,$$

with a similar value for dy . Substituting these values for dx and dy in the preceding equations we shall have the equations of condition for ds and sdp . If we make

$$\begin{aligned}m \sin M &= f' \cos (F' + v), & n' \sin N' &= f' \cos (F' + P), \\m \cos M &= g' \cos (G' + v), & n' \cos N' &= g' \cos (G' + P); \\l \sin L &= f' \sin (F' + P), & k' \sin K' &= \frac{\rho_0}{\rho} \Delta \cos f, \\l \cos L &= g' \sin (G' + P), & k' \cos K' &= \frac{\rho_0}{\rho} \Delta \cos g.\end{aligned}$$

The values of the differential coefficients for ds and sdp are as follows, where $\sin \varphi'$ is put for e on the right-hand side of the equations.

For the distances;

$$\begin{aligned}\frac{ds}{dE} &= \frac{1}{\cos \varphi'} \cdot \left\{ m \cos (M - p) + n' \cos (N' - p) \sin \varphi' \right\} \\ \frac{ds}{edP} &= \frac{-1}{\cos \varphi'} \cdot \left\{ m \cos (M - p) (\cos \varepsilon \cos \varphi' + \text{tang } \frac{1}{2} \varphi') + n' \cos (N' - p) \right\} \\ \frac{ds}{de} &= \frac{1}{\cos \varphi'} \cdot \left\{ m \cos (M - p) \sin \varepsilon - l \cos (L - p) \cos \varphi' \right\}\end{aligned}$$

$$\begin{aligned}\frac{ds}{dA} &= \frac{s}{A} \\ \frac{ds}{\sin J dN} &= r \cdot \left\{ m \cos (M-p) \cos w \tan \frac{1}{2}i + k' \cos (K'-p) \cos u \right\} \\ \frac{ds}{dJ} &= -r \cdot \left\{ m \cos (M-p) \sin w \tan \frac{1}{2}i + k' \cos (K'-p) \sin u \right\}.\end{aligned}\tag{12}$$

For the angles:

$$\begin{aligned}\frac{sdp}{dE} &= \frac{1}{\cos \varphi'} \cdot \left\{ m \sin (M-p) + n' \sin (N'-p) \sin \varphi' \right\} \\ \frac{sdp}{dP} &= \frac{-1}{\cos \varphi'} \cdot \left\{ m \sin (M-p) (\cos \varepsilon \cos \varphi' + \tan \frac{1}{2}\varphi') + n' \sin (N'-p) \right\} \\ \frac{sdp}{de} &= \frac{1}{\cos \varphi'} \cdot \left\{ m \sin (M-p) \sin \varepsilon - l \sin (L-p) \cos \varphi' \right\} \\ \frac{sdp}{dA} &= 0 \\ \frac{sdp}{\sin J dN} &= -r \cdot \left\{ m \sin (M-p) \cos w \tan \frac{1}{2}i + k' \sin (K'-p) \cos u \right\} \\ \frac{sdp}{dJ} &= r \cdot \left\{ m \sin (M-p) \sin w \tan \frac{1}{2}i + k' \sin (K'-p) \sin u \right\}\end{aligned}\tag{13}$$

Generally in equations (12) and (13) we may put $\varphi'=0$, and these equations will become simpler.

The values of i , n and w will be found from the spherical triangle between the Equator, the Ecliptic and the orbit of the satellite. This triangle gives by means of the Gaussian equations, if we denote by ω the obliquity of the ecliptic;

$$\left. \begin{aligned}\cos \frac{1}{2}i \sin \frac{1}{2}(w+n) &= -\cos \frac{1}{2}N \cos \frac{1}{2}(\omega-J) \\ \cos \frac{1}{2}i \cos \frac{1}{2}(w+n) &= \sin \frac{1}{2}N \cos \frac{1}{2}(\omega+J) \\ \sin \frac{1}{2}i \sin \frac{1}{2}(w-n) &= \cos \frac{1}{2}N \sin \frac{1}{2}(\omega-J) \\ \sin \frac{1}{2}i \cos \frac{1}{2}(w-n) &= -\sin \frac{1}{2}N \sin \frac{1}{2}(\omega+J)\end{aligned}\right\}.\tag{14}$$

(3) In what precedes it is assumed that the elements of the orbit of the satellite are known, and the formulæ given are sufficient for comparing the observed positions of the satellite with those computed from the elements, and for forming the equations of condition necessary to correct the assumed values of these elements. In order to obtain an approximate knowledge of the orbit of a satellite various methods may be used, but in this case, the elaborate formulæ given for computing the orbit of a planet are not necessary; and in my own experience I have found it nearly useless to employ

ON UNSYMMETRICAL ADJUSTMENTS, AND THEIR LIMITS.

BY E. L. DE FOREST, A. M., WATERTOWN, CONN.

DENOTING any series of equidistant terms, unlimited in length, by

$$\dots u_{-2}, u_{-1}, u_0, u_1, u_2 \dots \&c.,$$

let it be adjusted, for example, by the formula

$$u'_0 = l_0 u_0 + l_1 u_1 + l_2 u_2 + l_3 u_3 + l_4 u_4 \\ + l_{-1} u_{-1} + l_{-2} u_{-2} + l_{-3} u_{-3} + l_{-4} u_{-4}, \quad (1)$$

which differs from the formulas heretofore used, in that the coefficients l equidistant from the middle, as l_3 and l_{-3} for instance, are not necessarily equal to each other, so that when they are all regarded as ordinates to a curve, the curve need not be symmetrical on each side of l_0 . If such a formula is to include more terms on one side of u_0 than on the other, we have simply to suppose that the coefficients of the absent terms are zero. Having obtained a number of adjusted terms

$$\dots u'_{-2}, u'_{-1}, u'_0, u'_1, u'_2, \dots$$

let them be adjusted again by another unsymmetrical formula,

$$u''_0 = L_0 u'_0 + L_1 u'_1 + L_2 u'_2 + L_{-1} u'_{-1} + L_{-2} u'_{-2}. \quad (2)$$

By a process of substitution quite similar to that followed in my article in the ANALYST for May 1878, p. 65, where the formulas were symmetrical, we here get the following expression;

$$u''_0 = \left. \begin{aligned} &(L_0 l_0 + L_1 l_{-1} + L_2 l_{-2} + L_{-1} l_1 + L_{-2} l_2) u_0 \\ &+ (L_0 l_1 + L_1 l_0 + L_2 l_{-1} + L_{-1} l_2 + L_{-2} l_3) u_1 \\ &+ (L_0 l_2 + L_1 l_1 + L_2 l_0 + L_{-1} l_3 + L_{-2} l_4) u_2 \\ &+ (L_0 l_3 + L_1 l_2 + L_2 l_1 + L_{-1} l_4) u_3 \\ &+ (L_0 l_4 + L_1 l_3 + L_2 l_2) u_4 \\ &\quad + L_1 l_4 + L_2 l_3) u_5 + L_2 l_4 u_6 \\ &+ \text{terms containing } u_{-1}, u_{-2}, \dots u_{-6}. \end{aligned} \right\} \quad (3)$$

The coefficients of $u_{-1}, u_{-2} \&c.$, will be the same as those of $u_1, u_2 \&c.$ respectively, except that all the sub-indices of L and l will have their signs changed. This formula becomes identical with (3) of the former article when the negative sub-indices of L and l are changed to positive. It is a resultant formula of adjustment, equivalent to (1) and (2) used in succession.

Let the coefficients of the several terms u in (1) and (2) become those of the powers of z in the two polynomials

$$l_{-4} + l_{-3}z + l_{-2}z^2 + l_{-1}z^3 + l_0z^4 + l_1z^5 + l_2z^6 + l_3z^7 + l_4z^8, \\ L_{-2} + L_{-1}z + L_0z^2 + L_1z^3 + L_2z^4.$$

When these two are multiplied together, it will be found that the coefficients of the several powers of z in the product are the same as those of the several terms u in the formula (3). Like results are obtained when formulas other than (1) and (2), that is, containing different numbers of terms, are successively used, and we generalize the fact thus. Let any two component formulas be

$$\left. \begin{aligned} u'_0 &= l_0 u_0 + l_1 u_1 + l_2 u_2 + \dots + l_m u_m \\ &\quad + l_{-1} u_{-1} + l_{-2} u_{-2} + \dots + l_{-m} u_{-m} \\ u'_0 &= L_0 u_0 + L_1 u_1 + L_2 u_2 + \dots + L_n u_n \\ &\quad + L_{-1} u_{-1} + L_{-2} u_{-2} + \dots + L_{-n} u_{-n} \end{aligned} \right\} \quad (4)$$

When the given series is adjusted by these two in succession, no matter which is used first, the result will be the same as that of a single adjustment made by a formula whose coefficients are those of the powers of z in the product of the two polynomials

$$\left. \begin{aligned} l_{-m} + l_{-m+1} z + l_{-m+2} z^2 + \dots + l_0 z^m + l_1 z^{m+1} + \dots + l_m z^{2m}, \\ L_{-n} + L_{-n+1} z + L_{-n+2} z^2 + \dots + L_0 z^n + L_1 z^{n+1} + \dots + L_n z^{2n}. \end{aligned} \right\} \quad (5)$$

In the resultant formula, the coefficient of the term u_0 is that one which includes the product $L_0 l_0$. It may be noticed that the coefficients in the product of the polynomial factors are independent of the particular exponents assigned to z in the two factors, provided that each series of exponents is an arithmetical progression, and that both progressions increase by the same constant difference. We might therefore use instead

$$\left. \begin{aligned} l_{-m} z^{-m} + \dots + l_{-1} z^{-1} + l_0 z^0 + l_1 z + \dots + l_m z^m, \\ L_{-n} z^{-n} + \dots + L_{-1} z^{-1} + L_0 z^0 + L_1 z + \dots + L_n z^n, \end{aligned} \right\} \quad (6)$$

where the indices of z are the same as the sub-indices of l , L and u .

It follows from the foregoing that if a series is adjusted k times in succession by the single formula

$$u'_0 = \lambda_0 u_0 + \lambda_1 u_1 + \lambda_2 u_2 + \dots + \lambda_m u_m \\ + \lambda_{-1} u_{-1} + \lambda_{-2} u_{-2} + \dots + \lambda_{-m} u_{-m}, \quad (7)$$

and if the equivalent or resultant formula is denoted by

$$u_0^{(k)} = l_0 u_0 + l_1 u_1 + l_2 u_2 + \dots + l_{km} u_{km} \\ + l_{-1} u_{-1} + l_{-2} u_{-2} + \dots + l_{-km} u_{-km}, \quad (8)$$

then the coefficients l will be the same as the coefficients of the powers of z in the expansion of

$$(\lambda_{-m} + \lambda_{-m+1} z + \lambda_{-m+2} z^2 + \dots + \lambda_0 z^m + \lambda_1 z^{m+1} + \dots + \lambda_m z^{2m})^k. \quad (9)$$

A convenient general form of this expansion, known as the Multinomial Formula, may be found in Davies and Peck's *Dictionary of Mathematics*. (For a demonstration, but in less convenient form, see Price's *Calculus*, Vol. I, p. 161.) It is such that if we write

$$(a_0 + a_1 z + a_2 z^2 + \dots + a_n z^n)^k = B_0 + B_1 z + B_2 z^2 + \dots + B_{kn} z^{kn}, \quad (10)$$

then any group of $n + 1$ coefficients B in the expansion, taken from B_{r-n} up to B_r inclusive, are evidently connected by the relation

$$B_r = \frac{1}{a_0 r} \left\{ (k-r+1)a_1 B_{r-1} + (2k-r+2)a_2 B_{r-2} + \dots + (nk-r+n)a_n B_{r-n} \right\} \quad (11.)$$

This relation, by the way, holds also for the expansion extended backward to terms in z^{-1} , z^{-2} &c., provided that their coefficients B_{-1} , B_{-2} &c., are made zero, so that, remembering that $B_0 = a_0^k$, we can find the other coefficients B_1 , B_2 &c., successively from (11), by giving to r the values 1, 2, 3 &c., in succession. Formulas (10) and (11) may therefore be regarded as expressing the multinomial theorem in a concise form.

Now let us take $n = 2m$, and change the notation for the coefficients to correspond with (8) and (9).

For $a_0, a_1, a_2 \dots a_n$ write $\lambda_{-m}, \lambda_{-m+1} \dots \lambda_{-1}, \lambda_0, \lambda_1 \dots \lambda_m$.

For $B_0, B_1, B_2 \dots B_{kn}$ write $l_{-km}, l_{-km+1} \dots l_{-1}, l_0, l_1 \dots l_{km}$.

Also let the place of the last one of any group of $2m + 1$ coefficients in the expansion be denoted, not by its rank r reckoned from the first term, but by its rank $i + m$ reckoned from the middle term l_0 ; so that l_i is the middle term of the group, and i is its rank reckoned from l_0 . This gives

$$r = km + i + m,$$

and the group

$$B_r, B_{r-1} \dots B_{r-n}$$

is now

$$l_{i+m}, l_{i+m-1} \dots l_{i+1}, l_i, l_{i-1}, \dots l_{i-m}.$$

Substituting the above in (11), we get

$$\begin{aligned} & -(mk + i + m)\lambda_{-m} l_{i+m} - [(m-1)k + i + m - 1]\lambda_{-m+1} l_{i+m-1} - \dots \\ & -(k + i + 1)\lambda_{-1} l_{i+1} - i\lambda_0 l_i + (k - i + 1)\lambda_1 l_{i-1} + (2k - i + 2)\lambda_2 l_{i-2} + \dots \\ & + (mk - i + m)\lambda_m l_{i-m} = 0, \end{aligned}$$

and consequently

$$\frac{(\lambda_1 l_{i-1} - \lambda_{-1} l_{i+1}) + 2(\lambda_2 l_{i-2} - \lambda_{-2} l_{i+2}) + \dots + m(\lambda_m l_{i-m} - \lambda_{-m} l_{i+m})}{\lambda_0 l_i + (\lambda_1 l_{i-1} + \lambda_{-1} l_{i+1}) + (\lambda_2 l_{i-2} + \lambda_{-2} l_{i+2}) + \dots + (\lambda_m l_{i-m} + \lambda_{-m} l_{i+m})} = \frac{i}{k+1} \quad (12.)$$

This, like (11), is simply a form of the multinomial theorem. Our changes of notation have not affected its generality, for if the number of terms in the first member of (10) were an even number $2m$, we could always make it the odd number $2m + 1$ without altering its value, by adding a term with coefficient zero. As has been seen, (12) expresses not only the relation between any $2m + 1$ consecutive coefficients l of the powers of z in the expansion of (9), but also the relation between any $2m + 1$ consecutive coefficients

l , in the resultant adjustment formula (8), which is the equivalent of k successive applications of (7). When we suppose (7) to be symmetrical, so that $\lambda_{-1} = \lambda_1$, $\lambda_{-2} = \lambda_2$ &c., then (12) becomes identical with (3) of my article in ANALYST for Sept., 1878, and a demonstration is found for the property which was there inferred only from a number of isolated cases.

We will now proceed to determine the limiting curve followed by the coefficients l when the number of applications of the unsymmetrical formula (7) becomes infinite. This is a general problem, including the special case of a symmetrical formula. The process will be similar to that employed for the special case, in my article just mentioned.

Let the finite differences of the series l_i, l_{i+1} &c., be

$$\begin{aligned} \Delta_1 &= l_{i+1} - l_i, & \Delta_3 &= l_{i+3} - 3l_{i+2} + 3l_{i+1} - l_i, \\ \Delta_2 &= l_{i+2} - 2l_{i+1} + l_i, & \Delta_4 &= l_{i+4} - 4l_{i+3} + 6l_{i+2} - 4l_{i+1} + l_i, \\ \Delta_5 &= l_{i+5} - 5l_{i+4} + 10l_{i+3} - 10l_{i+2} + 5l_{i+1} - l_i, \end{aligned}$$

and let differences higher than Δ_5 be neglected, so that Δ_5 is regarded as constant for any six consecutive coefficients l . Thus we find

$$\begin{aligned} l_{i+1} &= l_i + \Delta_1, & l_{i-1} &= l_i - \Delta_1 + \Delta_2 - \Delta_3 + \Delta_4 - \Delta_5, \\ l_{i+2} &= l_i + 2\Delta_1 + \Delta_2, & l_{i-2} &= l_i - 2\Delta_1 + 3\Delta_2 - 4\Delta_3 + 5\Delta_4 - 6\Delta_5, \\ l_{i+3} &= l_i + 3\Delta_1 + 3\Delta_2 + \Delta_3, & l_{i-3} &= l_i - 3\Delta_1 + 6\Delta_2 - 10\Delta_3 + 15\Delta_4 - 21\Delta_5, \\ &\text{\&c.,} & &\text{\&c.} \end{aligned}$$

The law of these expressions is such that we have in general

$$\left. \begin{aligned} l_{i+n} &= l_i + n\Delta_1 + \frac{1}{2}n(n-1)\Delta_2 + \frac{1}{2 \cdot 3} n(n-1)(n-2)\Delta_3 \\ &\quad + \frac{1}{2 \cdot 3 \cdot 4} n(n-1)(n-2)(n-3)\Delta_4 + \text{\&c.}, \\ l_{i-n} &= l_i - n\Delta_1 + \frac{1}{2}n(n+1)\Delta_2 - \frac{1}{2 \cdot 3} n(n+1)(n+2)\Delta_3 \\ &\quad + \frac{1}{2 \cdot 3 \cdot 4} n(n+1)(n+2)(n+3)\Delta_4 - \text{\&c.}, \end{aligned} \right\} \quad (13)$$

and they will be the same if differences higher than Δ_5 are taken into account. Substitute them in (12), and write for brevity

$$\left. \begin{aligned} b_0 &= \lambda_0 + (\lambda_1 + \lambda_{-1}) + (\lambda_2 + \lambda_{-2}) + \dots + (\lambda_m + \lambda_{-m}) \\ b_1 &= 1(\lambda_1 - \lambda_{-1}) + 2(\lambda_2 - \lambda_{-2}) + \dots + m(\lambda_m - \lambda_{-m}) \\ b_2 &= 1^2(\lambda_1 + \lambda_{-1}) + 2^2(\lambda_2 + \lambda_{-2}) + \dots + m^2(\lambda_m + \lambda_{-m}) \\ b_3 &= 1^3(\lambda_1 - \lambda_{-1}) + 2^3(\lambda_2 - \lambda_{-2}) + \dots + m^3(\lambda_m - \lambda_{-m}) \\ b_4 &= 1^4(\lambda_1 + \lambda_{-1}) + 2^4(\lambda_2 + \lambda_{-2}) + \dots + m^4(\lambda_m + \lambda_{-m}) \end{aligned} \right\} \quad (14)$$

&c., &c.

It will be seen that b_2, b_4 &c., here are not the same as they were at p. 130 of my former article, since they include the λ with negative sub-indices.

Denoting the numerator and denominator of the first member of (12) by N and D , we get

$$\left. \begin{aligned} N &= b_1 l_i - b_2 \Delta_1 + \frac{1}{2}(b_2 + b_3) \Delta_2 - \frac{1}{2 \cdot 3}(2b_2 + 3b_3 + b_4) \Delta_3 \\ &\quad + \frac{1}{2 \cdot 3 \cdot 4}(6b_2 + 11b_3 + 6b_4 + b_5) \Delta_4 - \&c., \\ D &= b_0 l_i - b_1 \Delta_1 + \frac{1}{2}(b_1 + b_2) \Delta_2 - \frac{1}{2 \cdot 3}(2b_1 + 3b_2 + b_3) \Delta_3 \\ &\quad + \frac{1}{2 \cdot 3 \cdot 4}(6b_1 + 11b_2 + 6b_3 + b_4) \Delta_4 - \&c., \\ \frac{N}{D} &= \frac{i}{k+1}. \end{aligned} \right\} (15)$$

It will be found that in the coefficient of any difference Δ_m the numerical coefficients of the b within the parentheses are the same as those of the powers of n in the product of the factors

$$(1+n), (2+n), \dots [(m-1)+n],$$

so that the above expressions for N and D might be readily extended as far as desired. Now when k becomes infinite, and the successive values of l are regarded as consecutive ordinates to a limiting curve, we have

$$l_i = y, \quad \Delta_1 = dy, \quad \Delta_2 = d^2y, \quad \Delta_3 = d^3y, \&c.$$

Let l_0 be taken as the axis of Y . The abscissa x corresponding to any y , which before was $x = i\Delta x$, becomes $x = idx$ at the limit, when the ordinates are set close together so as to be consecutive. Thus (15) gives

$$\frac{b_1 y - b_2 dy + \frac{1}{2}(b_2 + b_3)d^2y - \frac{1}{2 \cdot 3}(2b_2 + 3b_3 + b_4)d^3y + \&c.}{b_0 y - b_1 dy + \frac{1}{2}(b_1 + b_2)d^2y - \frac{1}{2 \cdot 3}(2b_1 + 3b_2 + b_3)d^3y + \&c.} = \frac{x}{(k+1)dx}, \quad (16)$$

as the differential equation of the limiting curve. The order of this equation and the nature of the curve will depend upon the values of the constants b .

To show their significance, suppose that the series of terms $u_0, u_1 \&c.$, to which the adjusting process is applied, is of algebraic form and therefore represented by equidistant ordinates u to the curve

$$u = A_0 + A_1 x + A_2 x^2 + A_3 x^3 + \&c., \quad (17)$$

whose degree is equal to the order of the series. The origin may be taken at any term we please, and we take it at u_0 , so that $u_0 = A_0$. The constant interval between the ordinates may be the unit of x . Thus we have

$$\begin{aligned} u_1 &= A_0 + A_1 + A_2 + \&c., & u_{-1} &= A_0 - A_1 + A_2 - \&c., \\ u_2 &= A_0 + 2A_1 + 2^2 A_2 + 2^3 A_3 + \dots, & u_{-2} &= A_0 - 2A_1 + 2^2 A_2 - 2^3 A_3 + \dots, \\ u_3 &= A_0 + 3A_1 + 3^2 A_2 + 3^3 A_3 + \dots, & u_{-3} &= A_0 - 3A_1 + 3^2 A_2 - 3^3 A_3 + \dots, \\ &\&c., & &\&c., \end{aligned}$$

and the adjustment formula (7) becomes by substitution of these expressions, and by using the notation of (14),

$$u'_0 = A_0 b_0 + A_1 b_1 + A_2 b_2 + A_3 b_3 + \&c. \quad (18)$$

If the series to be adjusted is of the zero order, so that all its terms are equal, then in (17) $A_1, A_2, \&c.$, will be zero, and in order that the adjusted term u'_0 shall be the same as the corresponding given term $u_0 = A_0$, it is necessary and sufficient that in (18) we shall have $b_0 = 1$. If the series is of the first order, or an arithmetical progression, then in (17) $A_2, A_3 \&c.$, will be zero, and since A_0 and A_1 may have any values, it is necessary and sufficient for making $u'_0 = u_0$ that in (18) we should have $b_0 = 1$ and $b_1 = 0$. If the series is of the second order, $A_3, A_4 \&c.$, will be zero, and to make $u'_0 = u_0$ we must have $b_0 = 1, b_1 = 0, b_2 = 0$. So in general, if the given series is of the n th order, the conditions necessary and sufficient to make the adjusted series identical with the given one are

$$b_0 = 1, \quad b_1 = 0, \quad b_2 = 0, \quad \dots \quad b_n = 0. \quad (19)$$

This is Schiaparelli's theorem differently proved, and extended to include the case of unsymmetrical coefficients. Since after any number k of successive adjustments the final adjusted series, in the cases supposed, will be the same as the original one, we see that if the conditions (19) are satisfied by the coefficients λ in the original formula (7), they will also be satisfied by the l in the resultant formula (8), and consequently also by the y in the limiting curve (16). The numerical coefficients $1^n, 2^n, 3^n \&c.$, of the λ in (14) will become the n th powers of the number i which denotes the rank of the l in (12) and of the y in (16). Thus the conditions $b_0 = 1$ and $b_n = 0$ for the adjustment formula (7), imply for (8) and (12) the conditions

$$\Sigma l = 1, \quad \Sigma i^n l = 0. \quad (20)$$

Every adjustment formula must obviously be such that if we apply it to a series of given terms which are all equal, the adjusted terms will be equal to the given ones. This requires that the sum of all the λ 's in (7) shall be unity, so that we always have $b_0 = 1$. As for b_1 , we can reduce it to zero in (16) by changing the place of the origin of coordinates on the axis of X . The coefficients λ in (7) may be regarded as a system of parallel forces acting perpendicularly at intervals of unity along the axis, which is itself regarded as a mathematical lever with fulcrum at λ_0 . Thus any term $n\lambda_n$, in the expression for b_1 in (14) is the moment of λ_n about the fulcrum, and b_1 is the moment of the whole system. If the fulcrum coincides with the centre of parallel forces, there is equilibrium, and $b_1 = 0$. In this case, putting $b_0 = 1$ and $b_1 = 0$ in (16), and neglecting the $d^2y, d^3y \&c.$, in the denominator, in comparison with y , we get

$$\begin{aligned} & -b^2\left(\frac{dy}{y}\right) + \frac{1}{2}(b_2 + b_3)\left(\frac{d^2y}{y}\right) - \frac{1}{2.3}(2b_2 + 3b_3 + b_4)\left(\frac{d^3y}{y}\right) + \\ & \qquad \qquad \qquad = \frac{1}{(k+1)} \end{aligned}$$

Taking the problem of adjustment in its most general form, supposing the λ to have any arbitrarily assigned values consistent with $b_0 = 1$, the centre of parallel forces will of course be in general not at λ_0 , but at some other place. The values of the coefficients in a resultant formula after k applications, however, are independent of the position of the origin or zero point. For example, let the adjustment formula include six terms, whose coefficients are $-\frac{1}{7}, \frac{2}{7}, \frac{4}{7}, \frac{3}{7}, \frac{1}{7}, -\frac{2}{7}$, and let us put λ_0 equal first to the $\frac{1}{7}$, and secondly to the $\frac{3}{7}$, or two intervals to the right of its first place. We thus have the two formulas

$$u'_0 = \frac{1}{7}(-u_{-1} + 2u_0 + 4u_1 + 3u_2 + u_3 - 2u_4),$$

$$u'_0 = \frac{1}{7}(-u_{-3} + 2u_{-2} + 4u_{-1} + 3u_0 + u_1 - 2u_2),$$

whose coefficients are the same and in the same order, but differently situated with reference to the origin. If the given series is adjusted twice by each of these, the coefficients of the resultant formulas will in both cases be those of the powers of z in the expansion of

$$\left(\frac{-1 + 2z + 4z^2 + 3z^3 + z^4 - 2z^5}{7} \right)^2,$$

so that the two resultant formulas are

$$u''_0 = \frac{1}{49}(u_{-2} - 4u_{-1} - 4u_0 + 10u_1 + 26u_2 + 32u_3 + 9u_4 - 10u_5 - 11u_6 - 4u_7 + 4u_8),$$

$$u''_0 = \frac{1}{49}(u_{-6} - 4u_{-5} - 4u_{-4} + 10u_{-3} + 26u_{-2} + 32u_{-1} + 9u_0 - 10u_1 - 11u_2 - 4u_3 + 4u_4).$$

The coefficients are the same in both, but the origins or places of u_0 are separated by four intervals, being twice as far as in the original formulas. If each formula were applied k times in succession, the origins in the final resultants would be k times as far apart as they were at first, but both sets of resultant coefficients would still be alike. If therefore we suppose the origin or fulcrum in (7) to be transferred to the center of parallel forces, that is, shifted through an interval b_1 to the right of its first position, the limiting curve will be unchanged, and will still be characterized by the differential equation (16), only its origin will be transferred through a distance $b_1 k dx$ to the right, and the constants b_1, b_2, b_3 &c., of (14) and the i in (20) must now be reckoned from the new origin or center of parallel forces, and we have $b_1 = 0$. Thus equation (16) has lost nothing of its generality by being put in the form (21). The principle of continuity implies that a shifting of the origin of the resultant formula through an interval of $b_1 k$ units will correspond analytically to a shifting of the origin in the original formula through b_1 units, even though b_1 should not be a whole number. In such a case, the numbers i become fractional.

Now if b_2 estimated from the centre of parallel forces, is not zero, neglecting in (21) the differentials of a higher order in comparison with those

of a lower one, we have this differential equation of the first order,

$$-b_2 \left(\frac{dy}{y} \right) = \frac{xdx}{(k+1)(dx)^2}, \quad \therefore \frac{dy}{dx} = \frac{-xy}{b_2(k+1)(dx)^2}. \quad (22)$$

The denominator $b_2(k+1)(dx)^2$ is a finite constant, and k is an infinity of the second order. (ANALYST, May, 1879, pp. 67 and 72.)

The constant of integration will be determined by the condition $b_0 = 1$, which, as (20) shows, is in the limiting curve equivalent to $\Sigma y = 1$, that is,

$$\frac{1}{dx} \int_{-\infty}^{\infty} y dx = 1. \quad (23)$$

The condition $b_1 = 0$ is not used for determining a constant. The value of b_1 cannot affect the dimensions or form of the curve, but only its place, or the place of λ_0 , along the axis of X .

Next, suppose that besides $b_0 = 1$ we have $b_2 = 0$; then (21) reduces to

$$\frac{1}{2} b_3 \left(\frac{d^2 y}{y} \right) = \frac{xdx}{(k+1)(dx)^2}, \quad \therefore \frac{d^2 y}{dx^2} = \frac{2xy}{b_3(k+1)(dx)^2}. \quad (24)$$

Here k must be an infinity of the third order, if x is finite and if $d^2 y$ is an infinitesimal of the second order as compared with y . There will be two constants of integration, determined by the two conditions, $b_0 = 1$ and $b_2 = 0$, which (20) shows to be equivalent at the limit to $\Sigma y = 1$ and $\Sigma i^2 y = 0$, that is to say, since $i = x \div dx$,

$$\frac{1}{dx} \int_{-\infty}^{\infty} y dx = 1, \quad \int_{-\infty}^{\infty} x^2 y dx = 0. \quad (25)$$

Likewise when $b_0 = 0$, $b_2 = 0$, $b_3 = 0$, (21) becomes

$$-\frac{1}{2 \cdot 3} b_4 \left(\frac{d^3 y}{y} \right) = \frac{xdx}{(k+1)(dx)^2}, \quad \therefore \frac{d^3 y}{dx^3} = \frac{-2 \cdot 3xy}{b_4(k+1)(dx)^4} \quad (26)$$

with the three conditions,

$$\frac{1}{dx} \int_{-\infty}^{\infty} y dx = 1, \quad \int_{-\infty}^{\infty} x^2 y dx = 0, \quad \int_{-\infty}^{\infty} x^3 y dx = 0, \quad (27)$$

and so on when also $b_4 = 0$, $b_5 = 0$ &c. Denoting the coefficients of xy in (22), (24), (26) &c., by a , we have this general proposition; that the limiting curve is always characterized by a linear differential equation of the form

$$a = \frac{1 \cdot 2 \cdot 3 \dots n(-1)^n}{b_{n+1}(k+1)(dx)^{n+1}}, \quad \frac{d^n y}{dx^n} = axy, \quad (28)$$

where k is an infinity of the $(n+1)$ th order, and the n constants of integration are to be determined by the n conditions

$$\left. \begin{aligned} \frac{1}{dx} \int_{-\infty}^{\infty} y dx &= 1, \\ \int_{-\infty}^{\infty} x^2 y dx &= \int_{-\infty}^{\infty} x^3 y dx = \dots = \int_{-\infty}^{\infty} x^n y dx = 0. \end{aligned} \right\}$$

The origin of coordinates is at the centre of parallel forces for the given coefficients λ , and the constants b are reckoned from that point. The above proposition holds good for the ultimate resultant coefficients of any adjustment formula, and also for the coefficients in the expansion of any given polynomial to the power $k = \infty$, when the sum of the given coefficients is unity. It has long been known that when p and q are positive fractions whose sum is unity, the terms in the expansion of $(p+q)^{\infty}$ take the form of the probability curve, answering to the case $n = 1$ in (28). These terms are the same as the coefficients of the powers of z in the expansion of the binomial $(p+qz)^{\infty}$. But I am not aware that any writer has hitherto shown that the limiting form of the coefficients in the expansion of a polynomial is also, in general, the probability curve, while in special cases it may be one of the higher curves included under the form (28) and (29).

The order n of the differential equation will always be less than the number of terms which compose the given polynomial, for the conditions $b_0 = 1, b_1 = 0, b_2 = 0$ &c., are $n + 1$ in number, and to satisfy them all, we must have at least as many as $n + 1$ disposable coefficients λ . It may happen that there will be no limiting curve at all, the consecutive resultant coefficients diverging from each other as k increases, so that the finite differences in (15) do not become differentials. Such cases appear to be distinguished by the constants of integration taking imaginary values. (ANALYST, Sept., 1878, p. 140.)

We have said in connection with (28), that k is an infinity of the $(n+1)$ th order. The whole number of terms in a resultant formula is $2km + 1$, as is shown in (8). Since dx is the interval between consecutive terms, and is contained ∞ times in a finite distance on the axis of X , the whole series of resultant coefficients must apparently be regarded as reaching, at the limit, to a distance comparable to the n th power of infinity, which would be the full extent of the limits of integration in (29). Can we say that the whole axis of X may be regarded as reaching not only to the distance ∞ , but to ∞^n ? If so, it would seem that the higher the order n of the curve, the more remote will the limit be, and that for a given finite value of k , the limiting form will be approached more nearly if n is small than if it is large.

(To be continued in No. 6.)

See p. 161

REMARKS ON THE RESOLUTION IN INTEGERS OF THE
EQUATION $X(X+1)(X+2) \dots (X+N) = Y^2$ (1),
AND SOLUTION OF PROB. 262.

TRANSLATED FROM "NOUVELLES ANNALES DE MATHÉMATIQUES", FIRST
SERIES, VOL. XIX, BY PROF. ASHER B. EVANS, LOCKPORT, N. Y.

EVERY divisor common to any two of the integral numbers $x, (x+1), (x+2), \dots (x+n)$, since it must divide their difference, is necessarily one of the numbers $1, 2, 3, 4, \dots n$. Therefore every prime number that is a divisor of any two of the integral numbers $x, (x+1), (x+2), \dots (x+n)$ is necessarily one of the prime numbers $2, 3, 5, \dots p$, where p is the greatest of the prime numbers comprised in the series $1, 2, 3, 4, \dots n$.

If the product $x(x+1)(x+2) \dots (x+n)$ is an exact square, as equation (1) supposes, and if the exponent of the highest power of a prime number dividing one of its factors is odd, that prime number will belong to the series $2, 3, 5, \dots p$; for it will divide at least two of the factors of the product: hence we conclude that:

1°. If a factor of the product $x(x+1)(x+2) \dots (x+n)$ is not divisible by any of the numbers $2, 3, 5, \dots p$, it will be, for this reason, an exact square.

2°. No one of the factors $x, (x+1), (x+2), \dots (x+n)$ can admit as divisors all the numbers $2, 3, 5, \dots p$.

We now proceed to demonstrate 2°. Let us in the first place suppose that one of the two extreme factors, for example x , is separately divisible by $2.3.5 \dots p$. Then $(x+1)$, which is prime to x , not admitting any one of these divisors, will be an exact square (1°); and as x is, by hypothesis, a multiple of $(2.3.5 \dots p)$, the factors $(x+2), (x+4)$ will be divisible by 2 and will not have another divisor comprised in the series $2, 3, 5, \dots p$. Moreover, neither of the factors $(x+2), (x+4)$, is an exact square, for $(x+1)$ being the square of an integer greater than unity, no other number greater than $x+1$ can be the square of a whole number unless it exceeds $x+1$ by at least two units. It is necessary then (1°) that $(x+2), (x+4)$, should be whole numbers of the form $2a^2, 2b^2$, and this requires that $b^2 - a^2 = 1$. We have therefore arrived at this conclusion, that if x is a multiple of $(2.3.5 \dots p)$, the difference of the squares of two whole numbers will be unity.

The same demonstration applies to the other extreme factor $(x+n)$.

With reference to the intermediate factors $(x+1), (x+2), \dots (x+n-1)$ if one of them, $(x+1)$ for example, were a multiple of $(2.3.5 \dots p)$, the factors x and $(x+2)$ which include it, would be squares (1°); which is entirely impossible, since they differ by only two units.

APPLICATIONS OF THE FOREGOING.

[If $x(x+1)(x+2)(x+3)(x+4) = y^2$ (2), then $p = 3$, $(2.p) = 6$. and (2°) , $x = 6m+1$; and (1°) , $x = 6m+1 = a^2$ and $x+4 = 6m+5 = b^2$; whence $b^2 - a^2 = 4$; which is evidently impossible in integers. Translator.]

[Solution of Problem IV., *Mathematical Monthly*, Vol. I., No. II.]-If $x(x+1)(x+2)(x+3)(x+4)(x+5) = y^2$ (3), $p = 5$, $(2.3.p) = 30$.

We will proceed to show that if (3) admits of a solution in integers, it produces the absurdity that, of six consecutive integers, no one is divisible by 6.

If $x = 6a$, the whole number a is not divisible by 5 (2°). Now the equation $x = 6a$ gives $x+5 = 6a+5$: moreover we see that $x+5$ is prime to 2, 3, 5; then $(x+5)$ is a square (1°). It follows that $x(x+1)(x+2)(x+3) \times (x+4)$, the product of five consecutive integers, would be a square; that which never takes place.

The same demonstration applies to the other extreme factor $(x+5)$.

It remains to consider the remaining intermediate factors $(x+1)$, $(x+2)$, $(x+3)$, $(x+4)$. If one of them, $(x+2)$ for example, is a multiple of 6 it is evident that the two factors $(x+3)$, $(x+1)$ which comprise $(x+2)$, will be prime to 6; and as they cannot be both multiples of 5, one of them will not admit of any of the divisors 2, 3, 5, and will therefore be the square, a^2 , of a whole number.

The other factor, $a^2 \pm 2$, not being a square ought to be divisible by 5 (1°). We shall then have $a^2 \pm 2 = 5m$; now this equation cannot exist, for we know that the formula for squares that are not multiples of 5, is $a^2 = 5m \pm 1$. It follows from what precedes that the product of six consecutive integers cannot be a square.

[Solution of Problem 262, *Analyst*, Vol. VI., No. 3.]-If $x(x+1)(x+2) \times (x+3)(x+4)(x+5)(x+6) = y^2$ (4), admitted of a solution in integers, no one of the seven numbers x , $(x+1)$, $(x+2)$, $(x+3)$, $(x+4)$, $(x+5)$, $(x+6)$ would be divisible by 6. In reality p being still equal to 5, we will demonstrate as in the preceding case, that it is impossible that one of the factors $(x+1)$, $(x+2)$, $(x+3)$, $(x+4)$, $(x+5)$ can be a multiple of 6.

We will now demonstrate that the extreme factors x , $(x+6)$ cannot be multiples of 6. Supposing $x = 6m$, we shall have $x+6 = 6(m+1)$; and neither of the members m , $(m+1)$ will be divisible by 5 (2°). Now, the equalities $x = 6m$, $x+6 = 6(m+1)$ give $x+1 = 6(m+1)-5$ and $x+5 = 6m+5$. These last show that the factors $(x+1)$, $(x+5)$ are not divisible by any of the three prime factors 2, 3, 5. Then (1°) we must have $x+1 = a^2$, $x+5 = b^2$, whence $b^2 - a^2 = 4$; an equality which is impossible, since it requires that the difference of the squares of two integers b , a , greater than unity, be less than 5. It is also proved that the proposed equation (4) does not admit of a solution in integers.

DEMONSTRATION OF A PROPOSITION (SEE QUERY, P. 55).

BY R. J. ADCOCK, MONMOUTH, ILL.

Let ABC represent the triangle. Draw AP perpendicular to CD , the radius of the sphere. In the plane DBC , perpendicular to CD , draw PG = sine of BC .

Then twice the area of APG , the orthographic projection of the plane triangle ABC upon the plane through A perpendicular to CD , = $\sin a \sin b \sin C$; and likewise the projections of the plane triangle ABC upon planes through B and C , perpendicular to DB and DA are $\sin a \sin c \sin B$, $\sin b \sin c \sin A$, which projections are equal, being upon planes equally inclined to the plane ABC .



Also, $\sin AO \sin BO \sin AOB$ is twice the projection of the plane triangle AOB upon a plane perpendicular to OD , the inclination of which plane to APG is the arc OC . Hence the projection of $\sin AO \sin BO \sin AOB$ upon APG is $\sin AO \sin BO \sin AOB \cos OC$.

Through A , draw AK , the sine of AO , KH perpendicular to CD , and HL parallel and equal to PG ; then the triangle LKH is the projection of the plane triangle OBC upon a plane perpendicular to CD , and $LH \times HK \times \sin C = \sin a \cos AO \sin CO \sin C = \sin BO \sin CO \sin BOC \cos AO$.

In like manner the projection of the plane triangle AOC upon a plane perpendicular to CD is equal to $\sin AO \sin CO \sin AOC \cos BO$. Hence $\sin a \sin b \sin C = \sin a \sin c \sin B = \sin b \sin c \sin A = \sin AO \sin BO \times \sin AOB \cos CO + \sin BO \sin CO \sin BOC \cos AO + \sin AO \sin CO \sin AOC \cos BO = \sin AO \sin BO \sin CO (\cot AO \sin BOC + \cot BO \sin AOC + \cot CO \sin AOB)$.

NOTE ON THE SOLUTION OF PROBLEM 260, BY THE EDITOR.—In the solution of this problem (pp. 121–22), the equations from Routh were incorrectly written, and should be corrected as follows: In (1), for —, read +, and in (2), for “sin”, read cos; also, in (3), for “cos” read sin.

By introducing these corrections in the solution at p. 121 we get

$$\frac{d^2\varphi}{dt^2} = \frac{5g}{7(R-r)} \cos \varphi = \frac{\frac{5}{7}g}{l} \cos \varphi, \quad (6)$$

where l is put for $R-r$. Integrating once, and reducing, we find

$$dt = \sqrt{\left(\frac{l}{\frac{5}{7}g}\right)} \cdot \frac{d\varphi}{\sqrt{(\sin \varphi)}}.$$

If we put $l \sin \varphi = x$, we have

$$d\varphi = \frac{dx}{\sqrt{(l^2 - x^2)}}; \therefore dt = \frac{l}{\sqrt{(\frac{5}{7}g)}} \cdot \frac{dx}{\sqrt{(x^2 - x^2)}}$$

This value of dt is (see Brande's Encyc., art. pend.) the differential of the time of a semi-oscillation of a pendulum whose length is l , through an arc $= \pi$, or a semi-circumference; the accelerating force being $\frac{5}{7}g$.

We are under obligations to Mr. Adcock and Mr. Siverly for calling our attention to the errors above alluded to. The above corrections being introduced, this solution agrees with Mr. Adcock's solution, he writes, "by the principle that twice the work of gravity equals *vis viva* due to translation plus that due to rotation."

"If t' is the time of a simple pendulum through the same arc φ , $t = t' \sqrt{\frac{7}{5}}$; length of pendulum $= R-r$."

SOLUTIONS OF PROBLEMS IN NUMBER FOUR.

SOLUTIONS of problems in No. 4 have been received as follows:

From R. J. Adcock, 271, 272; G. M. Day, 275; Geo. Eastwood, 267; Prof. Edgar Frisby, 266; H. Heaton, 266, 271; W. E. Heal, 270, 274; Prof. A. Hall, 275; Chas. H. Kummell, 270, 271, 273, 274, 275; Prof. D. J. McAdam, 269, 270, 274, 275; Artemas Martin, 271; P. Richardson, 270; E. B. Seitz, 267, 269, 270, 271, 272, 273, 274, 275.

We have also received from Prof. W. P. Casey, of San Francisco, Cal., a very elaborate and elegant solution of problem 85, and a geometrical construction of problem 126; also a very simple construction of problem 234; all of which we would be pleased to publish if our space permitted.

At the time of the publication of problem 85, we received several solutions of it, but as all the solutions that have yet been received, except for particular cases, involve equations of a higher degree than the second, we think the quadrilateral has not yet been "constructed". Nevertheless, as the solution by Prof. Casey, above alluded to, is perhaps the most complete and elegant that has been received, it will be inserted as soon as our space will permit; and we hope also to be able, at some future time, to present the geometrical construction of 126, above referred to.

266. "Sum the series

$$\cos 2\theta \cos \theta + \frac{\cos^2 2\theta \cos 3\theta}{2 \cdot 3} + \frac{1 \cdot 3}{2 \cdot 4} \cos^3 2\theta \frac{\cos 5\theta}{5} + \dots \text{to infinity.}"$$

SOLUTION BY PROF. EDGAR FRISBY, NAVAL OBSERVATORY, WASH., D. C.

$$\text{Let } x^{\pm n} = e^{\pm n\theta} \sqrt{-1} = \cos n\theta \pm \sqrt{-1} \sin n\theta, \quad (1)$$

$$x^n + x^{-n} = 2 \cos n\theta. \quad (2)$$

$$\begin{aligned} 2S &= \cos 2\theta(x + x^{-1}) + \frac{1}{2} \cos^2 2\theta \left(\frac{x^3 + x^{-3}}{3} \right) + \frac{1}{2} \cdot \frac{3}{4} \cos^3 2\theta \left(\frac{x^5 + x^{-5}}{5} \right) + \&c. \\ &= \sqrt{(\cos 2\theta)} \left\{ \begin{array}{l} x \cos^{\frac{1}{2}} 2\theta + \frac{1}{2} x^3 \cdot \frac{1}{3} (\cos^{\frac{3}{2}} 2\theta + \frac{1}{2} \cdot \frac{3}{4} x^5 \cdot \frac{1}{5} (\cos^{\frac{5}{2}} 2\theta + \&c.) \\ + x^{-1} \cos^{\frac{1}{2}} 2\theta + \frac{1}{2} x^{-3} \cdot \frac{1}{3} (\cos^{\frac{3}{2}} 2\theta + \frac{1}{2} \cdot \frac{3}{4} x^{-5} \cdot \frac{1}{5} (\cos^{\frac{5}{2}} 2\theta + \&c.) \end{array} \right\} \\ &= \sqrt{(\cos 2\theta)} \cdot [\sin^{-1} x \sqrt{(\cos 2\theta)} + \sin^{-1} x^{-1} \sqrt{(\cos 2\theta)}] \\ &= \sqrt{(\cos 2\theta)} \cdot \sin^{-1} [x \sqrt{(\cos 2\theta)} \sqrt{(1 - x^2 \cos 2\theta)} + x^{-1} \sqrt{(\cos 2\theta)} \\ &\quad \times \sqrt{(1 - x^2 \cos 2\theta)}] \\ &= \sqrt{(\cos 2\theta)} \cdot \sin^{-1} \sqrt{(\cos 2\theta)} [\sqrt{(x^2 - \cos 2\theta)} + \sqrt{(x^{-2} - \cos 2\theta)}] \\ &= \sqrt{(\cos 2\theta)} \cdot \sin^{-1} \sqrt{(\cos 2\theta)} [\sqrt{(\sin 2\theta \sqrt{-1} + \sqrt{(-\sin 2\theta \sqrt{-1})})}], \text{ by (1).} \end{aligned}$$

But $\sqrt{(\pm \sin 2\theta \sqrt{-1})} = \sqrt{(\frac{1}{2} \sin 2\theta)} \pm \sqrt{(-\frac{1}{2} \sin 2\theta)}$,

$$\therefore 2S = \sqrt{(\cos 2\theta)} \cdot \sin^{-1} \sqrt{(\cos 2\theta)} \sqrt{(2 \sin 2\theta)} = \sqrt{(\cos 2\theta)} \cdot \sin^{-1} \sqrt{(\sin 4\theta)},$$

$$S = \frac{1}{2} \sqrt{(\cos 2\theta)} \cdot \sin^{-1} \sqrt{(\sin 4\theta)}.$$

[Mr. Heaton, by a similar process, obtains the same result under a slightly different form.]

267. "What was the duration of a building and loan association in which, for the first eight years, money was loaned at an average premium of \$45 per share (of \$200), interest paid being 6 per cent."

SOLUTION BY E. B. SEITZ, GREENVILLE, OHIO.

Let a = the value of a share, n = the number of shares, nd = the amt. of dues paid at the end of each interval of time t , p = the premium paid for a loan on a share, r = the rate of interest, and x = the required time in years. Then, the interest on the loans being payable monthly, we have

$$nd(1 + \frac{1}{12}r)^{12(x-t)}, nd(1 + \frac{1}{12}r)^{12(x-2t)}, nd(1 + \frac{1}{12}r)^{12(x-3t)}, \dots$$

for the compound amounts of the successive payments of dues; and the sum of these amounts must be equal to $n(a-p)$. Hence, since the amounts form a geometrical series whose least term is nd , and ratio, $(1 + \frac{1}{12}r)^{12t}$, we have

$$\frac{nd(1+\frac{1}{12}r)^{12x}-nd}{(1+\frac{1}{12}r)^{12x}-1} = n(a-p),$$

whence we find

$$x = \frac{\log \{ (a-p) [(1+\frac{1}{12}r)^{12x}-1] + d \} - \log d}{12 \log (1+\frac{1}{12}r)}.$$

In the question we have $a = \$200$, $p = \$45$, and $r = .06$.

In these associations the dues are generally payable weekly, and the amt. due on a share is $\frac{1}{48}$ of the yearly interest on the same; therefore $t = \frac{1}{52} \times 52$, and $d = \$1$. $\therefore x = 9.015$ years.

268. [No solution received.]

269. "Show that the equations

$$x = (r_1 - r_2) \cos \varphi + mr_2 \cos \left(\frac{r_1 - r_2}{r_2} \varphi \right),$$

$$y = (r_1 - r_2) \sin \varphi - mr_2 \sin \left(\frac{r_1 - r_2}{r_2} \varphi \right),$$

represent the prolate and curtate hypocycloids; and also that when $r_1 = 2r_2$ the curve becomes the ellipse

$$\frac{x^2}{(1+m)^2 r_2^2} + \frac{y^2}{(1-m)^2 r_2^2} = 1."$$

SOLUTION BY PROF. D. J. MCADAM, WASHINGTON, PA.

Take for axis of x the position of the common diameter of the two circles which passes through the generating point P , taken in the figure without the rolling circle.

Let $CA = r_2$, $OA = r_1$, $CP = mr_2$; $COB = \varphi$. Then $OF = x$, $OE + EF = OC \cos COB + CP \cos CPD = (r_1 - r_2) \cos \varphi + mr_2 \cos CPD$.

But $AG = AB = r_1 \varphi$, $\therefore \angle ACG = r_1 \varphi \div r_2$, and $ACG = CPD + \varphi$, $\therefore \angle CPD = r_1 \varphi \div r_2 - \varphi = \frac{r_1 - r_2}{r_2} \varphi$; $\therefore x = (r_1 - r_2) \cos \varphi + mr_2 \cos \left(\frac{r_1 - r_2}{r_2} \varphi \right)$,

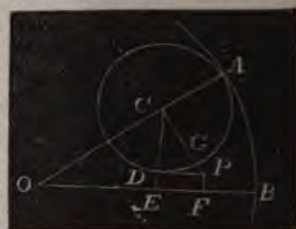
$$\text{and } CE - CD = y = (r_1 - r_2) \sin \varphi - mr_2 \sin \left(\frac{r_1 - r_2}{r_2} \varphi \right).$$

Making $r_1 = 2r_2$, $x = r_2 \cos \varphi + mr_2 \cos \varphi$; $y = r_2 \sin \varphi - mr_2 \sin \varphi$.

$$\therefore \cos \varphi = \frac{x}{r_2(1+m)} \quad (1); \quad \sin \varphi = \frac{y}{r_2(1-m)} \quad (2)$$

Squaring (1) and (2) and adding,

$$\frac{x^2}{r_2^2(1+m)^2} + \frac{y^2}{r_2^2(1-m)^2} = \sin^2 \varphi + \cos^2 \varphi = 1.$$



270. "Find the locus of the centres of circles tangent to a parabola and to the tangent at its vertex."

SOLUTION BY CHAS. H. KUMMELL, U. S. LAKE SURV., DETROIT, MICH.

Let $y'^2 = px'$ (1)

be the equation of the parabola, origin of coordinates at vertex. We obtain any point of the required locus if at the point (x', y') of the parabola a nor-

mal $y'' - y' = -\frac{2y'}{p}(x'' - x')$ (2)

is drawn and prolonged until its terminating point is as distant from the parabola as from the tangent at the vertex, and this distance is equal to the abscissa x of the required locus. We have then

$$\begin{aligned} x &= \sqrt{(x' - x)^2 + (y' - y)^2} = x' - x \cos [\tan^{-1}(2y' \div p)] \\ &= x' - \frac{px}{\sqrt{p^2 + 4y'^2}}, \end{aligned} \quad (3)$$

or since $y' = \sqrt{px'}$, by (1)

$$x = \sqrt{(x' - x)^2 + (y' - y)^2} = x' - \frac{x\sqrt{p}}{\sqrt{p + 4x'}}. \quad (4)$$

From the latter of these two equations, by transposing and squaring we find

$$(x' - x)^2(p + 4x') = px^2;$$

$\therefore 4x'^2 + (p - 8x)x' = 2x(p - 2x)$, and solving,

$$\begin{aligned} x' &= x - \frac{1}{8}p \pm \sqrt{(x - \frac{1}{8}p)^2 + \frac{1}{2}x(p - 2x)} \\ &= x - \frac{1}{8}p \pm \frac{1}{2}\sqrt{p(x + \frac{1}{16}p)}. \end{aligned}$$

Substituting this in the first of (4) we obtain

$$y = \sqrt{p} \left\{ x - \frac{1}{8}p \pm \frac{1}{2}\sqrt{p(x + \frac{1}{16}p)} \right\} + \sqrt{x^2 - \left\{ -\frac{1}{8}p \pm \frac{1}{2}\sqrt{p(x + \frac{1}{16}p)} \right\}^2}$$

The double sign refers to two loci, the external and internal.

271. "If three points be taken at random in the circumference of a circle, required the probability that the triangle formed by joining them will be acute."

SOLUTION BY HENRY HEATON, ATLANTIC, IOWA.

Let θ be the angle between the diameters drawn from two of the random points. Then, in order that the triangle be acute, the third point must fall upon the arc between the opposite extremities of these diameters. The probability of this is $\theta \div 2\pi$. The probability of θ having any particular value less than π is $d\theta \div \pi$. Hence the required probability is

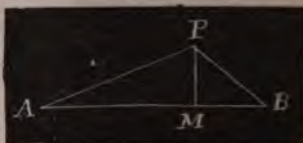
$$\frac{1}{2\pi^2} \int_0^\pi \theta d\theta = \frac{1}{4}.$$

272. "Two lines rotate, uniformly, in opposite directions about two fixed points, the velocity of one being n times that of the other; find the rectangular equation of their intersection."

SOLUTION BY E. B. SEITZ.

Let AP and BP be the two lines rotating about the fixed points A and B , the lines initially being coincident with AB .

Draw PM perpendicular to AD . Let $AM = x$, $PM = y$, $AB = a$, $\angle PAM = \theta$, and $\angle PBM = n\theta$. Then we have $\tan \theta = y \div x$, and $\tan n\theta = y \div (a-x)$. But



$$\begin{aligned} \tan n\theta &= \frac{n \tan \theta - \frac{n(n-1)(n-2)}{1.2.3} \tan^3 \theta + \frac{n(n-1)(n-2)(n-3)(n-4)}{1.2.3.4.5} \tan^5 \theta - \dots}{1 - \frac{n(n-1)}{1.2} \tan^2 \theta + \frac{n(n-1)(n-2)(n-3)}{1.2.3.4} \tan^4 \theta - \dots} \\ \therefore y \left[1 - \frac{n(n-1)}{1.2} \left(\frac{y}{x}\right)^2 + \frac{n(n-1)(n-2)(n-3)}{1.2.3.4} \left(\frac{y}{x}\right)^4 - \dots \right] \\ &= (a-x) \left[n \left(\frac{y}{x}\right) - \frac{n(n-1)(n-2)}{1.2.3} \left(\frac{y}{x}\right)^3 + \frac{n(n-1)(n-2)(n-3)(n-4)}{1.2.3.4.5} \left(\frac{y}{x}\right)^5 - \dots \right], \text{ the required equation.} \end{aligned}$$

273. "If m and n be the masses of the earth and moon, a the distance between their centers, r the radius of the earth, and if a body fall toward the earth from the point of equal attraction in the line joining their centers, find the time of falling from the height h to the earth's surface."

SOLUTION BY E. B. SEITZ, GREENVILLE, OHIO*.

Let E be the center of the earth, M that of the moon, P the point of eq'l attraction, N the position of the body at any time during its motion toward the earth; and let $EM = a$, $EN = x$, t = the time in seconds of falling from P to N , v = the velocity at N , and t_1 = the required time. Then

$$\begin{aligned} EP &= \frac{a\sqrt{m}}{\sqrt{m} + \sqrt{n}}, \quad g' = \frac{gr^2}{x^2} - \frac{gnr^2}{m(a-x)^2} \\ &= \text{force acting on the body at } N \text{ toward the center of the earth,} \\ vdv &= -g'dx, \quad dt = -\frac{dx}{v}. \end{aligned} \tag{1, 2}$$

*We have just learned that Mr. Seitz has accepted the position of Professor of Mathematics in the State Normal School, at Kirksville, Mo., and that after Aug. 28th his address will be Kirksville, Mo.—Ed.

Substituting the value of g' in (1), integrating, and observing that when $x = a\sqrt{m} \div (\sqrt{m} + \sqrt{n})$, $v = 0$, we have

$$v^2 = \frac{2gr^2 [a\sqrt{m} - (\sqrt{m} + \sqrt{n})x]^2}{am(ax - x^2)}. \quad (3)$$

Substituting the value of v in (2), and integrating, we have

$$t_1 = \left(\frac{am}{2gr^2} \right)^{\frac{1}{2}} \int_r^{h+r} \frac{\sqrt{(ax - x^2)} dx}{a\sqrt{m} - (\sqrt{m} + \sqrt{n})x}.$$

Put $x = \frac{ay^2}{1+y^2}$; then $\sqrt{(ax - x^2)} = \frac{ay}{1+y^2}$, $a\sqrt{m} - (\sqrt{m} + \sqrt{n})x = \frac{a(\sqrt{m} - y^2\sqrt{n})}{1+y^2}$, $dx = \frac{2aydy}{(1+y^2)^2}$, $y = \sqrt{\left(\frac{r}{a-r}\right)} = y_1$, when $x = r$, and $y = \sqrt{\left(\frac{h+r}{a-h-r}\right)} = y_2$, when $x = h+r$. Therefore

$$\begin{aligned} t_1 &= \frac{2a(am)^{\frac{1}{2}}}{r(2g)} \int_{y_1}^{y_2} \frac{y^2 dy}{(1+y^2)^2 (\sqrt{m} - y^2\sqrt{n})} = \frac{\sqrt{(am)}}{r\sqrt{(2g)}(\sqrt{m} + \sqrt{n})} \\ &\times \left\{ \sqrt{(ar - r^2)} - \sqrt{[a(h+r) - (h+r)^2]} + a \left(\frac{\sqrt{m} - \sqrt{n}}{\sqrt{m} + \sqrt{n}} \right) \sin^{-1} \sqrt{\left(\frac{h+r}{a}\right)} \right. \\ &- a \left(\frac{\sqrt{m} - \sqrt{n}}{\sqrt{m} + \sqrt{n}} \right) \sin^{-1} \sqrt{\left(\frac{r}{a}\right)} + \frac{a^4 \sqrt{(mn)}}{\sqrt{m} + \sqrt{n}} \log \left[\frac{\sqrt{(a-h-r)}^4 \sqrt{m} + \sqrt{(h+r)}^4 \sqrt{n}}{\sqrt{(a-h-r)}^4 \sqrt{m} - \sqrt{(h+r)}^4 \sqrt{n}} \right] \\ &\left. - \frac{a^4 \sqrt{(mn)}}{\sqrt{m} + \sqrt{n}} \log \left[\frac{\sqrt{(a-r)}^4 \sqrt{m} + \sqrt{r}^4 \sqrt{n}}{\sqrt{(a-r)}^4 \sqrt{m} - \sqrt{r}^4 \sqrt{n}} \right] \right\}. \end{aligned}$$

COR.—If $n = 0$, and $h = a - r$, we have

$$t_1 = \sqrt{[a \div (2gr^2)] [\sqrt{(ar - r^2)} + \frac{1}{2}\pi a - a \sin^{-1} \sqrt{(r \div a)}]},$$

the usual formula for the time of a body falling from a great distance.

274. "Required the shortest distance between two curves whose eq's are

$$\begin{aligned} 4x^2 + 9y^2 - 144 &= 0, \\ x'^2 + y'^2 - 26x' - 32y' + 25 &= 0." \end{aligned}$$

SOLUTION BY CHAS. H. KUMMELL, U. S. LAKE SURVEY.

The first curve is an ellipse, the semi axes of which are $a = 6$ and $b = 4$, the center being at the origin and the major axis coinciding with the x -axis. The shortest distance between this ellipse and any other curve must always be measured on a normal. Now the equation of the normal to (1) at the point (x, y) is

$$y'' - y = \frac{36y}{16x}(x'' - x). \quad (3)$$

Equation (2) is that of a circle, radius 20 and $x'_0 = 13$; $y'_0 = 16$ as co-ordinates of the center. The shortest line between the ellipse and circle

must pass through the center. Causing the normal (3) to pass through the center of the circle, we obtain the equation of that normal on which the shortest distance is measured, viz. ;

$$y'' = 16 + \frac{36y}{16x}(x''-13). \quad (4)$$

To find the intersection of this line with the ellipse, we have, substituting $\frac{x}{y}$ for $\frac{x''}{y''}$: $16x(y-16) = 36y(x-13)$, $\therefore y = \frac{64x}{117-5x}$. (5)

By (1) we have $y = \frac{2}{3}\sqrt{36-x^2}$; (6)

$$\therefore \frac{64x}{117-5x} = \frac{2}{3}\sqrt{36-x^2}, \text{ or } 25x^4 - 1170x^3 + 22005x^2 + 42120x - 492804 = 0.$$

The real roots of this eq'n are $x_1 = +4.238395$; $x_2 = -4.96895$.

The latter evidently belongs to the maximum distance and is not considered. By (5) or (6) we obtain $y_1 = +2.831258$.

We have now, for the shortest distance,

$$d_0 = \sqrt{[(x'_0 - x_1)^2 + (y'_0 - y_1)^2]} - r',$$

where $r' =$ radius of circle = 20. We have then, in numbers,

$$d_0 = 15.817126 - 20 = -4.182874.$$

275. "Find the moments of inertia of an elliptical disk: (1), about a right line in the plane of the disk and parallel to the axis of x ; (2), about a right line parallel to the axis of y , the equation of the disk being

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0."$$

SOLUTION BY PROF. HALL.

Solving the equation for x , we have

$$x = -\frac{by+d}{a} \pm \frac{1}{a}\sqrt{[(b^2-ac)y^2 + 2(bd-ae)y + d^2 - af]}.$$

If k be the mass of a unit of area, and q be the distance of the right line from the axis of x , the moment of inertia is

$$-\frac{2k}{a} \int_{y_1}^{y_2} (y-q)^2 \sqrt{[(b^2-ac)y^2 + 2(bd-ae)y + d^2 - af]} dy.$$

The values of y_1 and y_2 for the limits of integration are found by putting the quantity under the radical equal to zero. The integrals involved are all of known forms, but the reductions are complicated, and it is better to proceed as follows.

Let M be the mass of the disk, and if $2A$ and $2B$ are the principal axes of the disk we know that the moments of inertia about these axes are

$$\frac{1}{4}M \cdot B^2: \text{ and } \frac{1}{4}M \cdot A^2.$$

We have therefore only to resolve these moments along the direction of the given right line and add the quantities,

$$Mq^2: \quad \text{and } Mp^2,$$

q and p being now the distances of the right line from centre of the ellipse.

If the equation be transferred to the centre of the ellipse it becomes

$$ax^2 + 2bxy + cy^2 + f' = 0,$$

whence
$$-f' = \frac{b^2f + d^2c + e^2a - 2bde - acf}{ac - b^2} = \frac{\Delta}{ac - b^2}.$$

If θ be the angle that the axis of x makes with the major axis of the ellipse, and if we put $N = \sqrt{[(a-c)^2 + 4b^2]}$, we have

$$\cos \theta^2 = \frac{N + (a-c)}{2N}: \quad \text{and } \sin \theta^2 = \frac{N - (a-c)}{2N}.$$

We have also for the semi-axes

$$A^2 = \frac{\Delta}{ac - b^2} \cdot \frac{2}{a + c + N}: \quad B^2 = \frac{\Delta}{ac - b^2} \cdot \frac{2}{a + c - N}.$$

The moment of inertia about a line parallel to the axis of x and passing through the centre of the ellipse is $\frac{1}{4}M \cdot (B^2 \cos^2 \theta + A^2 \sin^2 \theta)$, which becomes on reduction

$$\frac{M}{4} \cdot \frac{a\Delta}{(ac - b^2)^2}: \quad \text{and the required moment is}$$

$$\frac{M}{4} \cdot \frac{a\Delta}{(ac - b^2)^2} + Mq^2.$$

For the moment about a line parallel to the axis of y we have, since the equation of the disk is symmetrical,

$$\frac{M}{4} \cdot \frac{a\Delta}{(ac - b^2)^2} + Mp^2.$$

PROBLEMS.

276. By *O. L. Mathies, Reistertown, Md.*—Given the chord AC of a circle, the side AB of a right angled triangle constructed on AC as hypotenuse, and the length of a perpendicular from A upon the line joining the right angle at B with the centre of the circle; to find the radius of the circle.

277. By *W. E. Heal, Wheeling, Indiana.*—Solve, algebraically the eq'n

$$x^{17} - 1 = 0.$$

278. By *Prof. E. J. Edmunds, New Orleans, La.*—A Chord AB of a circle, whose centre is at O , remains constantly parallel to itself; A and B are connected to the centre O , and on AO and BO , or those lines produced, M and N are taken so that $AM = BN = AB$. Required the locus of M and N .

279. By *Prof D. J. McAdam, Washington, Pa.*—If the earth be projected, when in perihelion, at right angles to a line joining the Earth and Sun, with a velocity sufficient to cause it to describe a parabola, how long would it take it to reach the orbit of Jupiter? The orbits of the Earth and Jupiter supposed in the same plane, both circles, and the radius of Jupiter's orbit five times that of the Earth.

280. By *Prof. J. Scheffer, Mercersb'g, Pa.*—Find the sum of the series

$$x \sin a - \frac{x^2 \sin 2a}{1.2} + \frac{x^3 \sin 3a}{1.2.3} - \frac{x^4 \sin 4a}{1.2.3.4} + \dots \text{to infinity,}$$

$$1 - x \cos a + \frac{x^2 \cos 2a}{2.2} - \frac{x^3 \cos 3a}{1.2.3} + \frac{x^4 \cos 4a}{1.2.3.4} - \dots \text{to infinity.}$$

PUBLICATIONS RECEIVED.

On Meteoric Fire-balls Seen in the United States during the year ending March 31, 1879.

By PROF. DANIEL KIRKWOOD. [Read before the American Philosoph. Soc., May, 1879.]

Thermodynamics. By HENRY T. EDDY, C. E., PH. D., University of Cincinnati. 132 pp. 12mo. D. Van Nostrand, Publisher. New York: 1879.

Key to the Universe, or a New Theory of its Mechanism. Founded upon, I. Continuous Orbital Propulsion, arising from the Velocity of Gravity and its consequent Aberration: II. A Resisting Ethereal Medium of Variable Density; with Mathematical Demonstrations and Tables. By ORSON PRATT, SEN. 118 pp. 8vo. Second Edition, from the first European Ed. On sale at the Historian's Office, Salt Lake City, Utah. Price, \$1.50.

The American Journal of Mathematics, Vol. II, No. 2. Baltimore, Maryland. June, 1879. 90 pages, 4to.—The leading article of this No. is a very interesting paper of 60 pages by EMORY MCCLINTOCK, F. I. A., of Milwaukee, Wis., in which the author presents the "Outlines", "Suggestions in Detail" and "Summary" of a new calculus, which he names the Calculus of Enlargement.

ERRATA.

On page 91, line 2, from bottom, for $\tan a(\beta-x)x^2$, read $\tan 2a(\beta-x)x^2$.

" " 122, " 14, for $(2 \div \sqrt{a}) \log (2 - \sqrt{2})$, read $(1 \div \sqrt{2a}) \log (1 - \sqrt{2})$.

" " 133, in the equations for r and u , 5th and 6th line from bottom, it is assumed that $s \div 2$ is reduced to the mean distance of the planet from the Earth.

" " 133, line 14, for equations, read equation.

" " 138, lines 2 and 3, transpose the signs before r , viz., read $-r$ &c., in line 2 and $+r$ &c., in line 3.

" " 147, line 19, for $b_0 = 0$, read $b_0 = 1$.

THE ANALYST.

VOL. VI.

NOVEMBER, 1879.

No. 6.

ON UNSYMMETRICAL ADJUSTMENTS, AND THEIR LIMITS.

BY E. L. DE FOREST.

(Continued from page 148.)

LET us now consider the simplest case under (28), putting $n=1$. If, as usually happens, b_2 is positive, (22) shows that a is negative. We will take

$$a = -2h^2, \quad \therefore h^2 = \frac{1}{2b_2(k+1)(dx)^2} \quad (30)$$

Integrating and applying the condition (23), and also writing $Y=y \div dx$, we find that the curve is the probability curve

$$y = \frac{hdx}{\sqrt{\pi}} e^{-h^2x^2}, \quad \therefore Y = \frac{h}{\sqrt{\pi}} e^{-h^2x^2}, \quad (31)$$

or, since $x = idx$, and putting $(hdx)^2 = g$,

$$g = \frac{1}{2b_2(k+1)}, \quad y = \sqrt{\left(\frac{g}{\pi}\right)} e^{-g^2}. \quad (32)$$

Although these formulas are strictly true only when k and i are infinities of the second and first orders, respectively, so that y is infinitesimal, yet they give approximately true results when these numbers are finite. Take for example eight applications of

$$u'_0 = \frac{1}{6}(u_{-1} + 3u_0 + 2u_1).$$

Here we have $\lambda_0 = \frac{2}{3}$, $\lambda_1 = \frac{2}{3}$, $\lambda_{-1} = \frac{1}{3}$, $b_0 = 1$, $k = 8$.

The place of the centre of parallel forces is

$$\frac{1(-1) + 3 \times 0 + 2 \times 1}{1 + 3 + 2} = \frac{1}{6},$$

or one sixth of the way from u_0 to u_1 . The value of b_2 estimated with reference to this centre as an origin is,

$$\frac{1}{6} \left\{ 1 \left(-\frac{7}{6} \right)^2 + 3 \left(-\frac{1}{6} \right)^2 + 2 \left(\frac{5}{6} \right)^2 \right\} = \frac{17}{36}.$$

Formula (32) then gives

$$g = \frac{2}{17}, \quad y = \sqrt{\left(\frac{2}{17\pi} \right)} e^{-\frac{1}{17}i^2},$$

$$\log y = 1.28671 - .051093i^2.$$

The origin or vertex of the curve will be placed at an interval of $k \times \frac{1}{4} = \frac{1}{4}$ to the right of u_0 , so that the resultant coefficient of u_0 is found by putting $i = -\frac{1}{4}$, and proceeding from this at intervals of unity we get the other coefficients y in Table I. They do not differ greatly from the true coefficients l of the resultant formula, which are also shown in the table, as given by expanding the polynomial

$$\left(\frac{1+3z+2z^2}{6} \right)^8,$$

and the approximation would be closer if k were greater.

TABLE I.

i	y	l	i	y	l	i	y	l
$-\frac{12}{8}$	.002	.001	$-\frac{7}{8}$	.102	.099	$\frac{8}{8}$	.084	.084
$-\frac{16}{8}$	.007	.005	$-\frac{4}{8}$	.157	.158	$\frac{11}{8}$	.040	.035
$-\frac{13}{8}$	.021	.018	$-\frac{1}{8}$	.191	.198	$\frac{14}{8}$	.015	.010
$-\frac{10}{8}$	.052	.049	$\frac{2}{8}$	.184	.194	$\frac{17}{8}$	.004	.002
			$\frac{5}{8}$	.140	.147			

We pass now to the second case under (28) where $n = 2$, and the equation and its conditions are

$$\frac{d^2y}{dx^2} = axy, \quad \frac{1}{dx} \int_{-\infty}^{\infty} y dx = 1, \quad \int_{-\infty}^{\infty} x^2 y dx = 0. \quad (33)$$

To integrate this by series, using the method of indeterminate coefficients, we assume

$$y = A + Bx + Cx^2 + Dx^3 + \&c.,$$

$$\therefore \begin{cases} \frac{d^2y}{dx^2} = 1.2C + 2.3Dx + 3.4Ex^2 + 4.5Fx^3 + \&c., \\ axy = Aax + Bax^2 + Cax^3 + \&c., \end{cases}$$

these last two being equal, so that

$$1.2C + (2.3D - Aa)x + (3.4E - Ba)x^2 + (4.5F - Ca)x^3 + \&c. = 0.$$

Coefficients of each power of x being separately zero,

$$C = 0, \quad D = \frac{Aa}{2.3}, \quad E = \frac{Ba}{3.4}, \quad F = 0,$$

$$G = \frac{Da}{5.6} = \frac{Aa^2}{2.3.5.6}, \quad H = \frac{Ea}{6.7} = \frac{Ba^2}{3.4.6.7}, \&c.,$$

and the equation of the curve is

$$y = A \left\{ 1 + \frac{ax^3}{2.3} + \frac{(ax^3)^2}{2.3.5.6} + \frac{(ax^3)^3}{2.3.5.6.8.9} + \&c., \right\} \\ + Bx \left\{ 1 + \frac{ax^3}{3.4} + \frac{(ax^3)^2}{3.4.6.7} + \frac{(ax^3)^3}{3.4.6.7.9.10} + \&c. \right\}. \quad (34)$$

We may write it thus,

$y = A[1 + H_1(ax^3) + H_2(ax^3)^2 + \&c.] + Bx[1 + K_1(ax^3) + K_2(ax^3)^2 + \&c.]$,
and tabulate the logarithms of H_n and K_n for the first 14 terms of each series, as below.

TABLE II.

n	$\log H_n$	$\log K_n$	n	$\log H_n$	$\log K_n$
1	1.2218487	2.9208188	8	17.5936919	18.7931718
2	3.7447274	3.2975695	9	20.7473548	21.9146500
3	5.8873949	5.3433270	10	23.8078355	24.9461671
4	7.7668210	7.1502024	11	26.7841716	27.8961742
5	9.4446017	10.7699912	12	29.6838011	30.7716700
6	12.9588803	12.2359651	13	32.5129529	33.5785454
7	14.3356310	15.5713231	14	35.2769198	36.3218277

The two conditions in (33) suffice to determine the two constants A and B . In making the integrations we consider as before (ANALYST, September, 1878, p. 135), that y will be so small as to be insensible when x equals or exceeds a moderately large finite value x_1 , so that it is not necessary to integrate up to infinity. We have then

$$\int_{-x_1}^{x_1} y dx = 2Ax_1 \left[1 + \frac{1}{7}H_2(ax_1^3)^2 + \frac{1}{13}H_4(ax_1^3)^4 + \frac{1}{19}H_6(ax_1^3)^6 + \&c. \right] \\ + 2Bx_1^2 \left[\frac{1}{5}K_1(ax_1^3) + \frac{1}{11}K_3(ax_1^3)^3 + \frac{1}{17}K_5(ax_1^3)^5 + \&c. \right], \\ \int_{-x_1}^{x_1} x^2 y dx = 2Ax_1^3 \left[\frac{1}{3} + \frac{1}{5}H_2(ax_1^3)^2 + \frac{1}{15}H_4(ax_1^3)^4 + \&c. \right] \\ + 2Bx_1^4 \left[\frac{1}{7}K_1(ax_1^3) + \frac{1}{13}K_3(ax_1^3)^3 + \&c. \right].$$

Denoting the sums of the terms in these four series by S_1, S_2, S_3, S_4 , the conditions (33) give

$$2Ax_1S_1 + 2Bx_1^2S_2 = dx, \quad AS_3 + Bx_1S_4 = 0,$$

from which we get

$$A = \frac{S_4 dx}{2x_1(S_1S_4 - S_2S_3)}, \quad \frac{B}{A} = -\frac{S_3}{x_1S_4}.$$

Putting $ax_1^3 = c$, we have $x_1 = \sqrt[3]{c \div a}$, and therefore

$$A = \frac{S_4 dx}{2(S_1 S_4 - S_2 S_3)} \left(\frac{a}{c}\right)^{\frac{1}{3}}, \quad B = -\frac{S_3}{S_4} \left(\frac{a}{c}\right)^{\frac{1}{3}}. \quad (35)$$

Since $S_1, S_2, \&c.$, are functions of c , they are constant so long as c is constant, though a may diminish and x_1 increase. Thus it appears that A is equal to $a^{\frac{1}{3}} dx$ multiplied by a constant number, and $B \div A$ is equal to $a^{\frac{1}{3}}$ multiplied by another constant number. Assuming any sufficiently large and convenient value of c we can compute $S_1, S_2 \&c.$, and consequently find the numbers in question. With $\log c = 2.7$, I found

$S_1 = 19302.99, S_2 = 3333.390, S_3 = 17550.55, S_4 = 3030.792$, and (35) became

$$A = .36a^{\frac{1}{3}} dx, \quad B = -.729012Aa^{\frac{1}{3}}.$$

An identical result had been reached with $\log c = 2.6$, except that the last figure of the second coefficient was 1 instead of 2. The first coefficient .36 contains only two figures because $S_1 S_4$ and $S_2 S_3$ in (35) are so nearly equal that when computed by logarithms to seven figures their difference contains only two figures. The result obtained is accurate enough to show the nature of the curve. We see that A and B vary as $a^{\frac{1}{3}}$ and $a^{\frac{1}{3}}$, respectively. If however a is essentially negative, S_2, S_4 and c will be also negative, and (35) will give

$$A = -.36a^{\frac{1}{3}} dx.$$

We have then the general values

$$A = \pm .36a^{\frac{1}{3}} dx, \quad B = \mp .262444a^{\frac{1}{3}} dx, \quad (36)$$

the upper or lower sign being used according as a is essentially positive or negative. Substituting in (34), the equation becomes

$$y = \pm .36a^{\frac{1}{3}} dx [1 + H_1(ax^3) + H_2(ax^3)^2 + \&c.] \\ \mp .262444a^{\frac{1}{3}} dx [1 + K_1(ax^3) + K_2(ax^3)^2 + \&c.]. \quad (37)$$

The presence of the factor dx shows that y is an infinitesimal of the first order. Writing $Y = y \div dx$, we have

$$Y = \pm .36a^{\frac{1}{3}} [1 + H_1(ax^3) + H_2(ax^3)^2 + \&c.] \\ \mp .262444a^{\frac{1}{3}} x [1 + K_1(ax^3) + K_2(ax^3)^2 + \&c.], \quad (38)$$

where the ordinate Y is finite, and is proportional to, but numerically infinitely greater than the elementary area $Ydx = y$ which represents the coefficient of the resultant adjustment formula at the limit. Again, if we write $g = a(dx)^3$, and remember that $x = idx$, (28) and (37) give

$$g = \frac{2}{b_3(k+1)}, \quad y = \pm .36g^{\frac{1}{3}} [1 + H_1(g^{\frac{1}{3}}) + H_2(g^{\frac{1}{3}})^2 + \&c.] \\ \mp .262444g^{\frac{1}{3}} i [1 + K_1(g^{\frac{1}{3}}) + K_2(g^{\frac{1}{3}})^2 + \&c.], \quad (39)$$

which can be used as (32) was, to give approximately the coefficients l when k is a finite number.

We will take, however, $a = .01$, and compute a series of values of Y from (38), and arrange them in Table III. The curve which they follow shows points of inflexion at $x = 0$ and at $y = 0$ as (33) requires.

TABLE III. ($a = \frac{1}{100}$.)

x	Y	x	Y	x	Y
-24	.0591	-8	.0826	8	.0115
-22	.0826	-6	.1122	10	.0060
-20	.0370	-4	.1156	12	.0029
-18	-.0358	-2	.1007	14	.0014
-16	-.0856	0	.0776	16	.0006
-14	-.0839	2	.0541	18	.0002
-12	-.0370	4	.0346	20	.0000
-10	.0278	6	.0206	22	&c.

For positive values of x in the table, Y is positive and the axis of X appears as an asymptote to the curve. For negative values of x , the curve consists of an infinite series of undulations lying alternately above and below the axis, only two of which, however, appear in the table. Since the equation contains only one arbitrary constant a , and the area

$$\int_{-\infty}^{\infty} Y dx = \frac{1}{dx} \int_{-\infty}^{\infty} y dx$$

is always unity, the curves for different values of a are similar figures in the same sense as all probability curves are similar. The dimensions in the direction of x vary inversely as those in the direction of Y , and since the latter vary as A and therefore as the absolute values of $a^{\frac{1}{k}}$ or $g^{\frac{1}{k}}$, it follows from (28) that the x dimensions vary directly as the cube root of b_3 without regard to sign, and also as the cube root of $k+1$. Since the tabulated values of Y are the same as those of y in (39) would be if we took $g = .01$ and gave i the same values which x has in the table, it results that the table can be used to compute approximately the coefficients for a finite number of applications of a given formula. Take for instance eight applications of

$$u'_0 = \frac{1}{8}(-u_{-1} + 6u_0 + 3u_1).$$

Here we have $\lambda_0 = \frac{6}{8}$, $\lambda_1 = \frac{3}{8}$, $\lambda_{-1} = -\frac{1}{8}$, $b_0 = 1$, $k = 8$.

The place of the centre of parallel forces is

$$\frac{-1(-1) + 6 \times 0 + 3 \times 1}{-1 + 6 + 3} = \frac{1}{2},$$

or half way from u_0 to u_1 . Taking this centre as an origin, b_2 is

$$\frac{1}{8}(-\frac{9}{4} + \frac{6}{4} + \frac{3}{4}) = 0, \quad \text{and } b_3 \text{ is}$$

$$\frac{1}{8}(\frac{27}{8} - \frac{6}{8} + \frac{9}{8}) = \frac{3}{8},$$

and (39) gives

$$g = \frac{16}{27}.$$

To find any resultant coefficient, we take the corresponding Y from the table, and multiply it by the ratio

$$\left(\frac{1}{27} \div \frac{1}{100}\right)^{1/2} = 3.90.$$

Multiplying this ratio into

$$\dots -2, -1, 0, 1, 2, 3, 4, \dots$$

we get the abscissas

$$-23.4, -19.5, -15.6, -11.7, -7.8, -3.9, 0, 3.9, 7.8, 11.7, 15.6, 19.5.$$

The corresponding ordinates, by interpolation in Table III., are

$$.075, .018, -.081, -.030, .085, .115, .078, .035, .012, .003, .001, .000.$$

Multiplying these by 3.90, we have the approximate coefficients of the resultant formula

$$u_0^{viii} = -.31u_0 - .12u_1 + .33u_2 + .45u_3 + .30u_4 + .14u_5 + .05u_6 + .01 + \&c. \\ + .07u_{-1} + .29u_{-2} - \&c.$$

Since in the original formula the centre of parallel forces lay at an interval of $\frac{1}{2}$ to the right of u_0 , the resultant coefficient .30 corresponding to $x = 0$ in the table belongs to the term u_4 , which is at an interval of $k \times \frac{1}{2} = 4$ to the right of u_0 . The above series of coefficients bears only a general resemblance to that of the true formula for eight applications.

$$u_0^{viii} = -.11u_0 - .11u_1 + .19u_2 + .42u_3 + .35u_4 + .16u_5 + .04u_6 + \&c. \\ + .04u_{-1} + .02u_{-2} - .02u_{-3} + .01u_{-4} - \&c.,$$

as found by expansion of the polynomial

$$\left(\frac{-1+6z+3z^2}{8}\right)^8,$$

but the approximation would be closer if k were greater. This and other trials which have been made indicate that unsymmetrical limits are not approached as closely as symmetrical ones are, for given finite values of k .

When the constant a in (33) is negative, the effect will be to reverse the position of the limiting curve, so that the undulations will be on the right hand of the origin. The sign of a depends upon that of b_3 , as (28) shows. If the order of the coefficients in the given adjustment formula were reversed, so that for instance in the example just considered, we should have

$$u'_0 = \frac{1}{8}(3u_{-1} + 6u_0 - u_1),$$

the resultant coefficients would evidently be the same as before, but in the reverse order. We should still have $b_0 = 1$, and $b_2 = 0$ when referred to the centre of parallel forces, but b_3 would become

$$\frac{1}{8}\left(-\frac{3}{8} + \frac{6}{8} - \frac{27}{8}\right) = -\frac{3}{8},$$

being the same as before only with sign changed. Thus a and g would remain unaltered in absolute value, but with negative signs. The form of the

equation of the limiting curve remains the same, for with given absolute values of a and x in (38), it makes no difference with the value of Y whether we use the upper signs with positive a , or the lower signs with negative a , provided we reverse the sign of x .

Passing now to the third case under (28), where $n=3$, and the equation and its conditions are

$$\frac{d^3y}{dy^3} = axy, \quad \frac{1}{dx} \int_{-\infty}^{\infty} y dx = 1, \quad \int_{-\infty}^{\infty} x^2 y dx = 0, \quad \int_{-\infty}^{\infty} x^3 y dx = 0, \quad (40)$$

there is not much to be added to what was said in the discussion of symmetrical formulas in my article already referred to. (ANALYST, Sept. 1878, p. 134, and May, 1879, p. 72.) As in the case of $n=1$, so here, the resultant or limiting curve will always be symmetrical with respect to the ordinate through the centre of parallel forces, whether the original coefficients λ be so or not. For the λ do not enter into the general differential equation (28) of these curves, except as they affect the constant b_{n+1} . The constants of even order, b_2, b_4, b_6 &c., are not altered by a reversing of the sequence of the λ . Take for example the adjustment formula

$$u'_0 = \frac{1}{81}(-4u_{-1} + 30u_0 + 60u_1 - 5u_2),$$

and its reverse

$$u'_0 = \frac{1}{81}(-5u_{-1} + 60u_0 + 30u_1 - 4u_2).$$

The centre of parallel forces in the first one is $\frac{2}{3}$ of the way from u_0 to u_1 , and consequently in the second one it is $\frac{1}{3}$ of the way from u_0 to u_1 . Since the first one satisfies the three conditions $b_0 = 1, b_2 = 0, b_3 = 0$, with reference to its centre of forces, the second also satisfies them with reference to its centre of forces, and the value of b_4 is in both cases

$$\frac{1}{81}[-4(\frac{2}{3})^4 + 30(\frac{2}{3})^4 + 60(\frac{1}{3})^4 - 5(\frac{4}{3})^4] = -\frac{40}{81},$$

so that a is the same for both. The equations of the two limiting curves will thus be alike in all respects; but one curve is the reverse of the other, therefore the curve must be symmetrical on either side of the axis of Y or ordinate passing through the centre of parallel forces. Its equation is

$$y = .40845a^{\frac{1}{2}}dx \left[1 + \frac{ax^4}{2.3.4} + \frac{(ax^4)^2}{2.3.4.6.7.8} + \frac{(ax^4)^3}{2.3.4.6.7.8.10.11.12} + \&c. \right] \\ - .1380517a^{\frac{1}{2}}x^2dx \left[1 + \frac{ax^4}{4.5.6} + \frac{(ax^4)^2}{4.5.6.8.9.10} + \&c. \right]. \quad (41)$$

We might put this into forms analogous to (38) and (39), taking

$$Y = y \div dx, \quad x = idx, \quad g = a(dx)^4.$$

The value of g will be given by (28),

$$g = \frac{-6}{b_4(k+1)}.$$

The significance of b_4 here must not be confounded with that which it had in my previous article (Sept. 1878, pp. 134-7), being somewhat different, as already stated in connection with (14) and (21). If b_4 is positive, a will be negative, the factors $a^{\frac{1}{4}}$ and $a^{\frac{3}{4}}$ in (41) will be imaginary, and there will be no limiting curve. This corresponds to the case under the probability curve (31), where $\sqrt{-a}$, being evidently a factor in the expression for y , becomes imaginary when b_2 is negative, making a essentially positive. Taking $a = .001$, the values of $Y = y \div dx$ in (41) are given in the subjoined table, for values of x from 0 to 36. They are the same as what were somewhat improperly called values of y , at p. 137 of the previous article referred to.

TABLE IV. ($a = \frac{1}{1000}$.)

x	Y	x	Y	x	Y	x	Y
0	.0726	10	.0196	20	— .0068	30	.0010
1	.0719	11	.0183	21	— .0059	31	.0010
2	.0696	12	.0077	22	— .0048	32	.0009
3	.0659	13	.0029	23	— .0037	33	.0008
4	.0610	14	— .0009	24	— .0026	34	.0006
5	.0550	15	— .0038	25	— .0016	35	.0004
6	.0483	16	— .0058	26	— .0007	36	.0002
7	.0412	17	— .0070	27	— .0000	37	etc.
8	.0338	18	— .0075	28	.0005	38	
9	.0265	19	— .0074	29	.0008	39	

Now let us take as an illustration the formula just considered,

$$u'_0 = \frac{1}{81}(-4u_{-1} + 30u_0 + 60u_1 - 5u_2),$$

for which $b_4 = -\frac{4}{81}$. With six applications we have $k = 6$, and consequently $g = \frac{243}{140}$. To find the resultant coefficients approximately we take the corresponding values of Y from the table and multiply them by the ratio

$$(\frac{243}{140} \div \frac{1}{1000})^{\frac{1}{6}} = 6.455.$$

First multiplying this ratio into 0, 1, 2, 3 &c., we have the abscissas

$$0, \quad 6.5, \quad 12.9, \quad 19.4, \quad 25.8, \quad 32.3.$$

The corresponding ordinates, by interpolation in Table IV, are

$$.0726, \quad .0448, \quad .0033, \quad - .0072, \quad - .0009, \quad .0009.$$

Multiplying these by 6.45, we get the approximate coefficients of the resultant formula

$$u_0^{vi} = .468u_4 + .289(u_3 + u_5) + .021(u_2 + u_6) - .046(u_1 + u_7) \\ - .006(u_0 + u_8) + .006(u_{-1} + u_9) - \&c.$$

In the original formula the centre of parallel forces was at an interval of $\frac{2}{3}$ to the right of u_0 , so that the resultant coefficient .468 corresponding to $x = 0$ in the table belongs now to the term u_4 which is at the interval $k \times \frac{2}{3} = 4$ to the right of u_0 . The true resultant formula in this case is

$$u_0^{vi} = .503u_4 + .318u_5 - .020u_6 - .049u_7 + .014u_8 - .002u_9 + \&c. \\ + .259u_3 + .006u_2 - .030u_1 - .002u_0 + .002u_{-1} + \&c.,$$

as found by expansion of

$$\left(\frac{-4 + 30z + 60z^2 - 5z^3}{81} \right)^6.$$

It does not agree very closely with the other, because $k = 6$ is not a large number.

Reasoning from the analogy of the cases here considered, as well as from the case of $n = 5$, discussed at p. 138 of my former article, we can make some general inferences respecting all the curves included under (28) and (29).

Since they contain only one parameter a , and their area is constant, the curves for a given order n , are similar figures in the sense already stated. To make this more clear, we may remark that in the probability curve (31), if we write

$$\omega = hx \sqrt{2} = (-a)^{1/2}x,$$

the equation of the curve becomes

$$Y = (-a)^{1/2} [(2\pi)^{-1/2} e^{-1/2 \omega^2}],$$

which may be written

$$Y (-a)^{1/2} f_1(\omega).$$

So also in (38), if we take $\omega = a^{1/2}x$, the equation of the curve becomes

$$Y = a^{1/2} \left\{ \pm .36 \left(1 + \frac{\omega^3}{2.3} + \frac{\omega^6}{2.3.5.6} + \&c. \right) \right. \\ \left. \mp .262444 \omega \left(1 + \frac{\omega^3}{3.4} + \frac{\omega^6}{3.4.6.7} + \&c. \right) \right\},$$

which may be written

$$Y = a^{1/2} f_2(\omega).$$

Again in (41), putting $\omega = a^{1/2}x$, we have the equation

$$Y = a^{1/2} \left\{ .40845 \left(1 + \frac{\omega^4}{2.3.4} + \frac{\omega^8}{2.3.4.6.7.8} + \&c. \right) \right. \\ \left. - .1380517 \omega^2 \left(1 + \frac{\omega^4}{4.5.6} + \frac{\omega^8}{4.5.6.8.9.10} + \&c. \right) \right\},$$

which we may write

$$Y = a^{1/2} f_3(\omega).$$

Thus it appears that for any curve whose order is n we shall have

$$x = \omega \div (\pm a)^{\frac{1}{n+1}}, \quad Y = (\pm a)^{\frac{1}{n+1}} f_n(\omega),$$

so that ω and $f_n(\omega)$ are what x and Y become when $\pm a$ is unity. Regarding ω as the abscissa and $f_n(\omega)$ as the ordinate of an initial curve, and taking ω for instance equal to 0, 1, 2, 3, &c., in succession, we can compute the corresponding values of $f_n(\omega)$, and these values will be independent of any particular value which a afterward may receive.

To construct the curve for a given value of a , taking the abscissa ω and ordinate $f_n(\omega)$ of any point in the initial curve, we have simply to divide the one and multiply the other by the same quantity $(\pm a)^{\frac{1}{n+1}}$; in order to get the abscissa x and ordinate Y of a point in the curve required. Likewise if we have given the abscissa x' and ordinate Y' of a curve whose parameter is a_1 , and wish to find the corresponding abscissa and ordinate of another curve whose parameter is a_2 , we shall simply have to multiply the x' and divide the Y' by the common ratio

$$(a_1 \div a_2)^{\frac{1}{n+1}},$$

or what comes to the same thing, divide the x' and multiply the Y' by

$$(a_2 \div a_1)^{\frac{1}{n+1}},$$

and this is what we have practically done in various examples. Thus we see that for all this class of curves, the dimensions in the direction of x vary inversely as those in the direction of Y , and the Y dimensions vary directly as the $(n+1)$ th root of $\pm a$. But by (28), a varies inversely as b_{n+1} and $k+1$, consequently the x dimensions of the curves vary directly as the $(n+1)$ th root of $\pm b_{n+1}$ and also as the $(n+1)$ th root of $k+1$. If the order n is odd, the curve will be symmetrical on either side of the axis of Y , but if n is even, the curve will be unsymmetrical and its position will be direct or reversed, according to the sign of b_{n+1} . Since the equation of the curve always contains the $(n+1)$ th root of $\pm a$ as a factor, it may be imaginary if n is odd, but is always real if n is even. This seems to indicate that the absence of any limiting curve can occur only when the order n is odd, so that the curve, if there were one, would be symmetrical.

(To be concluded in No. 1, Vol. VII.)

ON A THEOREM OF LAMBERT'S.

BY PROF. ASAPH HALL, NAVAL OBSERVATORY, WASHINGTON, D. C.

JOHN HENRY LAMBERT was born at Mülhausen, in Alsace, Aug. 26, 1728, and died at Berlin, Sept. 25, 1777. Lambert's father, a French refugee, was a tailor, and on account of poverty and a large family of children, he was able to give his son but little assistance in his early studies. Lambert, however, by means of his own wonderful industry and application became one of the most accomplished philosophers of his time. He was the author of many scientific and mathematical papers, and among the most important of his works may be mentioned the following:

(1). *Photometria, sive de mensura et gradibus luminis, colorum, et umbræ.* Augsburg, 1760.

(2). *Insigniores Orbitæ Cometarum Proprietates.* Augsburg, 1761.

(3). *The establishment of the Astronomisches Jahrbuch.* Berlin, 1774.

In the second of these works, Lambert gives the elegant theorem now known by his name and in common use for computing the orbit of a comet. This theorem had indeed been proven for the parabola by Euler in 1740, but it had been forgotten, and was re-discovered by Lambert and extended by him to the ellipse and hyperbola. Another formula given by Lambert in this work has been used by Encke in his excellent discussion of Olber's Method for computing the orbit of a comet. This formula gives the relation between the chord, the sum of the radii vectores and the time, by means of a rapidly converging series; and by tabulating this series for different values of the variable, and by giving a proper form to the computation, Encke has reduced the solution of Lamberts Equation by trial to a sure methodical process.

In the *Memoirs of the Berlin Academy* 1771, p. 352, Lambert gives a theorem on the apparent orbit of a comet which is interesting. This theorem may be stated as follows:

"Take two not very distant points on the apparent orbit of a comet, and draw through these points an arc of a great circle. If the apparent orbit between these points turns from this great circle toward the places of the Sun for the intermediate times, the comet is farther from the Sun than the Earth is, and in the other case, it is nearer. If the comet moves in a great circle, the comet and the Earth are at the same distance from the Sun."

This theorem is indeed of no great practical importance, since as soon as we know the real orbit of a comet we can compute its apparent geopath, but it gives us in a very simple manner some idea of the dis-

a comet. In his *Mécanique Céleste*, Tome I., p. 208, Laplace has given an analytical proof of this theorem, but it is interesting to consider the simple geometrical method by which Lambert establishes it.

Let S be the place of the Sun, MQN a small arc of the orbit of the comet, and ABC the corresponding places of the Earth in its orbit. Let Q and B be the middle points of the arcs MN and AC and draw the lines as in the figure. The curvature of the arcs MN and AC being a result of gravity, we shall have very nearly

$$qQ : bB :: \frac{k}{SQ^2} : \frac{k}{SB^2},$$

k being a constant depending on the attractive force of the Sun. If we suppose the Earth and comet to describe the chords AbC , MqN with uniform velocities the apparent orbit of the comet will be a great circle of the sphere. In the case of the actual orbits when the Earth is at B , the comet is seen at Q , but in the case of the rectilinear orbits when the Earth is at b the comet is seen at q . If the lines BQ and bq are parallel, these apparent places are the same. Since the points b and q are in the lines SB and SQ , if we suppose BQ parallel to bq , we have

$$Bb : Qq :: SB : SQ,$$

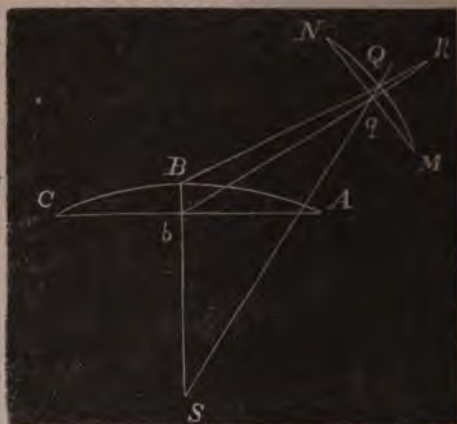
and hence, by means of the first proportion,

$$SQ = SB.$$

Thus when the right lines BQ and bq are parallel, the comet and the Earth are at the same distance from the Sun.

If we now suppose SQ greater than SB , then Bb will be greater than Qq , and the right lines BQ and bq intersect at a point R , beyond the comet. The angle SBR is smaller than the angle SbR , and the apparent place of the comet seen from the Earth is less removed from the Sun than when seen from the point b , or than in the case of its motion on a great circle.

Again, suppose SQ less than SB , then Bb is smaller than Qq , and the angle SBQ will be greater than the angle Sbq , or the apparent place of the comet will be more removed from the Sun than in the case of its motion on a great circle. If the comet moved exactly in the plane of the *Ecliptic*, its apparent orbit would be a great circle, but such an exceptional case has never been known to occur.



It will be remembered that a century ago the problem of computing the orbit of a comet was much discussed by mathematicians and astronomers, and almost every eminent man tried his hand on this problem. Kepler discussed this question, and assumed that the comets moved in right lines; and as he never wanted for an hypothesis, he also assumed that these erratic bodies were generated in the celestial spaces, and then were destroyed in some mysterious manner. Newton's discovery of the law of gravitation showed that the rectilinear motion of a comet was impossible, but this motion continued to be assumed for small portions of the orbit, and was used by Newton himself as a first approximation. It is worthy of notice that in the memoir referred to above, Lambert divides the chord MN at the point q in such a manner that the segments of this chord are proportional to the intervals of time between the observations. This assumption was afterward adopted by Olbers in his method of computing the orbit of a comet, a method which is now almost the only one in use.

AN ACCOUNT OF CAUCHY'S "CALCUL DES RESIDUS."

BY CHAS. H. KUMMELL, ASSISTANT U. S. LAKE SURVEY, DETROIT, MICH.

(Continued from page 46.)

§ 6.

THERE are other applications, more or less direct, of which a few examples may be given.

EX. 6. Assume in (6) $f(z) = e^{-cz^2} = e^{-c^2(x+yi)^2} = e^{-c^2(x^2-y^2)} (\cos 2c^2xy - i \sin 2c^2xy)$.

Since $e^{-c^2z^2} = \infty$ if $z = 0 \pm \infty i$ there would be a point of discontinuity if 0 was within the x -integration and $\pm \infty$ the upper or lower limit of the y -integration. But the correction would be 0 nevertheless for we have by (10'')

$$A_{0 \pm \infty i} = \pi i \left[\frac{z \mp \infty i}{e^{c^2z^2}} \right]_{z=\pm \infty i} = \pi i \left[\frac{1}{2c^2z e^{c^2z^2}} \right]_{z=\pm \infty i} = 0.$$

There is consequently no correction, and we have by (6)

$$\begin{aligned} & \int_{x_0}^{x_n} dx [e^{-c^2(x^2-y^2)} (\cos 2c^2xy_n - i \sin 2c^2xy_n) - e^{-c^2(x^2-y_0^2)} (\cos 2c^2xy_0 - i \sin 2c^2xy_0)] \\ &= i \int_{y_0}^{y_n} dy [e^{-c^2(x_n^2-y^2)} (\cos 2c^2x_ny - i \sin 2c^2x_ny) - e^{-c^2(x_0^2-y^2)} (\cos 2c^2x_0y - i \sin 2c^2x_0y)]. \end{aligned} \tag{g}$$

Let $x_0 = 0 = y_0$, then

$$\int_0^{x_n} dx \left[e^{-c(x^2 - y_n^2)} (\cos 2c^2 xy_n - i \sin 2c^2 xy_n) - e^{-cx^2} \right] \\ = i \int_0^{y_n} dy \left[e^{-c(x_n^2 - y^2)} (\cos 2c^2 x_n y - i \sin 2c^2 x_n y) - e^{-cy^2} \right], \quad (g')$$

which gives the following two relations:

$$\int_0^{x_n} dx \left[e^{-c(x^2 - y_n^2)} (\cos 2c^2 xy_n - e^{-cx^2}) \right] = \int_0^{y_n} dy e^{-c(x_n^2 - y^2)} \sin 2c^2 x_n y, \quad (g'_1)$$

$$\int_0^{x_n} dx e^{-c(x^2 - y_n^2)} \sin 2c^2 xy_n = - \int_0^{y_n} dy e^{cy^2} \left[e^{-cx_n^2} \cos 2c^2 x_n y - 1 \right]. \quad (g'_2)$$

In this let $x_n = \infty$, then

$$\int_0^{\infty} dx \left[e^{-c(x^2 - y_n^2)} \cos 2c^2 xy_n - e^{-cx^2} \right] = 0, \quad (g''_1)$$

$$\int_0^{\infty} dx e^{-c(x^2 - y_n^2)} \sin 2c^2 xy_n = \int_0^{y_n} dy e^{cy^2}. \quad (g''_2)$$

We have $\int_0^{\infty} dx e^{-cx^2} = \frac{\pi^{1/2}}{2c}$ a well known integral, therefore from (g''_1)

$$\int_0^{\infty} dx e^{-cx^2} \cos 2c^2 xy_n = \frac{\pi^{1/2}}{2c} e^{-cy_n^2}. \quad (g''_3)$$

Ex. 7. Let $f(z) = \frac{\varphi(z)}{e^z \pm e^{-z}}$, then we have in (6)

$$\int_{x_0}^{x_n} dx \left[\frac{\varphi(x + y_n i)}{e^{x+y_n i} \pm e^{-x-y_n i}} - \frac{\varphi(x + y_0 i)}{e^{x+y_0 i} \pm e^{-x-y_0 i}} \right] = i \int_{y_0}^{y_n} dy \left[\frac{\varphi(x_n + yi)}{e^{x_n+yi} \pm e^{-x_n-yi}} \right. \\ \left. - \frac{\varphi(x_0 + yi)}{e^{x_0+yi} \pm e^{-x_0-yi}} \right] - A. \quad (h)$$

Let $x_n = \infty$; $x_0 = -\infty$, then $\frac{\varphi(\pm\infty + yi)}{e^{\pm\infty+yi} \pm e^{\mp\infty-yi}} = 0$ if $\varphi(\pm\infty + yi) = 0$ or a finite quantity, or an infinity of a lower order than $e^{\pm\infty}$, and we have from (h)

$$\int_{-\infty}^{\infty} dx \left[\frac{\varphi(x + y_n i)}{e^{x+y_n i} \pm e^{-x-y_n i}} - \frac{\varphi(x + y_0 i)}{e^{x+y_0 i} \pm e^{-x-y_0 i}} \right] = -A. \quad (h')$$

Let us suppose $\varphi(z)$ continuous (if not, we may subsequently correct for it), then the points of discontinuity are given by the equation

$$e^z \pm e^{-z} = 0. \quad (i)$$

1. For the upper sign the roots are

$$z_{\pm\frac{1}{2}} = \pm \frac{\pi}{2} i; \quad z_{\pm\frac{3}{2}} = \pm \frac{3\pi}{2} i; \quad z_{\pm\frac{5}{2}} = \pm \frac{5\pi}{2} i; \dots z_{\pm\frac{2n-1}{2}} = \pm \frac{(2n-1)\pi}{2} i; \dots \quad (k)$$

The correction for discontinuity due the general value of these roots, as formula (10) gives it, is

$$\begin{aligned} d_{\pm \frac{2n-1}{2} \pi i} &= 2\pi i \varphi\left(\pm \frac{2n-1}{2} \pi i\right) \left[\frac{z \mp \frac{1}{2}(2n-1)\pi i}{e^z + e^{-z}} \right]_{z=\pm \frac{2n-1}{2} \pi i} \\ &= 2\pi i \varphi\left(\pm \frac{2n-1}{2} \pi i\right) \cdot \frac{1}{e^{\pm \frac{1}{2}(2n-1)\pi i} - e^{\mp \frac{1}{2}(2n-1)\pi i}} \\ &= \mp \pi e^{n\pi i} \varphi\left[\pm \frac{1}{2}(2n-1)\pi i\right]. \end{aligned} \quad (l)$$

By virtue of (10₃') one half this correction is taken if the discontinuous element coincides with one of the limits of the y -integration.

Let $y_n = n\pi$ and $y_0 = 0$, then by (h')

$$\int_{-\infty}^{\infty} dx \left[\frac{\varphi(x+n\pi i)}{e^{x+n\pi i} + e^{-x-n\pi i}} - \frac{\varphi(x)}{e^x + e^{-x}} \right] = \pi \sum_1^n \left\{ e^{s\pi i} \varphi\left(\frac{2s-1}{2} \pi i\right) \right\},$$

or, since $e^{n\pi i} = e^{-n\pi i} = (-1)^n$,

$$\int_{-\infty}^{\infty} dx \frac{e^{n\pi i} \varphi(x+n\pi i) - \varphi(x)}{e^x + e^{-x}} = \pi \sum_1^n \left\{ e^{s\pi i} \varphi\left(\frac{2s-1}{2} \pi i\right) \right\}. \quad (m)$$

Let $\varphi(z) = e^{-c^2 z^2}$ then since $\frac{e^{-c^2(\pm \infty + yi)^2}}{e^{\pm \infty + yi} + e^{\mp \infty - yi}} = 0$ we can use (m). We have

$$\int_{-\infty}^{\infty} dx \frac{e^{n\pi i - c^2(x+n\pi i)^2} - e^{-c^2 x^2}}{e^x + e^{-x}} = \pi \sum_1^n \left\{ e^{s\pi i + c^2 \left(\frac{2s-1}{2}\right)^2 \pi^2} \right\}. \quad (n)$$

Separating the possible from the impossible, we have

$$\int_{-\infty}^{\infty} \frac{e^{-c^2 x^2} dx}{e^x + e^{-x}} \left[e^{c^2 n^2 k^2} \cos[n\pi(1-2c^2 x)] - 1 \right] = \pi \sum_1^n \left\{ e^{s\pi i + c^2 \left(\frac{2s-1}{2}\right)^2 \pi^2} \right\}, (n'_1)$$

$$\int_{-\infty}^{\infty} \frac{e^{-c^2 x^2} dx}{e^x + e^{-x}} e^{c^2 n^2 k^2} \sin[n\pi(1-2c^2 x)] = 0, \quad (n'_2)$$

or let $\varphi(z) = e^{bz}$ then $\frac{e^{b(\pm \infty + yi)}}{e^{\pm \infty + yi} + e^{\mp \infty - yi}} = 0$, and from (m)

$$\int_{-\infty}^{\infty} \frac{dx}{e^x + e^{-x}} \left[e^{nki + b(x+nki)i} - e^{bxi} \right] = \pi \sum_1^n \left\{ e^{s\pi i - \frac{2s-1}{2} bk} \right\}. \quad (o)$$

The series on the right is a geometrical series of n terms. We have then

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{dx}{e^x + e^{-x}} e^{bxi} &= \frac{\pi}{e^{nki - bnk}} \cdot \frac{e^{(n+1)ki - \frac{1}{2}(2n+1)bk} - e^{ki - \frac{1}{2}bk}}{e^{k(i-b)} - 1} \\ &= \frac{\pi e^{-\frac{1}{2}bk}}{1 + e^{-bk}} = \frac{\pi}{e^{\frac{1}{2}bk} + e^{-\frac{1}{2}bk}}. \end{aligned} \quad (o')$$

$$\int_{-\infty}^{\infty} \frac{dx}{e^x + e^{-x}} \cos bx = \frac{\pi}{e^{\frac{1}{2}bk} + e^{-\frac{1}{2}bk}} (o'_1); \quad \int_{-\infty}^{\infty} \frac{dx}{e^x + e^{-x}} \sin bx = 0. \quad (o'_2)$$

It is also evident that

* For want of sorts, k is here, and throughout the rest of this article, written instead of the Agate Greek π .—Compositor.

** The same result would be found if in (o) we place $n=1$. It should also be noted that this form satisfies (11) and could be calculated by (12).

$$\int_0^{\infty} \frac{dx}{e^x + e^{-x}} \cos bx = \frac{\pi}{2(e^{\frac{1}{2}bk} + e^{-\frac{1}{2}bk})}. \quad (o'_2)$$

Because $\frac{x^p}{e^x + e^{-x}} \Big|_{x=\pm\infty} = 0$, any algebraical form may be chosen for $\varphi(z)$ in (m).

2. For the lower sign the roots of (i) are:

$$z_0 = 0; z_{+1} = \pm \pi i; z_{+2} = 2\pi i \dots z_{+n} = \pm n\pi i \dots \quad (p)$$

If we put $y_n = 2\pi$ and $y_0 = 0$, then $z_n = n\pi$ and $z_0 = 0$ coincide with these limits and formulæ (10'_2) and (10'_3) respectively are to be used for these roots, which is one half of what (10) gives. For $z_s = s\pi i$ we have the whole correction by (10),

$$A_{s,ki} = 2\pi i \varphi(s\pi i) \left[\frac{z - s\pi i}{e^z - e^{-z}} \right]_{z=ski} = 2\pi i \varphi(s\pi i) \frac{1}{e^{ski} + e^{-ski}} = \pi i e^{ski} \varphi(s\pi i). \quad (q)$$

We have then by (h')

$$\begin{aligned} \int_{-\infty}^{\infty} dx \left[\frac{\varphi(x + n\pi i)}{e^{x+nki} - e^{-x-nki}} - \frac{\varphi(x)}{e^x - e^{-x}} \right] &= -\pi i \left\{ \frac{1}{2} \varphi(0) + \sum_1^{n-1} \left[e^{ski} \varphi(s\pi i) \right] \right. \\ &\quad \left. + \frac{1}{2} e^{nki} \varphi(n\pi i) \right\}, \text{ or, since } e^{nki} = e^{-nki}, \\ \int_{-\infty}^{\infty} \frac{dx}{e^x - e^{-x}} \left[e^{nki} \varphi(x + n\pi i) - \varphi(x) \right] &= -\pi i \left\{ \frac{1}{2} \varphi(0) + \sum_1^{n-1} \left[e^{ski} \varphi(s\pi i) \right] \right. \\ &\quad \left. + \frac{1}{2} e^{nki} \varphi(n\pi i) \right\}. \quad (r) \end{aligned}$$

Let $\varphi(z) = e^{-\alpha z^2}$, then as we have already shown the condition for (h') to be true is satisfied and we have

$$\int_{-\infty}^{\infty} \frac{dx}{e^x - e^{-x}} [e^{nki - \alpha(x+nki)^2} - e^{-\alpha x^2}] = -\pi i \left[\frac{1}{2} + \sum_1^{n-1} (e^{ski + \alpha s^2 k^2}) + \frac{1}{2} e^{nki + \alpha n^2 k^2} \right]. \quad (s)$$

Separating the possible from the impossible, we obtain :

$$\int_{-\infty}^{\infty} \frac{dx e^{-\alpha x^2}}{e^x - e^{-x}} \{ e^{\frac{1}{2}n^2 k^2} \cos [n\pi(1 - 2\alpha^2 x)] - 1 \} = 0, \quad (s'_1)$$

$$\int_{-\infty}^{\infty} \frac{dx e^{-\alpha x^2}}{e^x - e^{-x}} \sin [n\pi(1 - 2\alpha^2 x)] = -\pi e^{-\frac{1}{2}n^2 k^2} \left[\frac{1}{2} + \sum_1^{n-1} (e^{ski + \frac{1}{2} s^2 k^2}) + \frac{1}{2} e^{nki + \frac{1}{2} n^2 k^2} \right]. \quad (s'_2)$$

Also let $\varphi(z) = e^{bzi}$, then in (r)

$$\int_{-\infty}^{\infty} \frac{dx}{e^x - e^{-x}} [e^{nki + b(x+nki)i} - e^{bxi}] = -\pi i \left[\frac{1}{2} + \sum_1^{n-1} (e^{ski - sbk}) + \frac{1}{2} e^{nki - nbk} \right]. \quad (t)$$

Place $n = 1$, then

$$\int_{-\infty}^{\infty} \frac{e^{bxi} dx}{e^x - e^{-x}} = \frac{\pi i}{e^{-bk} + 1} \left(\frac{1}{2} - \frac{1}{2} e^{-bk} \right) \quad (t'), \text{ and } \int_{-\infty}^{\infty} \frac{\cos bx dx}{e^x - e^{-x}} = 0, \quad (t'_1)$$

$$\int_{-\infty}^{\infty} \frac{\sin bx dx}{e^x - e^{-x}} = 2 \int_0^{\infty} \frac{\sin bx dx}{e^x - e^{-x}} = \frac{\pi}{2} \cdot \frac{e^{bk} - 1}{e^{bk} + 1}. \quad (t'_2)$$

This may suffice to show at least a glimpse of the method for the form $z = x + yi$. Other forms may be developed in a similar manner which give rise to new relations. The method requires extreme caution with regard to the conditions to be satisfied by the element function.

[illegible]

Differentiating the assumed values in (1), we have

$$\left. \begin{aligned} \frac{dx_1}{dt} &= -\frac{m}{a_1} A_{1,1} e^{-\frac{mt}{a_1}} \\ \frac{dx_2}{dt} &= -\frac{m}{a_1} A_{1,2} e^{-\frac{mt}{a_1}} - \frac{m}{a_2} A_{2,2} e^{-\frac{mt}{a_2}} \\ \frac{dx_3}{dt} &= -\frac{m}{a_1} A_{1,3} e^{-\frac{mt}{a_1}} - \frac{m}{a_2} A_{2,3} e^{-\frac{mt}{a_2}} - \frac{m}{a_3} A_{3,3} e^{-\frac{mt}{a_3}} \\ &\vdots \\ \frac{dx_n}{dt} &= -\frac{m}{a_1} A_{1,n} e^{-\frac{mt}{a_1}} - \frac{m}{a_2} A_{2,n} e^{-\frac{mt}{a_2}} - \dots - \frac{m}{a_n} A_{n,n} e^{-\frac{mt}{a_n}} \end{aligned} \right\} \cdot (4)$$

If now we substitute the assumed values (1) in equations (2), and compare the results with equations (4), the first of these is verified; and, by equating the coefficients of the exponentials in the n th equations, we have

$$\left. \begin{aligned} -\frac{m}{a_1} A_{1,n} &= \frac{m}{a_{n-1}} A_{1,n-1} - \frac{m}{a_n} A_{1,n} \\ -\frac{m}{a_2} A_{2,n} &= \frac{m}{a_{n-1}} A_{2,n-1} - \frac{m}{a_n} A_{2,n} \\ &\vdots \\ -\frac{m}{a_r} A_{r,n} &= \frac{m}{a_{n-1}} A_{r,n-1} - \frac{m}{a_n} A_{r,n} \\ &\vdots \\ -\frac{m}{a_n} A_{n,n} &= -\frac{m}{a_n} A_{n,n} \end{aligned} \right\} \quad (5)$$

the last of which is an identity. The general equation in (5) reduces to

$$A_{r, n} = \frac{a_r a_n}{a_{n-1} (a_r - a_n)} A_{r, n-1},$$

by substituting the corresponding expression for $A_{r,n-1}$, we obtain

$$A_{r,n} = \frac{a_r^2 a_n}{a_{n-2} (a_r - a_{n-1}) (a_r - a_n)} A_{r,n-2},$$

etc. etc.; the $(n-r)$ th equation being

$$A_{r,n} = \frac{a_r^{n-r-1} a^n A_{r,r}}{(a_r - a_{r+1})(a_r - a_{r+2}) \dots (a_r - a_n)}. \quad (6)$$

By means of this relation and the group (3) the values of the constants

may be determined. Thus from (3) we have $A_{1,1} = a$, hence from (6)

$$A_{1,n} = \frac{a_1^{n-2} a_n a}{(a_1 - a_2)(a_1 - a_3) \dots (a_1 - a_n)} \quad (7)$$

Again, making $n = 2$ in (7),

$$A_{1,2} = \frac{a_2 a}{a_1 - a_2},$$

substituting in the second of group (3), we derive

$$A_{2,2} = \frac{a_2 a}{a_2 - a_1},$$

whence, from (6),

$$A_{2,n} = \frac{a_2^{n-2} a_n a}{(a_2 - a_1)(a_2 - a_3) \dots (a_2 - a_n)} \quad (8)$$

The form of equations (7) and (8) suggests that in general

$$A_{r,n} = \frac{a_r^{n-2} a_n a}{(a_r - a_1)(a_r - a_2) \dots (a_r - a_n)} \quad (9)$$

(the subscripts of the second letter in the binomials taking all values from 1 to n except r), and this expression will be demonstrated when it is shown that it satisfies the n th equation of group (3); that is, the equation

$$A_{1,n} + A_{2,n} + A_{3,n} + \dots + A_{n,n} = 0.$$

Making the substitutions, dividing by $a_n a$, and multiplying by the expression

$$(a_1 - a_2)(a_1 - a_3) \dots (a_1 - a_n) \times (a_2 - a_3)(a_2 - a_4) \dots (a_2 - a_n) \times (a_3 - a_4) \dots (a_{n-1} - a_n),$$

the equation to be verified becomes

$$\left. \begin{aligned} &+ a_1^{n-2} (a_2 - a_3) \dots (a_2 - a_n) \cdot (a_3 - a_4) \dots (a_3 - a_n) \dots (a_{n-1} - a_n) \\ &- a_2^{n-2} (a_1 - a_3)(a_1 - a_4) \dots (a_1 - a_n) \cdot (a_3 - a_4) \dots (a_{n-1} - a_n) \\ &+ a_3^{n-2} (a_1 - a_2)(a_1 - a_4) \dots (a_1 - a_n) \dots \\ &\dots \dots \dots \end{aligned} \right\} = 0,$$

in which the coefficient of a_r^{n-1} is the product of all the differences which do not involve a_r . Using the determinant expressions for these products, the equation takes the form

$$a_1^{n-2} \begin{vmatrix} 1, & 1, & \dots, & 1 \\ a_n, & a_{n-1}, & \dots, & a_2 \\ a_n^2, & a_{n-1}^2, & \dots, & a_2^2 \\ \vdots & \vdots & & \vdots \\ a_n^{n-2}, & \dots, & a_2^{n-2} \end{vmatrix} - a_2^{n-2} \begin{vmatrix} 1, & 1, & \dots, & 1, & 1 \\ a_n, & \dots, & a_3, & a_1 \\ a_n^2, & \dots, & a_3^2, & a_1^2 \\ \vdots & & \vdots \\ a_n^{n-2}, & \dots, & a_1^{n-2} \end{vmatrix} + a_3^{n-2} \begin{vmatrix} 1, & 1, & \dots \\ a_n, & \dots \\ a_n^2, & \dots \text{etc.} \\ \vdots & \\ a_n^{n-2}, & \dots \end{vmatrix} = 0.$$

Now the right hand member of this equation is equivalent to the determinant

$$\begin{vmatrix} 1 & , & 1 & , \dots & 1 & , & 1 \\ a_n & , & a_{n-1} & , \dots & a_2 & , & a_1 \\ a_n^2 & , & a_{n-1}^2 & , \dots & a_2^2 & , & a_1^2 \\ \vdots & & \vdots & & \vdots & & \vdots \\ \vdots & & \vdots & & \vdots & & \vdots \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_n^{n-2} & , & a_{n-1}^{n-2} & , \dots & a_2^{n-2} & , & a_1^{n-2} \\ a_n^{n-2} & , & a_{n-1}^{n-2} & , \dots & a_2^{n-2} & , & a_1^{n-2} \end{vmatrix}$$

which is identically = 0, because the last two rows are identical. Hence the solution of the problem is given by (1), the coefficients being determined by (9).

It is a remarkable result of this solution that the value of the fraction $x_n \div a_n$, or ratio of the wine to the whole contents of the last cask, is wholly independent of the order of the casks.

NOTE ON A FORM OF THE EQ. OF THE TANG. TO A CONIC.

BY PROF. W. W. HENDRICKSON, U. S. NAVAL ACADEMY.

THE envelope of any line $Aa^2 + Ba + C = 0$, where a is an arbitrary parameter, and A , B and C are functions of x and y , is $B^2 = 4AC$; it follows that if the equation to any conic be put in the form $B^2 = 4AC$, the equation to the tangent may be written either as $Aa^2 + Ba + C = 0$ or $Ca^2 + Ba + A = 0$.

The equations to the conic sections and their tangents may be written as follows:

$$\begin{array}{lll} \text{Circle,} & y^2 = (a+x)(a-x), & \text{tangent,} & (a-x)a^2 + 2ya + a + x = 0. \\ \text{Parabola,} & y^2 = 4ax, & \text{"} & a^2 + ya + x = 0. \\ \text{Ellipse,} & a^2y^2 = b^2(a+x)(a-x), & \text{"} & b(a-x)a^2 + 2aya + b(a+x) = 0. \\ \text{Hyperb.,} & a^2b^2 = (bx+ay)(bx-ay), & \text{"} & (bx-ay)a^2 + 2aba + bx + ay = 0. \end{array}$$

This form of the equation to the tangent will be found useful in certain problems of Analytical Geometry; thus, suppose it required to find the tangents to the conic

$$y^2 - 2xy + 2x^2 + 8y + 6x + 49 = 0,$$

which pass through $(-14, -22)$. The equation may be put in the form $(y - x + 4)^2 + (x + 11)(x + 3) = 0$, and the equation to the tangent is $(x + 11)a^2 + 2(y - x + 4)a - x - 3 = 0$; substituting the co-ordinates of the given point, we have $3a^2 + 8a - 11 = 0$, whence $a = 1$ or $-\frac{11}{3}$, and the equations to the required tangents are $y - x + 8 = 0$, and $89x - 33y + 520 = 0$.

PROBLEM RELATIVE TO THE MOVE OF THE KNIGHT AT THE GAME OF CHESS.

BY T. P. STOWELL, ROCHESTER, NEW YORK.

"The Knight being placed on any given square of the chess board, it is required, at sixty-three successive moves, to cause it to move over the remaining sixty-three squares."

This curious question is perhaps well known to most of those who are familiar with the game of chess. It was considered of sufficient importance in the time of Euler for him to furnish an extended article on it to the St. Petersburg Academy.

I propose to show that the Knight may be placed on any square of the chess board and, by sixty-three or sixty-four moves, may land on any other designated square of the board, moving in its course over every square on the chess board.

There is a great variety of courses which the Knight may take, starting from any particular square, and move over all the remaining sixty-three squares at sixty-three successive moves. If we can once obtain what may be termed a reëntering course, such as is shown in figures 1 and 2, the last move being but one move from the starting point, it is plain that we may from this course obtain as many courses as we please.

Suppose, for instance, we wish to have the Knight end his course on No. 28, Fig. 1. All we have to do is to reverse the numbers, putting 64 in place of 28, 63 for 29, 62 for 30, 61 for 31, &c., until we arrive at 64, which will be 28; then, afterward, from 27 down to 1 the moves will be unchanged.

In the same manner, if we wished the Knight to end his course on No. 56, reversing the numbers to 64, then from 55 to 1 the course is unchanged. Pursuing the same course, we may, in a very few transformations, have the Knight end his course on any one of the squares represented by even Nos.

FIG. 1.

52	7	24	31	50	35	12	33
25	30	51	8	11	32	49	36
6	53	10	23	48	13	34	59
29	26	47	54	9	60	37	14
46	5	28	41	22	15	58	61
27	2	45	16	55	42	21	38
4	17	64	43	40	19	62	57
1	44	3	18	63	56	39	20

FIG. 2.

45	32	9	18	47	30	7	20
10	17	46	31	8	19	48	29
33	44	55	58	51	62	21	6
16	11	52	61	54	57	28	49
43	34	59	56	63	50	5	22
12	15	64	53	60	25	38	27
35	42	13	2	37	40	23	4
14	1	36	41	24	3	26	39

It will be observed that, diagonally across the board in either direction, the rows of figures are alternately odd and even, and, from the nature of the moves of the Knight, it would be impossible for it, in sixty-three moves, to commence and land on any square in the odd rows. When a new course is found in the way proposed another new course may be obtained from this one, and, commencing with Fig. 1, in the corner, the Knight may be made to move successively over all the remaining sixty-three squares and end his journey on any one of the squares in the alternate rows represented by even numbers. In Fig. 2, the Knight is supposed to commence on another square. Here, as in the 1st figure, each alternate row, diagonally, has odd and even numbers and the Knight can be made to end his journey on any square represented by even numbers, the same as in Fig. 1, in sixty-three moves.

It may further be observed that, in either Fig. 1 or Fig. 2, if we wish to have the Knight *end on any square on the board*, regardless of where the starting point is, it can be done, by one transformation, in this manner:

Suppose in Fig. 2 we want the Knight to end his course on No. 29. Instead of 29, 28, 27, &c., write 64, 63, 62, &c., down to No. 1 on the board. The whole course will be 29 to 1, written 64 to 36, and 64 to 30 written 35 to 1. The starting point of this new course will be 30, ending at 29. And in this way courses may be found, without number, beginning and ending within one move of each other; each course being a reëntering course.

The Knight, from the nature of his moves, cannot, at sixty-three moves, begin and end his course on any of the squares represented by odd or even numbers; if he commences on an odd row he must end on an even row, and vice versa. So that by placing the Knight on any square represented by an odd number, he may finish his course ending on either of the thirty-two even numbers, and vice versa; and again these courses, in each case, may be varied in many ways. The Knight may thus in reëntering courses begin or end on any particular square that may be named. The recreation is a pleasant one to practice, and upon once getting started is quite easy to perform.

Euler's article was mainly to show how to obtain by trial and changes what I have here termed a reëntering course.

CORRESPONDENCE.—EDITOR OF ANALYST: Problem 258, published in ANALYST, No. 2, Vol. VI, probably admits of no purely mathematical solution, as several numbers of the ANALYST have since appeared and no solution has been given.

The only attempt at an exact determination of the arc required, that I know of, is by Prof. Phillips of the Polytechnic school at Paris; the result

of which I give in the following extract from Saunier's "Traité D' Horlogerie Moderne."

"M. Phillips a démontré, dans un Mémoire présenté à l' Académie des sciences, que l' étendue des oscillations du balancier, ou le défaut d' équilibre s'apercevra le moins au réglage dans les différentes positions inclinées, est de $439^{\circ}28'$, soit en nombre rond 440° , un peu moins de 1 tour $\frac{1}{4}$."

Whether this demonstration was analytical or experimental I cannot say, not having seen the Memoir referred to. In the course of twenty years experience in rating and adjusting watches and pocket chronometers, many opportunities have occurred to verify Prof. Phillips' conclusion.

The arc mentioned seems to be independent of the amount of error of poise, as experiments with a balance only slightly out of poise, and with the same balance after removing a whole screw from one portion of its rim will show.

In a marine chronometer, the balance vibrates in a horizontal plane, and is not so susceptible to errors of poise, but if the balance is made to vibrate as near to this arc as possible, any induced polarity, caused by the magnetism of iron ships, will not cause so much irregularity in the rate as it would if the balance had either a larger or smaller arc of vibration.

J. M. ARNOLD.

Boston, Mass.

FURTHER REMARKS ON THE SOLUTION OF PROBLEM 260, BY THE EDITOR.—The solution of problem 260, given at pp. 121–122, as corrected by the Note at p. 151, is, we trust, a sufficient discussion of that question, as announced at p. 95. It will be seen, however, that equations (2) and (3) as written at the top of p. 122 (with a slight alteration of signs), are adapted to the solution of a different question, viz., Prob. 8, ANALYST, Vol. I, page 35.

Writing $R + r$ instead of $R - r$ in (2) and changing all the signs in (3), equation (1) retaining the negative sign, as written at p. 121, we have, for the equivalent of (6) on p. 122,

$$\left(\frac{d\varphi}{dt}\right)^2 = \frac{10}{7} \frac{g}{R+r} (1 - \cos \varphi); \quad (6)$$

and writing $R' = 0$ in (3) it becomes

$$\left(\frac{d\varphi}{dt}\right)^2 = \frac{g}{R+r} \cos \varphi. \quad (7)$$

Equating the right hand members of (6) and (7) and reducing, we get $\cos \varphi = \frac{10}{7}$. (See foot-note, p. 35, ANAL. Vol. I.)

If, instead of quitting the sphere when $R' = 0$, the rolling sphere is constrained to move on the surface of the fixed sphere until $\cos \varphi = 0$, then will the solution of (6) give the time t , in which the rolling sphere will descend from the summit of the fixed sphere over one fourth of its circumf., viz.;

$$t = \frac{1}{\sqrt{a}} \left[\frac{2}{\sqrt{2}} \log \frac{\sqrt{1+\cos \varphi} - \sqrt{2}}{\sqrt{1-\cos \varphi}} \right]_0^{\frac{\pi}{2}},$$

where $a = 10g \div 7(R+r)$.

This value of t , is not $[2 \div \sqrt{2a}] \log(1 - \sqrt{2})$, as indicated at p. 122, $[2 \div \sqrt{2a}] \log(1 - \sqrt{2})$ being the value of the integral at the superior limit only. At the inferior limit we have

$$\frac{\sqrt{1+\cos \varphi} - \sqrt{2}}{\sqrt{1-\cos \varphi}} = \frac{0}{0}, \text{ a vanishing fraction, whence}$$

$$\frac{-\sqrt{1-\cos \varphi}}{\sqrt{1+\cos \varphi}} = \frac{-0}{\sqrt{2}} = -0.$$

$$\therefore t = \frac{2}{\sqrt{2a}} \log \frac{1-\sqrt{2}}{-0} = \frac{2}{\sqrt{2a}} \log \frac{\sqrt{2}-1}{0} = \infty.$$

That is, the time required, to roll from the *summit* of the fixed sphere over one fourth of its circumference, is infinite.

If, instead of starting from the summit of the fixed sphere, the rolling sphere should start from a point, say $1''$ from the summit, the time, obviously, would be finite. In that case the time would be, as calculated by Mr. Kummell, $t = \sqrt{2+a} \times 12.7418368$.

SOLUTIONS OF PROBLEMS IN NUMBER FIVE.

SOLUTIONS of problems in No. 5 have been received as follows:

From R. J. Adcock, 277; Alex. S. Christie, 276, 278, 279, 280; Prof. W. P. Casey, 278; G. M. Day, 276, 278; Prof. E. J. Edmunds, 276, 277, 278; Henry Gunder, 276, 278, 280; Henry Heaton, 276, 277, 278, 279, 280; Chas. H. Kummell, 276, 277, 278, 279, 280; Prof. D. J. McAdam, 279; O. L. Mathiot, 276; W. L. Marcy, 276, 278, 279; K. S. Putnam, 276; P. Richardson, 276; Prof. E. B. Seitz, 276, 278, 279, 280; Prof. J. Scheffer, 278, 280; Prof. D. Trowbridge, 278, 279, 280.

276. "Given the chord AC of a circle, the side AB of a right angled triangle constructed on AC as hypotenuse, and the length of a perpendicular from A upon the line joining the right angle at B with the centre of the circle; to find the radius of the circle."

277. "Solve, algebraically, the equation $x^{17} - 1 = 0$."

SOLUTION BY R. J. ADCOCK, MONMOUTH, ILLINOIS.

All the roots of this equation are contained in the form

$$x = \cos \frac{1}{17}(2n\pi) \pm \sin \frac{1}{17}(2n\pi)\sqrt{-1},$$

where n may be any entire number not greater than the half of 17, that is $n = 0, 1, 2, 3, 4, 5, 6, 7, 8$, giving the seventeen roots,

$$x = 1,$$

$$x = \cos \frac{2}{17}\pi \pm \sin \frac{2}{17}\pi\sqrt{-1},$$

$$x = \cos \frac{4}{17}\pi \pm \sin \frac{4}{17}\pi\sqrt{-1},$$

$$x = \cos \frac{6}{17}\pi \pm \sin \frac{6}{17}\pi\sqrt{-1},$$

$$x = \cos \frac{8}{17}\pi \pm \sin \frac{8}{17}\pi\sqrt{-1},$$

$$x = \cos \frac{10}{17}\pi \pm \sin \frac{10}{17}\pi\sqrt{-1},$$

$$x = \cos \frac{12}{17}\pi \pm \sin \frac{12}{17}\pi\sqrt{-1},$$

$$x = \cos \frac{14}{17}\pi \pm \sin \frac{14}{17}\pi\sqrt{-1},$$

$$x = \cos \frac{16}{17}\pi \pm \sin \frac{16}{17}\pi\sqrt{-1}.$$

Hence the binomial and quadratic factors of $x^{17} - 1 = 0$, are

$$\begin{aligned} x^{17} - 1 = 0 &= (x-1)(x^2 - 2x \cos \frac{2}{17}\pi + 1)(x^2 - 2x \cos \frac{4}{17}\pi + 1) \\ &\quad (x^2 - 2x \cos \frac{6}{17}\pi + 1)(x^2 - 2x \cos \frac{8}{17}\pi + 1)(x^2 - 2x \cos \frac{10}{17}\pi + 1) \\ &\quad (x^2 - 2x \cos \frac{12}{17}\pi + 1)(x^2 - 2x \cos \frac{14}{17}\pi + 1)(x^2 - 2x \cos \frac{16}{17}\pi + 1). \end{aligned}$$

Similar expressions are given, Art. 40, p. 19, of Peirce's *Curves, Functions and Forces*, Vol. II. 1846.

SOLUTION BY HENRY HEATON, ATLANTIC, IOWA.

$$\text{Factoring we have } (x-1)(x^{16} + x^{15} + x^{14} + \dots + x + 1) = 0, \quad (1)$$

$$\therefore x = 1, \text{ or } x^{16} + x^{15} + x^{14} \dots + x + 1 = 0. \quad (2)$$

Since $x^{17} = 1$, $x^{15} = x$, $x^{13} = x^2$, &c.; $\therefore$ we have

$$\begin{aligned} (x + x^2 + x^4 + x^8 + x^{16} + x^{15} + x^{13} + x^9)(x^3 + x^6 + x^{12} + x^7 + x^{14} + x^{11} + x^5 + x^{10}) \\ = 4(x^{16} + x^{15} + x^{14} + \dots + x) = -4; \quad (3) \end{aligned}$$

$$\therefore x + x^2 + x^4 + x^8 + x^{16} + x^{15} + x^{13} + x^9 = -\frac{1}{2} \pm \frac{1}{2}\sqrt{17}, \quad (4)$$

$$x^3 + x^6 + x^{12} + x^7 + x^{14} + x^{11} + x^5 + x^{10} = -\frac{1}{2} \mp \frac{1}{2}\sqrt{17}, \quad (5)$$

$$(x + x^4 + x^{16} + x^{13})(x^2 + x^5 + x^{15} + x^9) = -1. \quad (6)$$

Combining equations (4) and (6), we find

$$x + x^4 + x^{16} + x^{13} = -\frac{1}{4} \pm \frac{1}{4}\sqrt{17} \pm \frac{1}{4}\sqrt{(34 \mp 2\sqrt{17})}, \quad (7)$$

$$x^2 + x^5 + x^{15} + x^9 = -\frac{1}{4} \pm \frac{1}{4}\sqrt{17} \mp \frac{1}{4}\sqrt{(34 \pm 2\sqrt{17})}. \quad (8)$$

Similarly we find

$$x^3 + x^{12} + x^{14} + x^5 = -\frac{1}{4} \mp \frac{1}{4}\sqrt{17} \pm \frac{1}{4}\sqrt{(34 \pm 2\sqrt{17})}, \quad (9)$$

$$x^6 + x^7 + x^{11} + x^{10} = -\frac{1}{4} \mp \frac{1}{4}\sqrt{17} \mp \frac{1}{4}\sqrt{(34 \pm 2\sqrt{17})}, \quad (10)$$

$$(x + x^{16})(x^4 + x^{13}) = x^5 + x^{14} + x^3 + x^{12} = -\frac{1}{4} \mp \frac{1}{4}\sqrt{17} \pm \frac{1}{4}\sqrt{(34 \pm 2\sqrt{17})}. \quad (11)$$

Combining (7) and (11) we find

$$x + x^{16} = \frac{1}{8}[-1 \pm \sqrt{17} \pm \sqrt{(34 \pm 2\sqrt{17})}] \pm \frac{1}{4}[17 \pm 3\sqrt{17} \mp 2\sqrt{(34 \pm 2\sqrt{17})} \mp \sqrt{(34 \mp 2\sqrt{17})}]^{\frac{1}{2}}.$$

$$\begin{aligned} \text{Since } x^{17} = 1, x^{16} = 1/x; x = \frac{1}{16}[-1 \pm \sqrt{17} \pm \sqrt{(34 \mp 2\sqrt{17})}] \\ \pm \frac{1}{8}[17 \pm 3\sqrt{17} \mp 2\sqrt{(34 \pm 2\sqrt{17})} \mp \sqrt{(34 \mp 2\sqrt{17})}]^{\frac{1}{2}} \pm \frac{1}{8}\{-34 \pm 2\sqrt{17} \\ \mp 2\sqrt{(34 \mp 2\sqrt{17})} \pm 4[17 \pm 3\sqrt{17} \pm 2\sqrt{(34 \pm 2\sqrt{17})} \pm \sqrt{(34 \mp 2\sqrt{17})}]^{\frac{1}{2}}\}^{\frac{1}{2}}. \end{aligned}$$

278. "A Chord AB of a circle, whose centre is at O , remains constantly parallel to itself; A and B are connected to the centre O , and on AO and BO , or those lines produced, M and N are taken so that $AM = BN = AB$. Required the locus of M and N ."

SOLUTION BY PROF. W. P. CASEY, SAN FRANCISCO, CAL.

Let SR and HE represent two diameters at right angles to each other, SR being parallel to AB . Let $\angle POM = \theta$; $PO = x$ and $PM = y$. Then $OM = \sqrt{(x^2 + y^2)}$, $AO \times \cos \theta = AD$, or $2r \cos \theta = 2AD = AM = r + \sqrt{(x^2 + y^2)}$; therefore $(2 \cos \theta - 1)r = \sqrt{(x^2 + y^2)}$, or $(2 \cos \theta - 1)^2 r^2 = x^2 + y^2$, which is the equation of the locus of M . By giving in succession values to θ , and taking the corresponding values of AM , the curve which is the locus of M will be traced. The curve will pass through the points E , O and R . The locus of N will be a similar curve and will pass through the points E , O and S .



SOLUTION BY D. TROWBRIDGE, A. M., WATERBURGH, N. Y.

Put the radius of the circle $= a$, co-ordinates of $M = x, y$, and of $A = x', y'$. We now have $AB = AO \pm OM$, or $2x' = a \pm \sqrt{(x^2 + y^2)}$, and $xy' = x'y = x\sqrt{(a^2 - x'^2)}$, $x'^2(x^2 + y^2) = a^2x^2$, and $2x' = 2ax \div \sqrt{(x^2 + y^2)} = a \pm \sqrt{(x^2 + y^2)}$. Put $x^2 + y^2 = r^2$, and $x = r \cos \theta$, then $2a \cos \theta = a \pm r$,
 $r = a(2 \cos \theta - 1), \quad r = a(1 - 2 \cos \theta).$

These are the eq'ns of the trisectrix. (See Hadden's Diff. Cal., p. 110.)

279. "If the earth be projected, when in perihelion, at right angles to a line joining the Earth and Sun, with a velocity sufficient to cause it to describe a parabola, how long would it take it to reach the orbit of Jupiter? The orbits of the Earth and Jupiter supposed in the same plane, both circles, and the radius of Jupiter's orbit five times that of the Earth."

SOLUTION BY PROF. D. J. MC ADAM, WASHINGTON, PA.

As in problem 50, ANALYST, Vol. II, let the circle $DEBF$ be the orbit of Jupiter, then the area ASD (D being the point of exit from Jupiter's orbit)

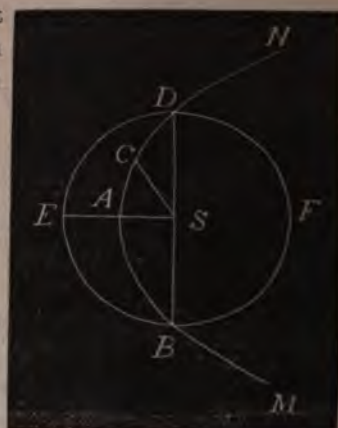
$$= R^2(\tan \frac{1}{2}\theta + \frac{1}{3} \tan^3 \frac{1}{2}\theta) = \frac{1}{3} R^2.$$

(Where $\theta = \angle ASC$.)

Then, putting $\tan \frac{1}{2}\theta = 2$ (see ANALYST, p. 30, Vol. II),

$$\begin{aligned} \sqrt{(10R)} : \sqrt{(4R)} &:: 25\pi R^2 : 50\pi R^2 \\ &= \text{the area described by the earth in one of} \\ &\text{Jupiter's years.} \end{aligned}$$

Therefore the time it will take the Earth to describe the portion of its orbit within Jupiter's orbit will $= \frac{1}{3} R^2 \div 50\pi R^2$, of Jupiter's year, $= 389.3$ days.



280. "Find the sum of the series

$$\begin{aligned} x \sin a - \frac{x^2 \sin 2a}{1.2} + \frac{x^3 \sin 3a}{1.2.3} - \frac{x^4 \sin 4a}{1.2.3.4} + \dots \text{to infinity,} \\ 1 - x \cos a + \frac{x^2 \cos 2a}{1.2} - \frac{x^3 \cos 3a}{1.2.3} + \frac{x^4 \cos 4a}{1.2.3.4} - \dots \text{to infinity.} \end{aligned}$$

SOLUTION BY CHAS. H. KUMMELL, U. S. LAKE SURV., DETROIT, MICH.

$$\text{We have } s_2 \pm s_1 i = 1 - x e^{\pm ai} + \frac{x^2}{1.2} e^{\pm 2ai} - \frac{x^3}{1.2.3} e^{\pm 3ai} + \dots$$

$$= e^{-x e^{\pm ai}} \quad (\text{where } i = \sqrt{-1}), \quad (1)$$

$$\text{then} \quad \sqrt{s_2^2 + s_1^2} = e^{-x \cos a}, \quad (2)$$

$$\text{and} \quad \sqrt{\frac{s_2 + s_1 i}{s_2 - s_1 i}} = e^{+x \sin a}, \quad (3)$$

or, taking logarithms,

$$\begin{aligned} x i \sin a &= i \left[\frac{s_1}{s_2} - \frac{1}{3} \left(\frac{s_1}{s_2} \right)^3 + \frac{1}{5} \left(\frac{s_1}{s_2} \right)^5 - \dots \right] \\ &= i \tan^{-1} (s_1 \div s_2). \end{aligned}$$

$$\therefore s_1 = s_2 \tan (x \sin a). \quad (4)$$

Substituting this in (2) we obtain

$$s_2 = e^{-x \cos a} \cos (x \sin a),$$

and by (4)

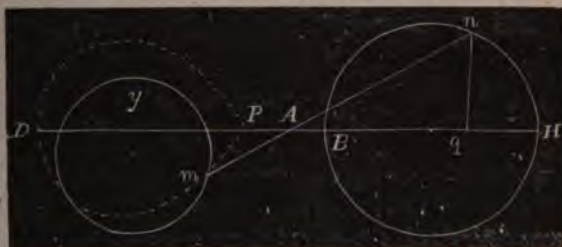
$$s_1 = e^{-x \cos a} \sin (x \sin a).$$

[Mr. Kummell has also sent us a solution of prob. 266, done by a method similar to the above.]

SOLUTION OF PROB. 85 (SEE P. 193, VOL. II) BY PROF. W. P. CASEY.

Let mn be the given line, passing through the given point A , BnH and any the given circles.

Through the given p't A , draw the diameter BH , and make HP and BD each $= mn$, $\therefore P$, and D are given points, and are in the locus of the point m .



Draw ng perpendicular to BH . Let $Ag = x$, $gn = y$, $BH = d$, $AB = a$, $mn = h$; then $ng^2 = Bg \times gH$, or $y^2 = (x-a)(a+d-x) = (x-a)(c-x)$ (if $c = a+d$), and $y^2 = -x^2 + (a+c)x - ac$. Let $An = r$, $\angle A = \theta$, then $y = r \sin \theta$, $x = r \cos \theta$; $\therefore$ by substitution $r^2 \sin^2 \theta = -r^2 \cos^2 \theta + (a+c)r \cos \theta - ac$, and $r^2 - (a+c)r \cos \theta = -ac$;

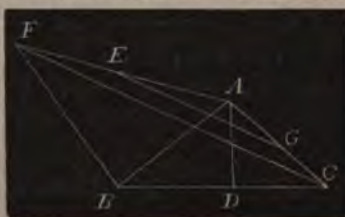
$$\therefore r = \frac{1}{2} \{ (a+c) \cos \theta \pm \sqrt{[(a+c) \cos^2 \theta - 4ac]} \}.$$

Because $Am = mn - An = h - r$, by giving successive values to θ , and taking the corresponding values of r , the curve, which is the locus of m , will be traced. The curve is an oval, whose axis $PD = BH = d$. The point m is therefore given and the line mn is in position.

[Prof. Casey also discusses the case when $An - Am$ is given, but our space will not permit its insertion.]

CONSTRUCTION OF THE METIAN RATIO BY PROF. CHASE.—Make $AB = 7$; $BC = 8$; $BD = 9$; $DF = 15$; $AE = AC$; AB perpendicular to CD and DF perpendicular to AD , and draw EG parallel to FC . $AF \div AG = 355 \div 113 = 3.14159292+$, π being 3.14159265 .

The error is less than one one hundred and sixteen thousandths of one per cent.



PROBLEMS.

281. By *Prof. Elias Schneider, Milton, Pa.* — Suppose the Earth to be a smooth sphere 25000 miles in circumference, and suppose a spherical ball, 5 miles in diameter, its mass being to that of the earth in proportion to the cubes of their diameters, to rest at a point *A*, on the earth's surface; and suppose further that the only influence acting on the ball and the earth is their mutual attraction, and that while the ball remains stationary upon the earth's surface at *A*, the earth rotates on its axis once in 24 hours.

Suppose now that the attraction of the sun acts upon the ball in the direction of the diameter *AE*, the earth having no other motion than rotation on its axis; when, from the rotation of the earth, the diameter *AE* ceases to point toward the sun, the ball will cease to be in equilibrium, and will roll over the earth's surface toward *E*. Required the relative velocity of the ball on the surface of the earth and the distance it will have advanced from the point *A*, when the point *A* shall have moved over 180° .

282. By *George Eastwood, Saxonville, Mass.* — Suppose a manufacturer of textile fabrics, who can operate sixty looms, ascertains by trial, (1) that when he makes his weavers sure of a dollar a day with constant work, then each loom, when it runs a whole day, will net three dollars; (2) that, from the nature of the work to be done, all the looms are never in running order at the same time; (3) that at any time it is equally possible that one, two, or three, or any number of them may be standing for repairs. Now, under these conditions, it is required to assign the least number of weavers to be placed under permanent pay, so that the average daily profits may be the greatest possible.

283. (Selected) By *Prof. Asaph Hall.* — If from any point in the plane of a parallelogram perpendiculars are let fall on the diagonal and on the two sides that contain this diagonal, the product of the diagonal by its perpendicular is equal to the sum of the products of the sides by their respective perpendiculars if the point falls outside of the parallelogram, or to their difference if it lies within the parallelogram.

Varignon's Theorem: *Mecanique Analytique*, Tome I, p. 13.

284. By *Prof. E. B. Seitz, Kirksville, Mo.* — *M*, *N* and *P*, *Q* are two pairs of random points within a circle, but on opposite sides of a given diameter *AB*: find the chance that the sum of the squares of the chords thro. *M*, *P* and *N*, *Q* is less than the square of the diameter.

285. By *W. E. Heal, Wheeling, Ind.* — If $E(e, x)$ denote the length of an elliptic arc, semi major axis unity and eccentricity e ; prove, without the aid of the Integral Calculus, that

$$d E(e, x) = dx \sqrt{\left(\frac{1 - e^2 x^2}{1 - x^2} \right)}.$$

286. By *Geo. M. Day, Lockport, N. Y.* — Three points are taken at random in a given circle, and a circle is drawn through them. What is the probability that the circle through the random points will be wholly in the given circle.

287. By *G. H. Harvill, Bonner, La.* — A ladder, 40 feet long, stands upright against a vertical wall, on a horizontal plane. A boy ascends the ladder, at a uniform rate, and at the same time the foot of the ladder slides out horizontally at the same uniform rate. How long is the boy's path?

288. By *Prof. E. J. Edmunds, New Orleans, La.* — Two circles are tangent to a given line AB ; from a point C , on the line AB the other tang'ts, to the two circles, CM and CN , are drawn. Find the envelop of MN , and the locus of its middle point.

Query. By *Chas. H. Kummell, Assist. U. S. Lake Survey, Detroit, Mich.* — Prove, otherwise than by induction, that

$$2^{n-1} = \sum_0^{n-1} \left\{ \frac{n+m-1}{1} \cdot \frac{n+m-2}{2} \dots \frac{n}{m} \left(\frac{1}{2} \right)^m \right\}$$

if m, n are positive integers, and give, if possible a value for

$$\sum_0^{n-1} \left\{ \frac{n+m-1}{1} \cdot \frac{n+m-2}{2} \dots \frac{n}{m} x^m \right\}$$

which reduces to 2^{n-1} for $x = \frac{1}{2}$.

ANNOUNCEMENT OF VOL. VII. — With this No. is completed the sixth annual volume of the ANALYST. And, though we have not been able to make any great improvement in its mechanical execution, we trust that, in respect to the subject-matter presented in its pages, it has not deteriorated in the late volumes, but that, on the other hand, it has steadily improved; and we feel flattered that many of our ablest contributors have expressed a desire that the publication may be continued. We take pleasure, therefore, in announcing that we have made definite arrangement for the publication of Vol. VII; and, if our health permits, we hope, and expect, to be able to publish other additional volumes.

As many of our readers are sufficiently interested in the problems proposed in each No. to furnish solutions in detail, of one or more, for publication, we embrace this opportunity to say that, in general, our publication is mailed about the 25th of the month preceding the one named under its title, and that only solutions that are rec'd on or before the 10th preceding the 25th above named, are included in the regular notice of solutions rec'd.

Though we do not desire to make the solution of questions a prominent feature of our publication, yet we think about one-fourth of its space may be profitably devoted to that purpose; we suggest, however, that the efforts of many of our contributors, in the solution of problems, should be *concentrated*. One solution well made is worth more, for our purpose, than half a dozen careless and imperfect solutions. And, when diagrams are required for illustration, they should be neatly drawn, of as small dimensions as is practicable, and detachable from the solutions, so that they can be sent to the engraver while the solution is in the hands of the compositor. Many solutions go into the waste basket for want of compliance with the above named obvious conditions.

EDITOR.

PUBLICATIONS RECEIVED.

The Elements of Co-ordinate Geometry. In Three Parts. I. Cartesian Geometry. II. Quaternions. III. Modern Geometry, and an Appendix. By DE VOLSON WOOD, Prof. of Mathematics and Mechanics in Stevens Institute of Technology. 329 pp., 8vo. New York: John Wiley and Sons. 1879.

In the first part of this work, comprising 228 pp., the subjects usually presented in works on Analytical Geometry are discussed and illustrated by numerous examples. The second part occupies 70 pages, and is devoted, as the title imports, to a discussion of the Elements of Quaternions: this part will be found especially interesting to students who have not read more extended discussions of the subject. "Modern Geometry" occupies 14 pages, and discusses briefly the various systems of co-ordinates which have been introduced since the Cartesian system was recognized. In the Appendix (13 pp.) we have a historical sketch of the development of the doctrine of Quaternions; from which it appears that, though the method has been elaborated mainly through the labors of Sir Wm. R. Hamilton, the seeds of the doctrine have germinated in many minds, stimulated by a desire to explain a recognized fact.

The Canada School Journal. 24 pp., 4to: Published monthly at Toronto, Canada; has a Mathematical Department edited by Alfred Baker, M. A.

ERRATA.

On page 259 [159], line 21, for $a\Delta$, read $c\Delta$.

" " 166, " 12, for $.01 + \&c.$, read $.01u^7 + \&c.$

THE ANALYST.

A JOURNAL OF

PURE AND APPLIED MATHEMATICS.

EDITED AND PUBLISHED BY

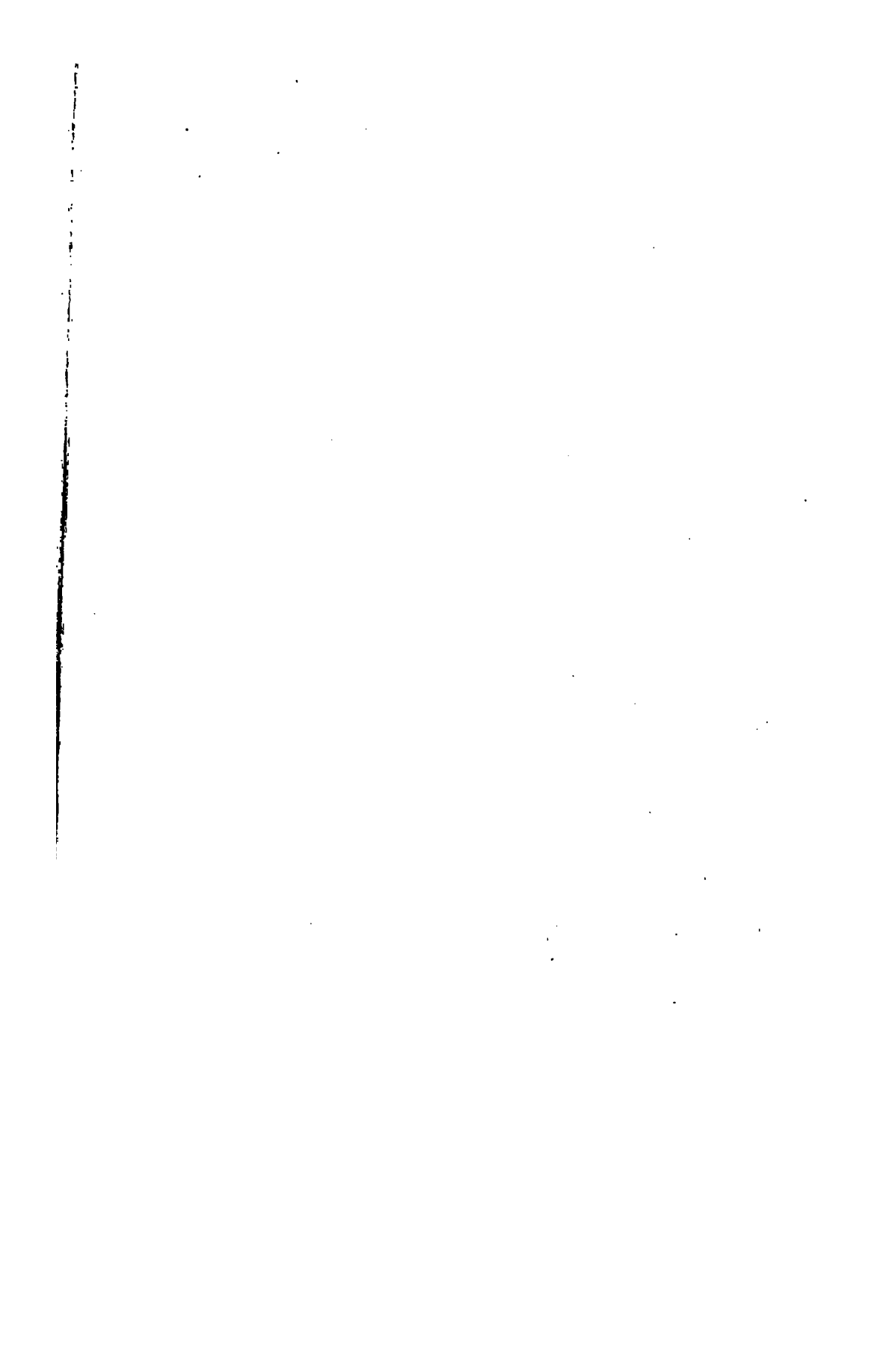
J. E. HENDRICKS, A. M.

VOLUME VII.

DES MOINES, IOWA:
MILLS & CO., BOOK AND JOB PRINTERS.
1880.

ALPHABETICAL INDEX.

	PAGE		PAGE
A Fourth Proportional, ...	128	Kirkwood, Prof. Daniel, - - -	9
A General Solution of Problem 292, ...	145	Ladd, Miss Christine, - - -	147
A Quasi proof of the Arithmetical Mean, ...	150	Mr. Glaisher's Enumeration of Primes, ...	118
Answers to Queries in No. 4, ...	164	Mechanics by Quaternions, - - -	137, 177
Announcement of Vol. VIII, ...	199	Macnie John, - - -	152, 190
Applegate, Jesse, ...	128	Mathiot Octavian, L. - - -	72
Adcock, R. J., ...	22, 189	Note on Schulenburg's Solution of the	
Barbour, Prof. L. G., ...	17	General Equation of the 5th Degree, ...	53
Beman, Prof. W. W., ...	115	New Demons. of the Binomial Theorem, ...	190
Correspondence, ...	54, 151	Note on the Computation of π , - - -	55
Continuation of Note at page 57, ...	88	On Unsymmetrical Adjustments and	
Curve of Pursuit, ...	89	their Limits, - - -	1
Criticisms, ...	185	On the Varia'n of the Length of the Day, ...	9
Christie, Alexander S., ...	185	On some Properties of Polyno's, 29, 73, ...	105
Craig, Dr. Thomas, ...	13	Origin of the name Quaternion, - - -	52
Casey, Prof. W. P.	120	On the Quadrature of Curves and the	
Determination of a Sphere which Cuts		Cubature of Solids by Infinite Series, ...	47
Five given Spheres at the Same Angle, ...	13	On the Phenomena of the Tides, - - -	154
Demonstration of the Cissoid, - - -	23	On a Theorem in Probability, - - -	169
Determination of the area of an Ellipse		Problems, ...	31, 63, 103, 135, 167, 199
Geometrically, - - -	72	Public's Re'cd, 32, 64, 104, 136, 168, ...	200
De Forest, E. L., ...	1, 39, 54, 73, 105, 169	Parallel Chords in an Ellipse, - - -	82
Extension of the Method of Least Sq's, ...	22	Proof of some Remarkable Properties	
Experimental Determination of the		in the Method of Least Squares, ...	84
Formula for Probable Error, - - -	189	Proof of a Proposition in Solid Geom.	154
Eastwood, George, - - -	158	Quaternions, - - -	11, 33, 65, 121
Errata, - - -	32, 64, 104, 136, 168, 200	Solutions of Problems in No. 6, Vol. VI, ...	24
Formula for Force of Mortality, - - -	54	in No. 1, 58, in No. 2, 97, in No 3, ...	132
Geometrical Solution of Problem 126, ...	120	in No. 4, 164, in No. 5, - - -	193
Heal, William E., - - -	53	Solution of Special Case of Prob. 288, ...	115
Hall, Prof. Asaph, - - -	82	Solution of a Prob. in Curves of Pursuit, ...	117
Harvill, George H., - - -	56	Solution of Prob. 292 with Cond. Ch'd, ...	160
Hyde, Prof. E. W., - - -	137, 157, 177	Siverly, Walter, - - -	54
Hoover, William, - - -	160	Scheffer Prof. J. - - -	55
Introduction to Differentiation, - - -	152	Stone, Ormond, ...	150, 162
Ingalls, Lieut. James M., - - -	89, 117	The Prunoid or Plum Curve, ...	17
Johnson, Prof. William Woolsey, - - -	52	The Nine-line Conic, ...	147
Kummell, Charles H., ...	84, 145, 160, 191	Trowdrige, Prof. David, ...	47



THE ANALYST.

. VII.

JANUARY, 1880.

No. 1.

UNSYMMETRICAL ADJUSTMENTS, AND THEIR LIMITS.

BY E. L. DE FOREST.

(Continued from page 170, Vol. VI.)

IE origin is always at the centre of parallel forces, and this is the position of the vertex, or the maximum ordinate, if the curve is a symmetrical

The centre of forces for the resultant of k applications of a given formula keeps the same position relatively to the two ends of the resultant formula, as the centre for the given formula had with reference to its two

To prove this, let us regard the coefficients of the given formula as coefficients of powers of z in the polynomial

$$a_0 + a_1z + a_2z^2 + \dots + a_nz^n, \quad (42)$$

the condition

$$a_0 + a_1 + \dots + a_n = 1. \quad (43)$$

These parallel forces a being supposed to act at right angles to the axis and at intervals equal to Δx , let h_1 denote the interval between a_0 and the centre of forces, in other words, the lever arm of the system about the position of a_0 as a fulcrum. Then we have as the condition of equilibrium the centre of forces,

$$b_1 \Delta x = a_0(-h_1) + a_1(-h_1 + \Delta x) + \dots + a_n(-h_1 + n\Delta x) = 0,$$

from this, with the condition (43), we get

$$h_1 = (a_1 + 2a_2 + \dots + na_n)\Delta x, \quad (44)$$

It is also evidently the statical moment of the system about the place

. For two applications of the given formula, the resultant coefficients will be those of the powers of z in the square of (42). To form the square first multiply (42) by a_0 , and in this product, taking account of the position (43), the moment of the coefficients about the place of the first is a_0h_1 . Multiplying next by a_1z we get a product with the moment

$a_1(h_1 + \Delta x)$. Likewise multiplying by a_2x^2 , the moment for the product is $a_2(h_1 + 2\Delta x)$, and so on, the total being

$$a_0h_1 + a_1(h_1 + \Delta x) + \dots + a_n(h_1 + n\Delta x),$$

which by virtue of (43) and (44) is equal to $2h_1$, or double the moment for the original polynomial (42). And since the sum of the coefficients in any power of (42) will remain unity, $2h_1$ is the lever arm of the square, or the distance of its centre of forces from its first term. Likewise for the third power, the lever arm can be shown to be $3h_1$, and so on. But to make the demonstration general, let us suppose that for the k power of (42) we have found that the lever arm is kh_1 . Then in the $(k + 1)$ th power the total moment about the place of its first coefficient will be

$$a_0kh_1 + a_1(kh_1 + \Delta x) + \dots + a_n(kh_1 + n\Delta x),$$

which by (43) and (44) reduces to $(k + 1)h_1$, and this is the expression for the lever arm. But we have shown that the lever arm is kh_1 when $k = 2$, therefore it is also kh_1 when $k = 3, k = 4$, and so on; so that always, the lever arm of the coefficients in the k power of a polynomial will be k times the lever arm of the coefficients of the polynomial. But the whole interval between the first and last term of the k power is k times the interval between the first and last term of the first power. Hence, as k increases, the lever arm of the product increases in the same ratio as the whole length of the product does, and the extremity of the lever arm, or the centre of parallel forces, divides the length of the product in a constant ratio. This is exemplified in the simple case of the expansion of $(p + q)^m$, or of $(p + qx)^m$, where p and q are probabilities whose sum is unity. The centre of parallel forces in the first power of this binomial, according to (44), is at the distance $h_1 = q\Delta x$ measured from the place of the first coefficient p , the whole length of the binomial being Δx . In the expansion to the m th power, the whole length of the expansion is $m\Delta x$, and by the foregoing principle, the distance from the first term of the expansion to its centre of parallel forces is $mh_1 = qm\Delta x$. But we already know that this distance is that of the maximum coefficient in the expansion whenever qm is a whole number (ANALYST, May 1879, p. 66), and that this maximum forms the vertex of the probability curve which is the limit of the series of coefficients when m becomes infinite. If qm is not a whole number, it of course cannot denote the exact rank of any term in the series of coefficients, and the rank of the greatest term will differ from it by a fraction. Formula (3) of the art. just cited is

$$\frac{\Delta y}{y} = \frac{(qm - p)\Delta x - x}{p(x + \Delta x)},$$

where y denotes any term, and x is its distance from the first term. If now

we regard y as a continuously varying function of x or ordinate to a curve, and put $dy = 0$, we get

$$x = (qm - p)dx,$$

showing that the two ordinates whose ranks are

$$qm - p, \quad \text{and} \quad qm - p + 1,$$

are always equal to each other. The greatest term must obviously fall between these two. The maximum ordinate to the curve, however, does not in general coincide with the greatest term of the series, nor does it have the precise rank qm . To show this, let us take the general term whose rank reckoned from the first term is i ,

$$y_i = \frac{1.2.3 \dots m}{1.2.3 \dots i.1.2.3 \dots (m-i)} p^{m-i} q^i,$$

and apply Stirling's formula

$$1.2.3 \dots m = (2\pi)^{\frac{1}{2}} m^{m+\frac{1}{2}} e^{-m},$$

which gives after reduction

$$y_i = \left(\frac{(2\pi)^{-\frac{1}{2}} m^{m+\frac{1}{2}} p^m}{i^{\frac{1}{2}+\frac{1}{2}} (m-i)^{m-\frac{1}{2}+\frac{1}{2}}} \right) \left(\frac{q}{p} \right)^i.$$

Neglecting constant factors and taking the Napierian logarithm, the second member becomes

$$f(i) = i \log'(q/p) - i \log' i - \frac{1}{2} \log' i - (m + \frac{1}{2}) \log'(m-i) + i \log'(m-i),$$

and the value of i which renders this a maximum will render y_i a maximum. It is given by the condition

$$\frac{df(i)}{di} = \log' \left(\frac{q}{p} \right) + \log'(m-i) - \log' i + \frac{1}{2(m-i)} - \frac{1}{2i} = 0.$$

When we put $i = qm$, this condition reduces to $q - p = 0$, indicating that the rank of the maximum ordinate cannot be precisely qm unless $p = q$. If we apply Newton's rule for finding a root of an equation, we get as the approximate rank of the maximum ordinate the value

$$qm + (q - p) \div \left\{ 2 - \frac{1}{m} \left(\frac{q+p}{p} \right) \right\},$$

or when m is very large

$$qm + \frac{1}{2}(q - p).$$

It is true that Stirling's formula is exact only when m is infinite, but it comes very near the truth when m has any large finite value.

Since the distance from the greatest term in the expansion of $(p+q)^m$ to any other term represents an error or deviation from the most probable value of an observed quantity, and since any term in the expansion is proportional to the number of times which the corresponding error may be expected to occur, our demonstration that the greatest term is at the centre

of parallel forces, whether m be finite or infinite, provided that the rank qm of that centre is an integer, is a proof that in this case the probable sum of all the errors on one side of the most probable value, is equal to the probable sum of all those on the other side. But this is the characteristic of the arithmetical mean, that the sum of the deviations in excess of the mean is equal to the sum of those in defect, or in other words, that the algebraic sum of all the deviations is zero. Thus the property which we have demonstrated gives support to the assumption, which some writers have accepted as an axiom, that in a series of observed values of a single quantity, when the observations are of equal weight, the most probable value of the quantity is the arithmetical mean of all the observed values. (Compare Merriman's *Least Squares*, pp. 127 and 196.)

From what we have seen in connection with (19) and (20), it follows that if the conditions $b_2 = 0, b_3 = 0, \dots b_n = 0$, or what comes to the same thing, $b_2(4x)^2 = 0, b_3(4x)^3 = 0, \dots b_n(4x)^n = 0$, are satisfied in the given polynomial when b_2, b_3 &c., are estimated from the centre of parallel forces, the same conditions will be satisfied for the centre of forces in the k power. To illustrate, take the polynomial

$$\frac{1}{81}(-4 + 30x + 60x^2 - 5x^3),$$

whose centre of parallel forces, by (44), is at the distance

$$h_1 = \frac{4x}{81}(30 + 2 \times 60 - 3 \times 5) = \frac{5}{3} 4x,$$

reckoned from the place of the first coefficient, being $\frac{2}{3}$ of the way from the 30 to the 60. For this centre we have not only $b_1 4x = 0$, but also in this instance

$$b_2(4x)^2 = \frac{(4x)^2}{81} \left\{ -4 \left(-\frac{5}{3} \right)^2 + 30 \left(-\frac{2}{3} \right)^2 + 60 \left(\frac{1}{3} \right)^2 - 5 \left(\frac{4}{3} \right)^2 \right\} = 0,$$

$$b_3(4x)^3 = \frac{(4x)^3}{81} \left\{ -4 \left(-\frac{5}{3} \right)^3 + 30 \left(-\frac{2}{3} \right)^3 + 60 \left(\frac{1}{3} \right)^3 - 5 \left(\frac{4}{3} \right)^3 \right\} = 0.$$

Now for example let $k = 3$. The third power of the polynomial is

$$\frac{1}{531441}(-64 + 1440x - 7920x^2 - 16440x^3 + 122400x^4 + 317700x^5 + 161700x^6 - 51750x^7 + 4500x^8 - 125x^9),$$

and by the foregoing theorem, its centre of forces should be at the distance $3(\frac{5}{3} 4x) = 5 4x$ from the first coefficient, and therefore coincident with the 317700. By actual calculation, we find at this centre

$$b_1 4x = \frac{4x}{531441} \left\{ -64(-5) + 1440(-4) - 7920(-3) - 16440(-2) + 122400(-1) + 161700(1) - 51750(2) + 4500(3) - 125(4) \right\} = 0,$$

$$b_2(\Delta x)^2 = \frac{(\Delta x)^2}{531441} \{ -64(-5)^2 + 1440(-4)^2 - 7920(-3)^2 - 16440(-2)^2 \\ + 122400(-1)^2 + 161700(1)^2 - 51750(2)^2 + 4500(3)^2 - 125(4)^2 \} = 0, \\ b_3(\Delta x)^3 = \frac{(\Delta x)^3}{531441} \{ -64(-5)^3 + 1440(-4)^3 - 7920(-3)^3 - 16440(-2)^3 \\ + 122400(-1)^3 + 161700(1)^3 - 51750(2)^3 + 4500(3)^3 - 125(4)^3 \} = 0,$$

so that the calculation agrees with the theory.

Let us now pass on to consider another property. In the polynomial (42) suppose that $b_2(\Delta x)^2$ is not zero, but has some other value. It is required to find its value for the k power of the polynomial. Let us denote the values of b_2 for the 1st, 2nd, 3rd &c., powers by b'_2, b''_2, b'''_2 &c., and let the lever arms of the several systems of coefficients about the first or left hand coefficients as fulcrums be h_1, h_2, h_3 &c. Again, let us suppose that the coefficients in each of these powers represent a system of material points attached to the axis of X and rotating in the plane of XY about the place of the left hand coefficient as an axis, and denote the radii of gyration of the systems for the 1st, 2nd, 3rd &c., powers by g_1, g_2, g_3 &c. This is merely for convenience of description and notation, since material points or masses are positive, while some of the coefficients a may be negative, a circumstance, however, which makes no difference in the analysis. We write then, for the first power,

$$g_1^2 = (1^2a_1 + 2^2a_2 + \dots + n^2a_n)(\Delta x)^2. \quad (45)$$

But from the nature of b_2 we have

$$b'_2(\Delta x)^2 = a_0(-h_1)^2 + a_1(-h_1 + \Delta x)^2 + a_2(-h_1 + 2\Delta x)^2 + \dots + a_n(-h_1 + n\Delta x)^2 \\ = h_1^2(a_0 + a_1 + \dots + a_n) - 2h_1\Delta x(a_1 + 2a_2 + \dots + na_n) \\ + (\Delta x)^2(1^2a_1 + 2^2a_2 + \dots + n^2a_n),$$

whence, by virtue of (43) (44) and (45),

$$b'_2(\Delta x)^2 = g_1^2 - h_1^2, \quad (46)$$

so that for any polynomial, the value of $b_2(\Delta x)^2$ is the excess of the square of the radius of gyration above the square of the lever arm, both being reckoned from the left hand term as an axis or fulcrum. In the second power of (42) therefore, we shall have likewise

$$b''_2(\Delta x)^2 = g_2^2 - h_2^2. \quad (47)$$

But h_2 is equal to $2h_1$, as already proved. To find g_2 , we remember that the sum of the coefficients in the square of (42) is unity, so that the moment of inertia of the system about its first term as an axis is g_2^2 , and this moment is equal to the sum of the moments for the portions of the product formed by multiplying (42) by a_0, a_1z, a_2z^2 &c., in succession. We have then

$$g_2^2 = a_0(1^2a_1 + 2^2a_2 + \dots + n^2a_n)(\Delta x)^2 + a_1[1^2a_0 + 2^2a_1 + \dots + (n+1)^2a_n] \\ \times (\Delta x)^2 + a_2[2^2a_0 + 3^2a_1 + \dots + (n+2)^2a_n](\Delta x)^2 + \dots \\ + a_n[n^2a_0 + (n+1)^2a_1 + (n+2)^2a_2 + \dots + (n+n)^2a_n](\Delta x)^2.$$

In the last or general term, the coefficient of $a_n(\Delta x)^2$, by expansion of the binomials in it, and by virtue of (43) (44) and (45) reduces to

$$n^2 + 2n \left(\frac{h_1}{\Delta x} \right) + \left(\frac{g_1}{\Delta x} \right)^2,$$

and assigning to n the values 0, 1, 2 &c., in succession, we get expressions for the coefficients of $a_0(\Delta x)^2$, $a_1(\Delta x)^2$ &c., and thus find

$$g_2^2 = a_0g_1^2 + a_1[1^2(\Delta x)^2 + 2.1h_1\Delta x + g_1^2] + a_2[2^2(\Delta x)^2 + 2.2h_1\Delta x + g_1^2] \\ + \dots + a_n[n^2(\Delta x)^2 + 2.nh_1\Delta x + g_1^2],$$

which by means of (43), (44) and (45), reduces to

$$g_2^2 = 2(g_1^2 + h_1^2). \quad (48)$$

Substituting this in (47) we get

$$b_2''(\Delta x)^2 = 2(g_1^2 - h_1^2),$$

and comparing it with (46), we see that the value of $b_2(\Delta x)^2$ for the second power of the given polynomial is twice as great as it is for the first power. We might proceed to show that for the third power it is three times as great as for the first, and so on. But to make the demonstration general for all powers, let us suppose that for the k power of (42) we have found, as in (48),

$$g_k^2 = k[g_1^2 + (k-1)h_1^2]. \quad (49)$$

Let us write also, as in (10),

$$(a_0 + a_1z + \dots + a_nz^n)^k = B_0 + B_1z + B_2z^2 + \dots + B_{kn}z^{kn}.$$

We have obviously, as in (43), (44) and (45),

$$\left. \begin{aligned} B_0 + B_1 + B_2 + \dots + B_{kn} &= 1, \\ h_k &= (B_1 + 2B_2 + \dots + knB_{kn})\Delta x, \\ g_k^2 &= [1^2B_1 + 2^2B_2 + \dots + (kn)^2B_{kn}](\Delta x)^2. \end{aligned} \right\} \quad (50)$$

In the $k+1$ power of (42) we have, for the moment of inertia about the place of the first term as an axis,

$$g_{k+1}^2 = a_0[1^2B_1 + 2^2B_2 + \dots + (kn)^2B_{kn}](\Delta x)^2 + a_1[1^2B_0 + 2^2B_1 + \dots \\ + (kn+1)^2B_{kn}](\Delta x)^2 + a_2[2^2B_0 + 3^2B_1 + \dots + (kn+2)^2B_{kn}](\Delta x)^2 + \dots \\ + a_n[n^2B_0 + (n+1)^2B_1 + \dots + (kn+n)^2B_{kn}](\Delta x)^2.$$

In the last or general term, the series is reducible, as before, by expansion of its binomials, and by virtue of (50), to

$$n^2 + 2n \left(\frac{h_k}{\Delta x} \right) + \left(\frac{g_k}{\Delta x} \right)^2.$$

Assigning to n the values 0, 1, 2 &c., in succession, we get equivalents for the series in the first, second and other terms, whence

$$g_{k+1}^2 = a_0 g_k^2 + a_1 [1^2 (\Delta x)^2 + 2.1 h_k \Delta x + g_k^2] + a_2 [2^2 (\Delta x)^2 + 2.2 h_k \Delta x + g_k^2] + \dots + a_n [n^2 (\Delta x)^2 + 2.n h_k \Delta x + g_k^2].$$

By means of (43), (44) and (45), this is reduced to

$$g_{k+1}^2 = g_k^2 + g_1^2 + 2h_1 h_k.$$

But we know that $h_k = k h_1$, and assigning to g_k^2 the value in (49), we have

$$g_{k+1}^2 = k g_1^2 + k(k-1) h_1^2 + g_1^2 + 2k h_1^2,$$

and finally

$$g_{k+1}^2 = (k+1) (g_1^2 + k h_1^2).$$

But this expression may be obtained from (49) by merely changing k into $k+1$, whence it appears that if (49) holds good for any one power of the given polynomial, it also holds good for the next higher power. And in (38) it was proved to be true for $k = 2$, therefore it is true for $k = 3$, for $k = 4$, and so on without limit.

Now as was shown in (46), so here, we shall have in the expanded k power

$$b_2^{(k)} (\Delta x)^2 = g_k^2 - h_k^2,$$

where $h_k = k h_1$ and g_k^2 has its value from (49), whence

$$b_2^{(k)} (\Delta x)^2 = k g_1^2 + k(k-1) h_1^2 - k^2 h_1^2,$$

and finally

$$b_2^{(k)} (\Delta x)^2 = k (g_1^2 - h_1^2).$$

Comparing this with (46) we see that the value of $b_2 (\Delta x)^2$ for the k power of the given polynomial is k times as great as it is for the first power. It may be noticed that, according to the mechanical analogy which we have employed, the quantity $b_2 (\Delta x)^2$ corresponds to the square of the radius of gyration, when the centre of parallel forces is taken as an axis.

As an illustration of the foregoing theorem, take the polynomial

$$\frac{1}{4} (2 + 3z - z^2),$$

where by (44) the centre of parallel forces is at the distance

$$h_1 = \frac{1}{4} \Delta x (3 - 2) = \frac{1}{4} \Delta x,$$

measured from the first coefficient, and therefore one fourth of the way from the 2 to the 3. We have here

$$b_2' (\Delta x)^2 = \frac{(\Delta x)^2}{4} \left\{ 2 \left(-\frac{1}{4} \right)^2 + 3 \left(\frac{3}{4} \right)^2 - 1 \left(\frac{7}{4} \right)^2 \right\} = -\frac{5}{16} (\Delta x)^2.$$

Now take for instance $k = 4$, and the fourth power of the polynomial is

$$\frac{1}{256} (16 + 96z + 184z^2 + 72z^3 - 111z^4 - 36z^5 + 46z^6 - 12z^7 + z^8).$$

According to the theorem we have established, the centre of parallel forces here is at the distance $4 \left(\frac{1}{4} \Delta x \right) = \Delta x$ from the first coefficient, that is,

it coincides with the 96. Also we ought to have

$$b_2^{(17)}(\Delta x)^2 = 4[-\frac{5}{18}(\Delta x)^2] = -\frac{5}{4}(\Delta x)^2.$$

And by actual computation, we find

$$b_2^{(17)}(\Delta x)^2 = \frac{1}{256}(\Delta x)^2[16(-1)^2 + 184(1)^2 + 72(2)^2 - 111(3)^2 - 36(4)^2 + 46(5)^2 - 12(6)^2 + 1(7)^2] = -\frac{5}{4}(\Delta x)^2,$$

so that the result confirms the previous work.

As a further illustration, take the binomial $p+q$ or $p+qz$, where p and q are probabilities whose sum is unity. As already shown, the centre of parallel forces is at the distance $q\Delta x$ measured from the first coefficient p . We have then

$$b_2'(\Delta x)^2 = (\Delta x)^2[pq^2 + qp^2] = pq(\Delta x)^2.$$

Now in the expansion of $(p+q)^m$ our theorem gives

$$b_2^{(m)}(\Delta x)^2 = pqm(\Delta x)^2.$$

Since $b_2^{(m)}(\Delta x)^2$ is estimated from the centre of parallel forces, which coincides, as we have before seen, with the greatest coefficient or maximum probability, and since it is the sum of the products formed by multiplying each term of the expansion into the square of its distance from the greatest term, we see that it is here identical with ϵ^2 , or the square of the quadratic (mean) error. Hence

$$\epsilon^2 = pqm(\Delta x)^2, \quad (51)$$

and we have a complete demonstration of the property which I had originally found only by induction from a number of particular cases. (ANALYST, May 1879, p. 71.) As was there shown, when Δx is reduced to dx , and m is increased to an infinity of the second order, ϵ still remains finite and has the well known value

$$\epsilon = dx \sqrt{pqm}.$$

In conclusion, it may be observed that the lever arm of a system of parallel forces, and the radius of gyration of a system of material points, will remain unchanged when all the forces, or the infinitesimal masses of all the material points, are increased or diminished in a constant ratio. Any rational and entire polynomial is equal to a polynomial the sum of whose coefficients is unity, multiplied by a numerical factor. Hence we shall always have

$$h_k = kh_1, \quad b_2^{(k)}(\Delta x)^2 = kb_2'(\Delta x)^2,$$

whether the condition (43) is fulfilled or not.

NOTE.—In that portion of the present paper which appeared in the November number, it ought perhaps to have been said respecting the curves given by equations (37) and (41), that the method of integrating between the finite limits $\pm x_1$, instead of $\pm \infty$, is not a very satisfactory one, and

should be taken only as provisional, enabling us to go on and examine the general properties of the curves. The ordinates beyond the limits x_1 are not zero, though they may be small enough to be neglected without material error. I think that the decimal coefficients in (41) are very nearly if not entirely correct, but am not so sure of those in (37), because so far as we can judge from Table III, the values of Y beyond the adopted lower limit $-x_1 = -37$ nearly, will not be very small. It is to be hoped that the integrals of (33) and (40) may yet be evaluated rigorously between the limits $\pm \infty$, so as to give the constants of integration for these curves in an exact form, as has been done in the simpler and well known case of the probability curve (31).

The last one of the three conditions in (40) is satisfied necessarily, because the curve is symmetrical with respect to the axis of Y , and thus its equation (41) shows only two constants of integration. In all cases where the order n of the general differential equation (28) is odd, the curve is symmetrical, the alternate conditions

$$\int_{-\infty}^{\infty} x^3 y dx = 0, \quad \int_{-\infty}^{\infty} x^5 y dx = 0, \quad \int_{-\infty}^{\infty} x^7 y dx = 0, \text{ \&c.,}$$

are necessarily satisfied, and the number of constants of integration to be determined is only $\frac{1}{2}(n+1)$.

ON THE VARIATION IN THE LENGTH OF THE DAY.

BY PROFESSOR DANIEL KIRKWOOD.

If we let m = the mass of a rotating globe in the process of condensation ;

t = its time of rotation ;

k = its principal radius of gyration ;

r = its radius, and

ω = its angular velocity ; then the principle of the preservation of areas gives us

$$m\omega k^2 = \frac{1}{2}m\omega r^2 = c = \text{a constant.}$$

$$\therefore \omega = \frac{5c}{2m} \cdot \frac{1}{r^2}; \text{ or, since } m \text{ is constant,}$$

$$\omega = \frac{c'}{r^2}; \text{ that is, the angular velocity varies inverse-}$$

ly as the square of the radius.

If now we let t' , r' and ω' represent the time, radius and angular velocity for any other epoch, we shall have

$$\begin{aligned}\omega &: \omega' :: r'^2 : r^2; \text{ and} \\ \omega &: \omega' :: t' : t; \\ \therefore t &: t' :: r^2 : r'^2;\end{aligned}$$

or, the time of rotation varies as the square of the radius. This proportion gives us

$$t' = t \left(\frac{r'}{r} \right)^2.$$

We may apply this formula to determine what was the length of the day when the earth's radius was 100 miles greater than at present.

$$t' = 23^{\text{h}}.933 \left(\frac{4056}{3956} \right)^2 = 25^{\text{h}}9^{\text{m}}29^{\text{s}}.$$

When the radius was about 21,500 ms. the day was equal to the moon's period of revolution, so that, during an indefinite period, the moon remained constantly on the same meridian. In this estimate the moon's mean distance from the earth is supposed to be constant and the effect of tidal retardation is disregarded. We find in the same manner that when the radius was about 75,000 miles the day and the year were equal. Finally, if the earth ever extended to the moon's orbit the period of rotation, by the same formula, was nearly ten years. So likewise when Mars filled the orbit of Phobos, his rotation period was $7^{\text{d}}.65$, or 24 times the orbital period of the satellite. It is probable, however, that in the gaseous form all parts of the mass would not rotate in the same time.

In the case of the earth or in that of any planet attended by satellites, the acceleration due to contraction would probably compensate, or more than compensate, any tidal retardation.* As each of these opposing forces, however, has evidently been variable, the determination of their relative effects is a problem of great difficulty.

[If it be true that the several planets of the solar system were detached from a central rotating gaseous mass, as is assumed in the Nebular Hypothesis, then it appears, by application of the above formula, that when the rotating mass extended to the orbits of the several planets the angular velocity of its equatorial surface must have been much less than that of its central portion, so that it is possible the rings, which subsequently formed several, or all, of the planets, may have been detached simultaneously.—Ed.]

*The effects of the tidal retardation have been ably discussed by Delaunay, Adams, G. H. Darwin, and others, but the data are insufficient to justify any definite conclusion.

QUATERNIONS.

BY PROF. DE VOLSON WOOD, HOBOKEN, NEW JERSEY.

1. QUATERNIONS is a system of mathematics invented by Sir William Rowan Hamilton, an Irish Mathematician, about the year 1843. He gave to it the above name because it involves four fundamental arbitrary units, viz. VECTOR, TENSOR, VERSOR and SCALAR. These terms will be defined hereafter.

The system grew out of an attempt to geometrize the imaginaries of algebra, of the form $a\sqrt{-1}$. During the early part of the present century there were many attempts in this direction, but most of them amounted to but little more than suggestions and expressions of opinion. There was, however, one notable exception in the labors of one M. Argand, a French mathematician, who, in 1806, published an essay on the manner of representing Imaginary Quantities, whose methods were substantially reproduced by a Mr. Warren, in England in 1828, and also during the same year by one M. Mourey, in Paris. These writers produced some algebraic expressions which are substantially the same as those used by Hamilton—also their modulus corresponds with Hamilton's tensor, and their *versor* is identical with his *versor* (the former being the French and the latter the English of the same word). Their method of multiplying together two directed lines in a plane—which consists in multiplying together their modulii and adding their versors—is apparently the same as Hamilton's. But Hamilton's fundamental conceptions of methods were evidently different from those of his predecessors and contemporaries. Thus, M. Argand's operations for multiplying two directed lines were in one plane, producing a third line in the same plane; but Hamilton's method produces a third line perpendicular to the plane of the two lines, and hence the operation involves the three rectangular dimensions of space. It is worthy of remark, that no method of treating imaginaries except Hamilton's, grew to a system; but this, by the hands of the inventor was so developed as to make a complete system of geometry, including its application to trigonometry and mechanics.

One of the earliest statements on record, in regard to the geometrical signification of the even roots of negative numbers, was by Dr. Wallis of Oxford, in his 'Treatise of Algebra', published in 1685; in which he states that *positive* implies one direction along a line, *negative* the opposite direction along the same line, but the $\sqrt{-a}$ cannot be found in the line but above the line, it may be in the same plane.

In the *Philosophical Transactions* (London) for 1806 is a paper by M. Abbe' Bnéé, in which the writer maintained that the $\sqrt{-1}$, as a symbol, denoted perpendicularity—a view which had been suggested by others, but no system ever grew out of it.

The negatives and imaginaries of algebra are not considered as quantities*; they are, primarily, symbols of unexecuted operations, but when they appear in the result of an analytical process, they are to be interpreted, and the interpretation sometimes leads to an enlargement of the language of the problem, and in other cases to a limitation of it. For instance, if the language implies that the results are to be all positive, but a solution shows that there are several sets of results, some of which are positive and others negative, it shows that the language need not be taken literally. Similarly, those conditions which would produce imaginary results, seem to limit the problem; as for instance in the equation to the circle, those values of the coordinates which give imaginary results do not belong to the locus.

The *positives* and *negatives* of algebra are readily geometrized, and since certain real quantities may be represented in forms involving imaginaries, it is natural to seek a geometrical expression for them. Such is the case with Euler's formulas—

$$2\sqrt{-1} \sin x = e^{x\sqrt{-1}} - e^{-x\sqrt{-1}} = 2Shx,$$

$$2 \cos x = e^{x\sqrt{-1}} + e^{-x\sqrt{-1}} = 2Chx,$$

(*Shx* and *Chx* being the modern notation) where $\sin x$ and $\cos x$ are real for all values of x .

The $\sqrt{-1}$ when deduced according to Hamilton's system, has, in that system, a real geometrical signification, and can be conceived of as definitely as an arc, or angle, or any other geometrical magnitude; yet his system does not pretend to geometrize the $\sqrt{-1}$ of algebra. It leaves that symbol, when it is the result of an algebraic operation, as imaginary as ever, as unreal as ever, as ungeometrical as ever, and as Hamilton says† it "represents no real line". I have emphasized this fact, because we occasionally meet with writers of recent date who boldly assert that the $\sqrt{-1}$ of algebra is real and may be represented by a geometrical line; although they are compelled to admit that it is impossible to find a quantity whose square is negative. It is not improbable that many have been misled by the identity of the symbol $\sqrt{-1}$ in the two systems, so that while they see its real character in quaternions, they do not understand why its application is not

*Hamilton's *Lectures on Quaternions*, preface, p. 2.

†Hamilton's *Lectures*, p. 638.

universal; but the expression is the result of very different operations in the two systems. Modern mathematics has given new meanings to old symbols, or, more properly speaking, it has used old symbols to represent new ideas, and thus the same symbol has a double meaning—and in some cases several meanings. Thus, in algebra, dx implies that x is multiplied by d , but in calculus d implies an operation; in the expression d^2x , the figure 2 may imply a square or a second operation; -1 in the expression a^{-1} implies a reciprocal, but in the expression $\sin^{-1}x$ an inverse function; the period (.) is used to denote a decimal fraction, multiplication, or a fluxion; $\beta = 0$ in algebra means that β is zero, but in modern geometry it may represent the equation of a line; &c. Conversely, the same symbol may be put under different forms; thus we have

$$a \div b = \frac{a}{b}; a^{-1} = \frac{1}{a}; dx = \dot{x}; \sqrt{-1} = (-1)^{\frac{1}{2}}; \&c.$$

Hamilton's system of quaternions and the imaginaries of algebra have a relation to each other similar to that between Chemistry and Alchemy. The false science of alchemy gave birth to the true science of chemistry,* but the latter, instead of solving the problem of the former—the transmutation of metals into gold—shows it to be all the more impossible, and has entered new fields of inquiry; so the imaginaries of algebra gave rise to the real system of Quaternions, but instead of geometrizing the former leaves them as imaginary as before and enters its own field of inquiry by developing new processes of analysis.

As soon as chemistry was established there was a deluge of pretenders in science who *attempted* to find the philosopher's stone; so after the establishment of the science of quaternions, there were, and still are, some who hold that *all* imaginaries are so only in name, but are in fact real quantities.

(To be continued.)

DETERMINATION OF A SPHERE WHICH CUTS FIVE GIVEN SPHERES AT THE SAME ANGLE.

BY DR. THOMAS CRAIG, WASHINGTON, D. C.

GENERAL Benjamin Alvord a few days ago attracted my attention to this problem as being one which had, at one time, possessed a peculiar interest

*J. Andrew, in the ninth edition of the *Encyclopædia Britannica*, says, "Alchemy was, we may say, the sickly but imaginative infancy through which modern chemistry had to pass before it attained its majority."

for him on account of its connection with the large class of problems on intersections and tangencies to which he has given so much attention, and from which he has obtained so many valuable results by the elegant processes of synthetic geometry. Gen. Alvord at the same time informed me that he had communicated the problem to the ANALYST, through the agency of Marcus Baker, Esq., and that Mr. Adcock, in the Sept. number of the ANALYST for 1877, had furnished the equations necessary for its solution, but did not give the solution.

Having given the necessary equations, I presume that Mr. Adcock thought it mere journeyman's work to obtain from them the required forms of the unknown quantities. The following solution is extremely simple but I have thought that it might not be destitute of interest to the readers of the ANALYST on account of the symmetry of the expressions and the ease with which the eliminations can be performed.

Call θ the angle of intersection and $(x_1, y_1, z_1), (x_2, y_2, z_2)$ &c., the coordinates of the centers of the given spheres, r_1, r_2, r_3, r_4, r_5 their radii, ρ the radius of the required sphere and (a, β, γ) the coordinates of its center, then we have by equating the expressions for the distances between the centers of any one sphere and the required sphere, in polar and rectangular coordinates

$$\cos \theta = \frac{(x_i - a)^2 + (y_i - \beta)^2 + (z_i - \gamma)^2 - r_i^2 - \rho^2}{2r_i \rho}, \quad (1)$$

where i has the values 1, 2, 3, 4, 5.

These are the expressions given by Mr. Adcock for the determination of the four unknown quantities (a, β, γ, ρ) ; after these are determined simple substitution will give a value of $\cos \theta$ for any given system of five spheres.

For the determination of (a, β, γ, ρ) we have the equations

$$\begin{aligned} r_2[(x_1 - a)^2 + (y_1 - \beta)^2 + (z_1 - \gamma)^2 - r_1^2 - \rho^2] &= r_1[(x_2 - a)^2 + (y_2 - \beta)^2 + (z_2 - \gamma)^2 - r_2^2 - \rho^2], \\ r_3[(x_1 - a)^2 + (y_1 - \beta)^2 + (z_1 - \gamma)^2 - r_1^2 - \rho^2] &= r_1[(x_3 - a)^2 + (y_3 - \beta)^2 + (z_3 - \gamma)^2 - r_3^2 - \rho^2], \\ r_4[(x_1 - a)^2 + (y_1 - \beta)^2 + (z_1 - \gamma)^2 - r_1^2 - \rho^2] &= r_1[(x_4 - a)^2 + (y_4 - \beta)^2 + (z_4 - \gamma)^2 - r_4^2 - \rho^2], \\ r_5[(x_1 - a)^2 + (y_1 - \beta)^2 + (z_1 - \gamma)^2 - r_1^2 - \rho^2] &= r_1[(x_5 - a)^2 + (y_5 - \beta)^2 + (z_5 - \gamma)^2 - r_5^2 - \rho^2]. \end{aligned} \quad (2)$$

Expand these expressions and for brevity write

$$\begin{aligned} r_1 r_2 (r_2 - r_1) + r_2 (x_1^2 + y_1^2 + z_1^2) - r_1 (x_2^2 + y_2^2 + z_2^2) &= P_1, \\ \vdots & \\ r_1 r_5 (r_5 - r_1) + r_5 (x_1^2 + y_1^2 + z_1^2) - r_1 (x_5^2 + y_5^2 + z_5^2) &= P_5, \end{aligned}$$

P_1, P_2 &c., being of course determinate constants. We find now

$$\begin{aligned} r_2-r_1)(\alpha^2+\beta^2+\gamma^2-\rho^2)-2[(r_2x_1-r_1x_2)\alpha+(r_2y_1-r_1y_2)\beta+(r_2z_1-r_1z_2)\gamma] &= P_1 \\ r_3-r_1)(\alpha^2+\beta^2+\gamma^2-\rho^2)-2[(r_3x_1-r_1x_3)\alpha+(r_3y_1-r_1y_3)\beta+(r_3z_1-r_1z_3)\gamma] &= P_2 \\ r_4-r_1)(\alpha^2+\beta^2+\gamma^2-\rho^2)-2[(r_4x_1-r_1x_4)\alpha+(r_4y_1-r_1y_4)\beta+(r_4z_1-r_1z_4)\gamma] &= P_3 \\ r_5-r_1)(\alpha^2+\beta^2+\gamma^2-\rho^2)-2[(r_5x_1-r_1x_5)\alpha+(r_5y_1-r_1y_5)\beta+(r_5z_1-r_1z_5)\gamma] &= P_4 \end{aligned} \quad (3)$$

Divide these through by r_2-r_1 , r_3-r_1 , r_4-r_1 , r_5-r_1 respectively and place

$$\frac{P_1}{r_2-r_1} = Q_1, \quad \frac{P_2}{r_3-r_1} = Q_2 \text{ \&c.};$$

then subtract the second equation from the first, the third from the second and so on: by this means we eliminate the quantity

$$\alpha^2 + \beta^2 + \gamma^2 - \rho^2$$

and in fact eliminate ρ altogether and have left three equations containing three unknown quantities α , β , γ .

The result of the first substitution is of the form

$$\frac{[r_1^2(x_3-x_2)+r_1r_2(x_1-x_3)+r_1r_3(x_2-x_1)]\alpha}{(r_3-r_1)(r_2-r_1)} + \dots = Q_1 - Q_2,$$

the terms in β and γ are obtained by merely changing x into y and z respectively. The next equations have their first terms in the forms

$$\frac{[r_1^2(x_4-x_3)+r_1r_3(x_1-x_4)+r_1r_4(x_3-x_1)]\alpha}{(r_3-r_1)(r_4-r_1)} + \dots = Q_2 - Q_3,$$

$$\frac{[r_1^2(x_5-x_4)+r_1r_4(x_1-x_5)+r_1r_5(x_4-x_1)]\alpha}{(r_4-r_1)(r_5-r_1)} + \dots = Q_3 - Q_4.$$

These equations can all be given in the determinant form as follows:

$$\begin{aligned} \begin{vmatrix} r_1 & r_2 & r_3 \\ x_1 & x_2 & x_3 \\ r_1 & r_1 & r_1 \end{vmatrix} \alpha + \begin{vmatrix} r_1 & r_2 & r_3 \\ y_1 & y_2 & y_3 \\ r_1 & r_1 & r_1 \end{vmatrix} \beta + \begin{vmatrix} r_1 & r_2 & r_3 \\ z_1 & z_2 & z_3 \\ r_1 & r_1 & r_1 \end{vmatrix} \gamma &= (Q_2 - Q_1)(r_2 - r_1)(r_3 - r_1), \\ \begin{vmatrix} r_1 & r_3 & r_4 \\ x_1 & x_3 & x_4 \\ r_1 & r_1 & r_1 \end{vmatrix} \alpha + \begin{vmatrix} r_1 & r_3 & r_4 \\ y_1 & y_3 & y_4 \\ r_1 & r_1 & r_1 \end{vmatrix} \beta + \begin{vmatrix} r_1 & r_3 & r_4 \\ z_1 & z_3 & z_4 \\ r_1 & r_1 & r_1 \end{vmatrix} \gamma &= (Q_3 - Q_2)(r_3 - r_1)(r_4 - r_1), \\ \begin{vmatrix} r_1 & r_4 & r_5 \\ x_1 & x_4 & x_5 \\ r_1 & r_1 & r_1 \end{vmatrix} \alpha + \begin{vmatrix} r_1 & r_4 & r_5 \\ y_1 & y_4 & y_5 \\ r_1 & r_1 & r_1 \end{vmatrix} \beta + \begin{vmatrix} r_1 & r_4 & r_5 \\ z_1 & z_4 & z_5 \\ r_1 & r_1 & r_1 \end{vmatrix} \gamma &= (Q_4 - Q_3)(r_4 - r_1)(r_5 - r_1). \end{aligned} \quad (4)$$

The coefficients of α , β , γ are constants and of course can be readily computed, for any particular case, by merely expanding these determinants; the same is true of the quantities on the right hand side of the equations, they are constants and readily computed. Call the values of the determinants in the first second and third equations, respectively,

$$\begin{aligned} A_1, B_1, C_1, \\ A_2, B_2, C_2, \\ A_3, B_3, C_3, \end{aligned}$$

and denote the second members of the equations by

$$D_1, D_2, D_3,$$

respectively, then these are

$$\begin{aligned} A_1\alpha + B_1\beta + C_1\gamma &= D_1, \\ A_2\alpha + B_2\beta + C_2\gamma &= D_2, \\ A_3\alpha + B_3\beta + C_3\gamma &= D_3; \end{aligned} \quad (5)$$

writing for brevity Δ = the determinants of the coefficients A, B, C , i. e.,

$$\Delta = \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix}$$

we have by direct solution

$$\left. \begin{aligned} \alpha &= \frac{1}{\Delta} \begin{vmatrix} D_1 & B_1 & C_1 \\ D_2 & B_2 & C_2 \\ D_3 & B_3 & C_3 \end{vmatrix}, \\ \beta &= \frac{1}{\Delta} \begin{vmatrix} C_1 & D_1 & A_1 \\ C_2 & D_2 & A_2 \\ C_3 & D_3 & A_3 \end{vmatrix}, \\ \gamma &= \frac{1}{\Delta} \begin{vmatrix} A_1 & B_1 & D_1 \\ A_2 & B_2 & D_2 \\ A_3 & B_3 & D_3 \end{vmatrix}. \end{aligned} \right\} \quad (6)$$

Nothing in the second members of these equations but known quantities, therefore the center of the sphere is determined. In order to find the radius ρ , it is only necessary to substitute these values of α, β, γ in any one of equations (2), and solve the result for ρ^2 . In fact we have

$$\rho^2 = \frac{r_2[(x_1 - \alpha)^2 + (y_1 - \beta)^2 + (z_1 - \gamma)^2] - r_1[(x_2 - \alpha)^2 + (y_2 - \beta)^2 + (z_2 - \gamma)^2]}{r_2 - r_1}.$$

The position of the center and the radius of the required sphere are now found. If it is desired to find at what angle the sphere intersects the five given ones, the quantities α, β, γ and ρ must be substituted in equation (1) and the result will be the cosine of the required angle.

It is worth while to observe that the coefficients of α, β, γ in the preceding linear equations are all derived from the same determinant, are in fact second minors of

$$V = \begin{vmatrix} r_1 & r_2 & r_3 & r_4 & r_5 \\ x_1 & x_2 & x_3 & x_4 & x_5 \\ y_1 & y_2 & y_3 & y_4 & y_5 \\ z_1 & z_2 & z_3 & z_4 & z_5 \\ r_1 & r_1 & r_1 & r_1 & r_1 \end{vmatrix}$$

This consideration would enable us to replace A, B, C by certain second differential coefficients of V , but it is not necessary to go into that.

THE PRUNOID OR PLUM CURVE.

BY PROFESSOR L. G. BARBOUR, RICHMOND, KENTUCKY.

IN a former number of the ANALYST (p. 111, Vol. VI) I had occasion to discuss the Polar Equation,

$$\rho = \frac{m + \cos \theta}{m^2 - 1}.$$

Making $m > 1$, we found as a result what was called the Curve of Capture. At the time, I did not intend to refer to this equation again, but some results of further consideration seem to me worthy the attention of mathematicians.

I. Let $m = 1$; then $\rho = \infty$ for every value of θ except 180° , and we have a circle of infinite radius for all values of θ other than 180° . For $\theta = 180^\circ$ $\rho = 0 \div 0$, i. e., is indeterminate. The true value, however, is easily obtained:

$$\rho = \frac{m - 1}{m^2 - 1} = \frac{1}{m + 1} = \frac{1}{2}. \text{ This gives an isolated point.}$$

II. A much more remarkable result emerges by taking $m < 1$. In drawing the curve it is necessary to observe the signs closely. For a particular case, take $m = \frac{1}{2}$. Let $\theta = 0$; then is

$$\rho = \frac{m + \cos \theta}{m^2 - 1} = \frac{1.5}{-.75} = -2.$$

Lay off $BA = 2$ in. to the left of B .

Let $\theta = 90^\circ$; $\therefore \rho = BF = -\frac{2}{3}$.

" $\theta = 180^\circ$; " $\rho = BC = \frac{2}{3}$.

" $\theta = 270^\circ$; " $\rho = BE = -\frac{2}{3}$.

" $\theta = 360^\circ$; " $\rho = BA = -2$.

For all values of θ between 0° and 90° we get points of the curve between A and F ; and for all values between 270° and 360° , we get points between E and A . Let us now determine the points B and G on $FBGC$.



At B , $\rho = \frac{m + \cos \theta}{m^2 - 1} = 0$; $\therefore m = .5 = -\cos \theta$; $\therefore \theta = 120^\circ$. Designating by G the highest point between B and C , let fall GN perpendicular to AB . GN is a maximum, $GB = \rho$. $GN = \rho \sin \angle GBN = \rho \sin \theta$.

Generalizing we get $\frac{m + \cos \theta}{m^2 - 1} \sin \theta$, for the distance of the highest or lowest

point of the curve from the line AD ; $\therefore m \sin \theta + \cos \theta \sin \theta =$ a maximum or minimum, $\therefore m \cos \theta + \cos^2 \theta - \sin^2 \theta = 0$. Therefore, for $m = .5$,

$$\cos^2 \theta + .25 \cos \theta = .5,$$

$$\therefore \cos \theta = .5931 \text{ or } -.8931;$$

$$\therefore \theta = 53^\circ 37' \text{ or } 147^\circ 28'.$$

The former value belongs to K ; the latter to G . It is evident that the curve is symmetrical in reference to the initial line AB ; for, by using negative angles, $-\theta$ for $+\theta$, we construct the curve from A through I , E , L , B and H to C , and $\cos(-\theta) = \cos \theta$, always. Hence I and H are symmetrical with K and G .

To find the greatest and least values of $\rho = \frac{m + \cos \theta}{m^2 - 1}$. Omit the denominator, and $m + \cos \theta$ is greatest when $\theta = 0$, viz., $.5 + 1 = 1.5$; it is least algebraically when $\theta = 180^\circ$, viz., $.5 - 1 = -.5$, and it is least numerically when $\theta = 120^\circ$ or 240° , viz., $.5 - .5 = 0$. The successive points are A , C and the double point B . The corresponding values of ρ are -2 , $+\frac{1}{2}$ and 0 .

To find the angles at which the curve or its tangent cuts the initial line AD . The trigonometrical tangent of the angle formed by a tangent line to the curve and the radius vector to the point of contact is

$$\frac{\rho d\theta}{d\rho} = \frac{m + \cos \theta}{-\sin \theta}.$$

This becomes infinite only when $\sin \theta = 0$; $\therefore$ for $\theta = 0^\circ$ or 180° , the tangent $= \infty$; and therefore the curve is perpendicular to AD at C and A , and no where else. Also, the tangent to the curve makes a right angle with the radius vector at no other points.

To find the point where the angle between the tangent line and the radius vector is zero. Let $\frac{m + \cos \theta}{-\sin \theta} = 0$; $\therefore \cos \theta = -m = -.5$, $\therefore \theta = 120^\circ$ or 240° . This is at B .

Let the radius vector start from the position BA . The tangent line at A is at right angles with the r. v. As the r. v. revolves through K , F , M , to B , the angle increases from 90° to 180° . Setting out anew at B and proceeding by way of G to C , the angle may be conceived as beginning with 0° and reaching 90° again at C . It then increases up to 180° or coincidence again at the multiple point B ; and once more setting out as from zero, goes back gradually to 90° at A .

While there are only two points, A and C , at which the tangent line is perp. to the r. v., it is manifest that there are two other points, as L and M , the tangent line through which is perpendicular to AD .

To find the point M . $DBQ' = \theta$. BMR is the angle between the tang. line and the radius vector. RM prolonged to D' is perpendicular to BD .

Tang. $BMR = -\text{tang. } BMD' = -\frac{1}{\tan MBD} = \frac{1}{\tan \theta}$. But we have already seen that

$$\tan BMR = -\frac{m + \cos \theta}{\sin \theta}; \therefore \frac{1}{\tan \theta} = \frac{\cos \theta}{\sin \theta} = -\frac{m + \cos \theta}{\sin \theta}.$$

$$\therefore 2 \cos \theta = -m = -.5, \therefore \cos \theta = -.25; \therefore \theta = 104^\circ 29'.$$

$$BM = \rho = \frac{m + \cos \theta}{m^2 - 1} = \frac{.5 - .25}{-.75} = -\frac{1}{3}.$$

Tests.—As it is always desirable to test our work, we notice that at the points I , G , H and K , the tangent line is parallel to the initial line AB . For example, let TI be the tang. line at I , and be parallel to BD . Then

$$\tan TIB = -\tan IBD = -\tan \theta;$$

$$\therefore -\frac{m + \cos \theta}{\sin \theta} = -\frac{\sin \theta}{\cos \theta}; \therefore \cos^2 \theta + m \cos \theta = \sin^2 \theta.$$

This is the previous result over again.

There is a neat test also for the points A , C , L and M . Let us suppose it undetermined whether the farthest point to the left of B is on the initial line. It might be, say, at W . Draw WV perpendicular to BA at V .

$$BV = \rho \cos \theta = \frac{m + \cos \theta}{m^2 - 1} \cos \theta; \therefore m \cos \theta + \cos^2 \theta = \text{a maximum.}$$

$$\therefore -m \sin \theta - 2 \cos \theta \sin \theta = 0.$$

This equation may be verified either by $\sin \theta = 0$, and therefore $\theta = 0^\circ$ or 180° , giving the points A and C ; or by $-m = -2 \cos \theta$, $\therefore \cos \theta = -.25$, giving L and M , as before.

$$\text{Subtangents.}—S. T. = \frac{\rho^2 d\theta}{d\rho} = -\frac{(m + \cos \theta)^2}{(m^2 - 1) \sin \theta}. \text{ When } \theta = 0^\circ \text{ or } 180^\circ$$

s. t. = ∞ . When $\cos \theta = -m = -.5$, $\theta = 120^\circ$ or 240° ; then s. t. = 0.

This is at B , where the tangent and the radius vector coincide. The propriety of testing this will appear in *The quadrature of the area*.

The differential of the area of a polar curve is $\frac{1}{2} \rho^2 d\theta$;

$$\therefore dA = \frac{m^2 + 2m \cos \theta + \cos^2 \theta}{2(m^2 - 1)^2} d\theta;$$

$$\therefore A = \frac{1}{2(m^2 - 1)^2} \left\{ m^2 \theta + 2m \sin \theta + \frac{\theta}{2} + \frac{\sin 2\theta}{4} \right\} + C.$$

When $\theta = 0$, the area = 0, $\therefore C = 0$.

$$\text{" } \theta = 120^\circ, A = \frac{3}{2}(2\pi + 3 \sin \theta) = 1.9736.$$

$$\text{" } \theta = 180^\circ, A = \frac{3}{2} \cdot 3\pi = \frac{9}{2}\pi = 2.0944.$$

The first of these areas is *BAKFM*. The second is *BAKFMBGC*, i. e., one half of the whole area and one half of the inner area *BGCH*. Hence one half of the inner area is .1208. (The recognized notation for this is

$$\int_{\theta=120^{\circ}}^{\theta=180^{\circ}} \frac{1}{2} \rho^2 d\theta = 2.0944 - C,$$

in which case $C = 1.9736$; area = .1208.) The whole area then is 3.9472, and the inner area is .2416.

Cubature of Volume.—Let any radius vector, as *BP*, be revolved about the initial line *AD*, describing the surface of a cone. Then let another radius vector from *B* to a point on the curve indefinitely near *P*, describe the surface of another cone. The volume between these surfaces will have a thickness of zero at *B*, and of $\rho d\theta$ at *P*. The base = $2\pi PQ\rho d\theta = 2\pi\rho^2 \sin\theta d\theta$, and the volume = $\frac{1}{3}\rho^2 2\pi\rho^2 \sin\theta d\theta = \frac{2}{3}\pi\rho^3 \sin\theta d\theta$. For all values of ρ from 0 to -2 its intrinsic sign is minus; and we have to integrate $-\frac{2}{3}\pi\rho^3 \sin\theta d\theta$,

$$\begin{aligned} &= \int -\frac{2\pi}{3(m^2-1)^{\frac{3}{2}}} (m^2 + 3m^2 \cos\theta + 3m \cos^2\theta + \cos^3\theta) \sin\theta d\theta \\ &= \frac{2\pi}{3(m^2-1)^{\frac{3}{2}}} \left(m^2 \cos\theta + 3m^2 \frac{\cos^2\theta}{2} + m \cos^3\theta + \frac{\cos^4\theta}{4} \right) + C \\ &= -\frac{128\pi}{81} \left(\frac{1}{8} \cos\theta + \frac{3}{8} \cos^2\theta + \frac{1}{2} \cos^3\theta + \frac{1}{4} \cos^4\theta \right) + C. \end{aligned}$$

For $\theta = 90^\circ$, $\cos\theta = 0$. Take the volume here = 0 and we get $C = 0$. For $\theta = 0^\circ$, $\cos\theta = 1$; $\therefore V = -\frac{128\pi}{81} \pi \left(\frac{1}{8} + \frac{3}{8} + \frac{1}{2} + \frac{1}{4} \right) = -\frac{160\pi}{81}$. The minus sign indicates that this volume is to the left of a plane passing through *EF* perpendicular to *AB*.

For $\theta = 120^\circ$, $\cos\theta = -\frac{1}{2}$; $\therefore V = -\frac{128\pi}{81} \pi \left(-\frac{1}{8} + \frac{3}{8} - \frac{1}{2} + \frac{1}{4} \right) = \frac{32\pi}{81}$. The plus sign shows that this volume is on the right of the aforesaid plane. The arithmetical sum of these two volumes is the entire volume described by *AFMB*, and is $= 2\pi =$ vol. of a cylinder whose altitude, and the diameter of whose base, $= 2 = AB$. Like the cylinder, its volume $= \frac{2}{3}$ of a sphere whose diameter is *AB*.

Interior Volume.—By this I mean the vol. described by *BGC*. We integrate between the limits $\theta = 120^\circ$, $\cos\theta = -\frac{1}{2}$ and $\theta = 180^\circ$, $\cos\theta = -1$.

$$\therefore \int_{\cos=-\frac{1}{2}}^{\cos=-1} = 0 - \frac{2\pi}{81} = -\frac{2\pi}{81}.$$

Hence the interior vol. lies wholly on the left of *EBF*, and is equal to the button-shaped vol. described by *BMF*.

Moreover, each of these two volumes is equal to the vol. of a cone whose altitude is *BC* or *BF*, i. e., $\frac{2}{3}$, and the diameter of whose base is the same; and the sum of the two volumes is equal to that of a sphere whose diameter equals *BC* or *BF*; so that the three round bodies are severally represented.

Test.—The following test is subjoined partly to certify the reader of the accuracy of the work thus far, and partly to present a neat method of integration which chances to offer itself. Resume the formulæ

$$dV = -\frac{2}{3}\pi\rho^3\sin\theta d\theta, \quad \rho = \frac{m + \cos\theta}{m^2 - 1}.$$

Differentiating, $\sin\theta d\theta = -(m^2 - 1)d\rho = \frac{2}{3}d\rho$;

$$\therefore dV = -\frac{1}{2}\pi\rho^3 d\rho; \therefore V = -\frac{1}{8}\pi\rho^4 + C.$$

Pursuing the same plan as before, we make $V=0$ for $\theta=90^\circ$; $\therefore \rho = -\frac{2}{3}$.

Hence the integral is $-\frac{2}{81}\pi + C$; $\therefore C = \frac{2}{81}\pi$.

For $\rho = -2$, $V = -2\pi + \frac{2}{81}\pi = -\frac{160}{81}\pi$. For $\theta = 120^\circ$, $\rho = 0$;

$\therefore V = C = \frac{2}{81}\pi$. For $\theta = 180^\circ$, $\rho = +\frac{2}{3}$; $\therefore V = -\frac{2}{81}\pi + \frac{2}{81}\pi$.

The former of these, $-\frac{2}{81}\pi$, is the interior volume, and the latter is the small exterior volume on the right of the plane passing through *EBF*.

The singular resemblance of this curve to a section of peach or plum inclusive of the seed suggests the name of Prunoid or Plum Curve.

NOTE.—Looking though Haddon's "Examples and Solutions in Diff. Calc." (Weale's Series), I find two curves which may be regarded as particular cases of the Prunoid.

One, which he calls the Trisectrix, has for its equation, $r = a(2\cos\theta \pm 1)$.

In the equation of the Prunoid, I originally introduced the constant a , but afterward for simplicity's sake treated it as unity. But as Haddon employs it, I restore this constant and write the equation of the Prunoid,

$$r = a \frac{\pm m + \cos\theta}{\pm m^2 \pm 1}.$$

Let $m = \frac{1}{2}$ and $a = \frac{2}{3}a'$; also let the sign of m^2 be minus, then

$$r = a'(2\cos\theta \pm 1).$$

Haddon's second curve has $r = a \cos \frac{1}{3}\theta$. Here $m = 0$, $\theta = \frac{1}{3}\theta'$; sign of 1 is plus; $\therefore$

$$r = a \cos \frac{1}{3}\theta'.$$

The former of these two curves seems, by a remarkable coincidence, identical with the Prunoid for $m = \frac{1}{2}$ as drawn and described in the foregoing article, except that the curve lies in an opposite direction. But even this discrepancy can be eliminated by writing, as Haddon suggests,

$$r = a(2\cos\theta - 1); \text{ also make } a = \frac{2}{3}.$$

For instance, Haddon finds the distance $AB = 3a$. I get it $-2a$. He gives no account of the origin of the equation in the first case, but in the second the equation grows out of this problem:

"Two points start from the opposite extremities of the diameter of a circle, and move with uniform velocity in the same direction round the circumference; their velocities are in the ratio of 2 : 1. Determine the locus

of the bisection of the chords which join the position of the two points, and find the polar subtangent of the curve."

The curve lies in a contrary direction to the figure in this article, and the pole is at *C*. $CA = 2CB$; therefore the curve is not identical with ours.

The equation of the Cardioid is $r = a(1 + \cos \theta)$. Write the Prunoid thus, $a \frac{m + \cos \theta}{m^2 + 1}$. Let $m = 1$ and $a = 2a'$; then $a \frac{m + \cos \theta}{m^2 + 1} = a'(1 + \cos \theta)$.

As its name indicates, the Cardioid is shaped like a heart.

NOTE BY E. L. DE FOREST.—SINCE the paper concluded at page 9 has been in the printer's hands, I have found that the theorems there given relative to the lever arm and the radius of gyration, are comprehended under two more general ones which may be briefly stated as follows.

If any number of polynomials are multiplied together, the lever arm of the coefficients in the product, about the first one as a fulcrum, is equal to the sum of the lever arms in all the factors; and the square of the radius of gyration in the product, about the centre of parallel forces as an axis, is equal to the sum of the squares of the radii of gyration in all the factors. Proof of these and some other properties must be reserved for a future article.

A correction should be made in the sentence at middle of page 8, which is so worded as to be not strictly true nor consistent with what had been said before.

The centre of parallel forces does not precisely "coincide" with the greatest coefficient unless the rank gm of that centre is a whole number. In all other cases, their positions differ by a fraction of the interval Δx . When this difference exists, the quadratic mean error ϵ is to be obtained by reckoning the distances of the terms from the centre of forces, and not from the greatest term. The difference vanishes when m is infinite and Δx becomes dx .

EXTENSION OF THE METHOD OF LEAST SQUARES TO ANY NUMBER OF VARIABLES.

BY R. J. ADCOCK, ROSEVILLE, ILL.

ANY point, line or surface is in its most probable position when the sum of the squares of the normals upon it, from the given points, is a minimum (ANALYST, Vol. IV, p. 184). That is, the most probable position

SOLUTIONS OF PROBLEMS IN NUMBER SIX, VOL. VI.

SOLUTIONS of problems in No. 6, Vol. VI, have been received as follows;

From R. J. Adcock, 284, 286; Prof. W. P. Casey, 283; George Eastwood, 282, 283, 287; Prof. E. W. Hyde, 283, 287; George H. Harvill, 287; O. L. Mathiot, 283; Prof. E. B. Seitz, 283, 284, 285, 286, 287.

Prof. Casey sent solutions of 276, 277 and 280, in time for the notice in last No., but they were accidentally overlooked; and W. E. Heal sent a good algebraic solution of 277 which was received too late for the notice in last No.

281. No solution has been received.

282. "Suppose a manufacturer of textile fabrics, who can operate sixty looms, ascertains by trial, (1) that when he makes his weavers sure of a dollar a day with constant work, then each loom, when it runs a whole day, will net three dollars; (2) that, from the nature of the work to be done, all the looms are never in running order at the same time; (3) that at any time it is equally possible that one, two, or three, or any number of them may be standing for repairs. Now, under these conditions, it is required to assign the least number of weavers to be placed under permanent pay, so that the average daily profits may be the greatest possible."

SOLUTION BY GEORGE EASTWOOD, SAXONVILLE, MASS.

Represent the number of weavers required by x and the average daily profits by y . Then,

If 1	loom is out of order, the profits are evidently	$3x$,
" 2	looms are " " " " " "	$3x$,
" $60-x$	" " " " " " " "	$3x$,
" $60-x+1$	" " " " " " " "	$3(x-1)-1$,
" $60-x+2$	" " " " " " " "	$3(x-2)-2$,
" $60-1$	" " " " " " " "	$3-(x-1)$,
" 60	" " " " " " " "	$-x$.

The sum of these divided by 60 gives, for the daily average,

$$y = \frac{1}{60}[3(60-x)x + \frac{3}{2}(x-1)x - \frac{1}{2}(x+1)x]$$

$$= \frac{1}{60}(178x - 2x^2);$$

$$\therefore dy = \frac{1}{60}d.(178x - 2x^2) = 0,$$

which gives $x = \frac{1}{2}(178) = 44 + \frac{1}{2}$; hence the least number of weavers to be kept on permanent pay is 44, and the average daily profits of the manufacturer is \$66.008 $\frac{1}{2}$.

Instead of 60 looms, let there be n looms, and let us suppose the profit on each loom, when it works a whole day, to be m times the pay of the weaver. Then, x and y being as before,

If 1 loom is not in working condition the profits are mx ,
 " 2 looms are " " " " " " " mx ,

 " $n-x$ " " " " " " " mx ,
 " $n-x+1$ " " " " " " " $m(x-1)-1$,
 " $n-x+2$ " " " " " " " $m(x-2)-2$,

 " $n-1$ " " " " " " " $m-(x-1)$,
 " n " " " " " " " $-x$.

Summing and dividing by n , the average profit will be

$$\begin{aligned} y &= \frac{1}{n} \left[m(n-x)x + \frac{m(x-1)x}{2} - \frac{(x+1)x}{2} \right] \\ &= \frac{1}{n} \left[\left(mn - \frac{m+1}{2} \right) x - \frac{m+1}{2} x^2 \right] \\ &= \frac{m+1}{2n} \left[\left(\frac{mn}{m+1} - \frac{1}{2} \right)^2 - \left(x - \frac{mn}{m+1} + \frac{1}{2} \right)^2 \right]. \end{aligned}$$

This is the greatest possible when

$$x = \frac{mn}{m+1} - \frac{1}{2},$$

or the integer nearest to this value of x . Hence the general value of y is

$$\frac{m+1}{2n} \left(\frac{mn}{m+1} - \frac{1}{2} \right)^2.$$

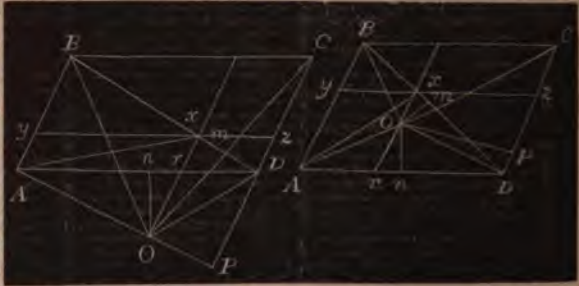
283. "If from any point in the plane of a parallelogram perpendiculars are let fall on the diagonal and on the two sides that contain this diagonal, the product of the diagonal by its perpendicular is equal to the sum of the products of the sides by their respective perpendiculars if the point falls outside of the parallelogram, or to their difference if it lies within the parallelogram.

Varignon's Theorem: *Mecanique Analytique*, Tome I, p. 13."

SOLUTION BY PROF. W. P. CASEY, SAN FRANCISCO, CAL.

Let $ABCD$ be the parallelogram and O the point; draw the perpendiculars Om , Op , On ; and draw AO , BO , CO and DO . Through O draw Ox

parallel to AB , and through x draw yz parallel to AD . Now parallelog'm Cx = parallelogram Ax . To each add the parallelogram Dx , and $\therefore$ par. Az = par. Cx , or triangle AxD = triangle DOC ; but triangle OBx = $\triangle OAx$, to each add triangle DOx , and therefore $\triangle BOD$ = quad. $AxDO$ = $\triangle AxD \pm \triangle AOD$ = $\triangle DOC \pm \triangle AOD$; that is, $BD \times Om = CD \times Op \pm AD \times On$. Q. E. D.



This theorem is of great use in the doctrine of statical moments.

SOLUTION (BY QUATERNIONS) BY PROF. E. W. HYDE, CIN., OHIO.

Let $AB = \beta$, $AD = \alpha$ and $AP = \rho$; then we have at once

$$Va\rho + VB\rho = V(a + \beta)\rho,$$

which contains the proof, since $Ua\rho = U\beta\rho = U(a + \beta)\rho$, and therefore

$$Ta\rho + T\beta\rho = T(a + \beta)\rho,$$

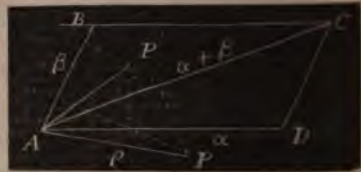
when P is outside the parallelogram; and

$$Ua\rho = -U\beta\rho = \pm U(a + \beta)\rho,$$

and therefore

$$Ta\rho - T\beta\rho = \pm T(a + \beta)\rho$$

when P is inside. These tensors are the products required.



SOLUTION BY O. L. MATHIOT, BALTIMORE, MARYLAND.

Referring to the marginal diagram, we find from the similar triangles ADH , MNV and HRV

$$MV = \frac{AH.MN}{AD},$$

$$RV = MR - MV = MR - \frac{AH.MN}{AD},$$

$$RH = \frac{AD.RV}{DH} = \frac{AD.RM - AH.MN}{DH}.$$

And from the similar triangles HDL , SHT we find

$$RS = (DL.RM) \div DH,$$

$$HS = HR - RS = \frac{AD.RM - AH.MN - DL.RM}{DH},$$



$$ST (=MZ) = \frac{DH.HS}{HL} = \frac{AD.RM - AH.MN - DL.RM}{HL}.$$

Substituting for AD its equivalent, $AL + DL$ we have

$$MZ = \frac{AL.RM + DL.RM - AH.MN - DL.RM}{HL},$$

$$\therefore MZ.HL = AL (=CH).RM - AH.MN,$$

$$\text{or } AH.MN = CH.RM - HL.MZ;$$

which was to be proved.

In a similar manner the prop. may be proved for an external point.

[Prof. Hall writes, in relation to this theorem, "Lagrange says: 'Varignon shows, by a very simple construction, that in forming the triangles which have the diagonal and the two sides for bases, and the given point for a common summit, the triangle formed on the diagonal is, in the first case, equal to the sum, and, in the second case, to the difference of the two triangles formed on the sides; that which is in itself a beautiful theorem of geometry, independently of its application to mechanics.'"]

At the risk of seeming tedious in the discussion of this proposition, we submit the following construction which presents a very direct and easy method of demonstration.

Let $ABCD$ represent the parallelogram, and let P be either an external or an internal point; also, let E denote the intersection of DP , or DP produced, with the side BC . Then, because the sum of the areas of the triangles ABP and CDP is half the area of the parallelogram, we may easily prove that the triangles BCP and AEP have equal areas.



Now if from the quadrilateral $DABP$ we take the triangle ABD , which is half the parallelogram, we have left the triangle BDP , which has for its base the diagonal BD and for its summit the point P . And if from the same quadrilateral, $DABP$, we take the triangle ADP $\mp$ the triangle AEP (which difference or sum is half the parallelogram), we have left the triangles $ABP \pm AEP$, which sum or difference is therefore equal to the triangle BDP . But we have shown that the triangle AEP is equal to the triangle BCP ; therefore the triangle BDP is equal to the triangle $ABP \pm$ the triangle BCP .—Ed.]

284. " M , N and P , Q are two pairs of random points within a circle, but on opposite sides of a given diameter AB : find the chance that the sum of the squares of the chords through M , P and N , Q is less than the square of the diameter."

SOLUTION BY PROF. E. B. SEITZ, KIRKSVILLE, MO.

Let CD and EF be the chords through M, P and N, Q , intersecting AB in R and S , and O the center of the circle. Draw OH and OK perpendicular to CD and EF .



Now the sum of the squares of CD and EF is less than the square of AB , if the sum of the angles COD and EOF is less than two right angles.

Let $OA = r$, $RM = w$, $RC = w'$, $RP = x$, $RD = x'$, $SN = y$, $SE = y'$, $SQ = z$, $SF = z'$, $\angle COH = \theta$, $\angle EOK = \varphi$, $\angle ROH = \psi$, and $\angle SOK = \mu$. Then we have

$$w' = r(\sin \theta + \cos \theta \tan \psi),$$

$$x' = r(\sin \theta - \cos \theta \tan \psi),$$

$$y' = r(\sin \varphi + \cos \varphi \tan \mu),$$

$$z' = r(\sin \varphi - \cos \varphi \tan \mu).$$

An element of surface at M is $r \sin \theta d\theta dw$; at P , $(w + x) d\psi dx$; at N , $r(\sin \varphi d\varphi dy$; and at Q , $(y + z) d\mu dz$. The limits of θ are 0 and $\frac{1}{2}\pi$; of φ , 0 and $\frac{1}{2}\pi - \theta$; of ψ , $-\theta$ and θ and doubled; of μ , $-\theta$ and θ , and doubled; of w , 0 and w' ; of x , 0 and x' ; of y , 0 and y' ; and of z , 0 and z' .

Hence, since the whole number of ways the four points can be taken is $\frac{1}{16}\pi^4 r^8$, the required chance is

$$\begin{aligned} P &= \frac{64}{\pi^4 r^8} \int_0^{\frac{1}{2}\pi} \int_0^{\frac{1}{2}\pi - \theta} \int_{-\theta}^{\theta} \int_{-\varphi}^{\varphi} \int_0^{w'} \int_0^{x'} \int_0^{y'} \int_0^{z'} r \sin \theta d\theta r \sin \varphi d\varphi \\ &\quad \times d\psi d\mu dw (w + x) dx dy (y + z) dz \\ &= \frac{64}{\pi^4} \int_0^{\frac{1}{2}\pi} \int_0^{\frac{1}{2}\pi - \theta} \int_{-\theta}^{\theta} \int_{-\varphi}^{\varphi} (1 - \cos^2 \theta \sec^2 \psi) (1 - \cos^2 \varphi \sec^2 \mu) \sin^2 \theta \sin^2 \varphi \\ &\quad \times d\theta d\varphi d\psi d\mu \\ &= \frac{256}{\pi^4} \int_0^{\frac{1}{2}\pi} \int_0^{\frac{1}{2}\pi - \theta} (\theta - \sin \theta \cos \theta) (\varphi - \sin \varphi \cos \varphi) \sin^2 \theta \sin^2 \varphi d\theta d\varphi \\ &= \frac{16}{\pi^4} \int_0^{\frac{1}{2}\pi} (\pi - 2\theta - \sin 2\theta)^2 (\theta - \sin \theta \cos \theta) \sin^2 \theta d\theta \\ &= \frac{1}{6} + \frac{2}{\pi^2} - \frac{320}{9\pi^4}. \end{aligned}$$

285. "If $E(e, x)$ denote the length of an elliptic arc, semi major axis unity and eccentricity e ; prove, without the aid of the Integral Calculus, that

$$dE(e, x) = dx \sqrt{\left(\frac{1 - e^2 x^2}{1 - x^2} \right)}."$$

SOLUTION BY PROFESSOR SEITZ.

Let $CA = a = 1$, $CB = b$, $CM = x$, $PM = y$,
are $BP = E(e, x)$, $Pp = dE(e, x)$, $np = dx$, $\angle PTM$
 $= Ppn = \theta$. Now by Analytical geometry we have

$$\tan \theta = \frac{b^2 x}{a^2 y} = \frac{x(1-e^2)}{y} = \frac{x\sqrt{1-e^2}}{\sqrt{a^2-x^2}} = \frac{x\sqrt{1-e^2}}{\sqrt{1-x^2}};$$

$$\therefore \sec \theta = \sqrt{1+\tan^2 \theta} = \sqrt{\left(\frac{1-e^2 x^2}{1-x^2}\right)},$$

$$\text{and } dE(e, x) = \sec \theta dx = dx \sqrt{\left(\frac{1-e^2 x^2}{1-x^2}\right)}.$$



236. "Three points are taken at random in a given circle, and a circle is drawn through them. What is the probability that the circle through the random points will be wholly in the given circle?"

SOLUTION BY R. J. ADCOCK, ROSEVILLE, ILL.

Let m represent the number of points on a unit of surface, then is

$\pi m r^2$ = the number of points on a circle, radius r , and

$\pi^3 m^3 r^6$ = " " of positions of three points on circle, radius r ;

$2\pi r \sqrt{m}$ = " " of points on the circumference $2\pi r$

= " " of positions of two points on circumf. and at a constant distance from each other.

$4\pi^2 r^2 m$ = " " of positions of two points on the circumf.;

$8\pi^3 r^3 m \sqrt{m}$ = " " " " of three points on circumf.

$\pi m(r-x)^2$ = " " " " wholly in the given circle, which

a circle of radius x , can have. Hence

$8\pi^4 m^2 \sqrt{m} (r-x)^2 x^3$ = the number of positions which three points can have on the circle r and at the same time on the circumference of the within circle of radius x . Therefore

$$\int_0^r 8\pi^4 m^2 \sqrt{m} (r-x)^2 x^3 \sqrt{m} dx = \frac{2}{15} \pi^4 m^3 r^6,$$

= the number of positions of three points, on circle r , through each of which positions a within circle may be passed. Therefore

$$\frac{\frac{2}{15} \pi^4 m^3 r^6}{\pi^3 m^3 r^6} = \frac{2\pi}{15} = \text{the required probability.}$$

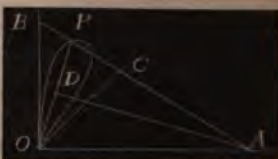
[Mr. Seitz has furnished a very elaborate solution of this problem and

obtains for the required probability $p = \frac{2}{3}$. We have not examined, critically, either solution, and cannot say whether the discrepancy in the results arises from error in reduction or error in logic. If it shall be found that there is error in the logic of the foregoing solution, we will publish Mr. Seitz's solution in No. 2.—Ed.]

287. "A ladder, 40 feet long, stands upright against a vertical wall, on a horizontal plane. A boy ascends the ladder, at a uniform rate, and at the same time the foot of the ladder slides out horizontally at the same uniform rate. How long is the boy's path?"

SOLUTION BY GEORGE EASTWOOD, SAXONVILLE, MASS.

In the annexed diagram, let $AB = a$ represent the ladder and P the position of the boy at any time. Join OP and draw AD perpendicular to it.



Put $OP = r$ and $\angle BOP = \theta$. Then, since AD bisects OP (because $AO = AP$, as per quest.) the $\angle OAD = \angle BOP = \theta$; $\therefore AO = a \cos 2\theta$, and $OP = r = 2OD$; $\therefore r = 2a \sin \theta \cos 2\theta$, the polar equation of the curve.

To find its length: We have

$$\begin{aligned} \frac{ds}{d\theta} &= \left[r^2 + \left(\frac{dr}{d\theta} \right)^2 \right]^{\frac{1}{2}} \\ &= 2a(1 - 12 \sin^2 \theta + 44 \sin^4 \theta - 32 \sin^6 \theta)^{\frac{1}{2}} \\ &= 2a(1 - 6 \sin^2 \theta + 4 \sin^4 \theta + 8 \sin^6 \theta + \dots) \end{aligned}$$

$$\begin{aligned} \text{Whence } s &= 2a \int (1 - 6 \sin^2 \theta + 4 \sin^4 \theta + 8 \sin^6 \theta + \dots) d\theta \\ &= 2a \left[2\theta + \dots - \cos(\sin \theta + \frac{8}{3} \sin^3 \theta + \frac{4}{5} \sin^5 \theta + \dots) \right], \end{aligned}$$

in which the limits of θ are 0 and 45° .

The area of the curve = $.0594a^2$.

[Prof. Hyde solves this problem also by quaternions and obtains for the Scalar equation to the curve

$$\rho^4 - 2l\rho^2 S a \rho - 4lS^3 a \rho = 0;$$

from which he deduces the Cartesian equation,

$$(x^2 + y^2)^2 - 2lx(x^2 + y^2) + 4lx^3 = 0.$$

Mr. Harvill finds the rectangular equation

$$\begin{aligned} (y^2 + x^2)^2 &= 2ax(y^2 - x^2), \text{ and the polar equation} \\ r &= 2a \sqrt{2}(\sin \theta \cos^2 \theta - \sin^3 \theta \cos \theta). \text{—Ed.} \end{aligned}$$

288. No solution of this problem has been received.

PROBLEMS.

289. (Selected) By Prof. M. L. Comstock.—In any spherical triangle, show that

$$\tan c = \frac{\cot A \cot a + \cot B \cot b}{\cot a \cot b - \cos A \cos B}.$$

290. By Geo. Lilley, A. M., Corning, Iowa.—Integrate

$$dy = \frac{dx}{n\epsilon^x - 2(2x+1)},$$

where ϵ is the Napierian base and n a constant.

291. By Prof. W. W. Johnson.—If $ABC, A'B'C'$ are two trirectangular triangles on the surface of a sphere (the letters being arranged in the inverse order of rotation); show that

$$\cos AA' = \cos BB' \cos CC' - \cos B'C \cos BC'.$$

292. By Chas. H. Kummell, U. S. Lake Survey, Detroit, Mich.—Two surveyors measure a plane quadrangular field, one measuring the four sides a, b, c, d with a chain and the other, the angles $(ab), (bc), (cd), (da)$ with a theodolite. From former experience it is known that the first is liable to a probable error of m inches per chain and the other to a probable error of n'' per angle. Required the weights of the linear and angular measurements, also conditions to be fulfilled by the measured quantities in approximate linear form and the analytical formation of the normal equations for determining the most probable corrections to the measured quantities.

293. By R. J. Adcock, Roseville, Ill.—Assuming that the surface of the earth is an ellipsoidal level surface, whose principal diameters are $2a, 2b, 2b_1$, the force of gravity at their extremities, p, q, r ; what then is the force of gravity at any other point whose latitude is l , and longitude, measured from the meridian of one extremity of $2a$, is L ?

294. By Alex. S. Christie, U. S. Coast Surv., Wash., D. C.—One curve rolls upon another; prove that a series of carried parallel curves envelope a series of parallel curves, or, involutes of the same evolute envelope involutes of the same evolute.

295. By Professor Hall.—Given the common astronomical equations

$$\begin{aligned}\tan(\lambda - \varrho) &= \cos i \tan u, \\ \sin \beta &= \sin i \sin u,\end{aligned}$$

eliminate u , and show in this manner that

$$\tan \beta = \tan i \sin(\lambda - \varrho).$$

PUBLICATIONS RECEIVED.

Mathematical Tables, Chiefly to Four Figures, First Series. By JAMES MILLS PEIRCE, University Professor of Mathematics in Harvard University. 8vo. 43 pp. Boston: Ginn and Heath. 1879.

This collection of Tables, consisting of table of logarithms, table of logarithms of sums and differences, table of logarithms of circular functions, table of inverse circular functions, table of logarithms of hyperbolic functions, table of natural sines and cosines, table of natural tangents and cotangents and table of natural secants and cosecants, is made specially interesting and valuable by the very full and lucid explanations of the tables, which occupy 17 pages of the book.

Elements of the Differential Calculus, with Examples and Applications. A Text Book: By W. E. BYERLY, PH. D., Assistant Professor of Mathematics in Harvard University. 258 pp., 8vo. Boston: Published by Ginn and Heath. 1879.

As we could not, by a hasty review, do justice to this very handsome volume, we quote the following paragraph from the Preface, which will sufficiently indicate the character of the book.

"Its peculiarities are the rigorous use of the Doctrine of Limits as a foundation of the subject, and as preliminary to the adoption of the more direct and practically convenient infinitesimal notation and nomenclature; the early introduction of a few simple formulas and methods for integrating; a rather elaborate treatment of the use of infinitesimals in pure geometry; and the attempt to excite and keep up the interest of the student by bringing in throughout the whole book, and not merely at the end, numerous applications to practical problems in geometry and mechanics."

An Elementary Treatise on the Differential Calculus, Founded on the Method of Rates or Fluxions. By JOHN MINOT RICE, Professor of Mathematics in the U. S. Navy, and WILLIAM WOOLSEY JOHNSON, Professor of Mathematics in St. John's College, Annapolis, Maryland. Revised Edition. 8vo. 469 pp. New York: John Wiley and Sons. 1879.

The notion of rates, which lies at the foundation of this treatise, was first brought to notice by the authors in a paper communicated to the Amer. Acad. of Arts and Sciences in 1873, and afterward revised and published as a pamphlet in 1875. The present treatise consists of an exposition of the new method of obtaining the differentials of functions, occupying 103 pp. Following this, we have, Evaluation of Indeterminate Forms, Maxima and Minima, the Development of functions, 125 pp.; Curve Tracing, 51 pp.; Higher plane Curves and Application of the Diff. Cal. to plane curves, 125 pp. The remaining 55 pp. are devoted to the consideration of functions of two or more variables.

It would be superfluous for us to recommend this book to readers of the ANALYST, who are conversant with the many elegant solutions, and mathematical papers, by Prof. Johnson (one of the authors of this book) which we have published; we may say, however, that upon the very interesting subject of Curve Tracing and the discussion of Higher Plane Curves, this work will be found especially interesting.

ERRATA.

On page 188 (Vol. VI), line 15, for $50\pi R^2$, read $50\pi R^2 + \sqrt{10}$.
" " " " " 20, " " " " " "
" " " " " 20, " 389.3 days, read 383.6 days.
" " 3, line 11, for q^2 read q^4 .
" " 5, " 23, for $2a_2 + \dots + a_n$, read $2a_2 + \dots + na^n$.

THE ANALYST.

VOL. VII.

MARCH, 1880.

No. 2.

QUATERNIONS.

BY PROF. DE VOLSON WOOD, HOBOKEN, NEW JERSEY.

(Continued from page 13.)

2. HAMILTON sought to establish a system which would, at the outset, give a clear interpretation to the square roots of negatives without introducing considerations so expressly geometrical as those which involve the idea of an angle, and was thus lead to consider Algebra as the SCIENCE OF PURE TIME; an essay upon which he read before the Royal Irish Academy in the year 1843.* From this as a starting point he proceeded by a logical process to the establishment of his new system.† But Hamilton, like many other inventors, was not fortunate in the popular presentation of his system. His lectures upon the subject—delivered in Trinity College, Dublin, in 1848 and subsequent years—were, in 1853, published in book form of 736 pages;‡ but the style is so peculiar, being diffuse and hesitating as if he *labored* to make his readers understand, and the arrangement being such as to separate parts of essential principles, that, probably, comparatively few of its readers, without other aids, have mastered its principles. Still the work is remarkably thorough, and remains a monument to the genius of the man. As these articles are designed for those who have little or no knowledge of the subject, we shall seek the most direct and simple manner of explaining its principles. Frequent references will be made to Hamilton's Lectures, for the convenience of those who have access to that work and who may desire to compare results and processes.

3. The system may be developed from the two following lemmas, here stated in the form of postulates:—

*Proceedings of the Royal Irish Academy, Vol. XVII, Part II, pp. 293-422.

†Hamilton's Lectures, Preface, p. 2.

‡Now out of print.

1.—Let it be granted, that the relations between the sides of a plane triangle, represented *in length and direction* respectively by the letters α, β, γ , may be expressed by the equation

$$\alpha = \beta + \gamma. \quad (1)$$

2.—Let it be granted that, if three mutually perpendicular lines be represented respectively by ai, bj, k ,—where i, j, k , are each unity in length—the operation of turning ai about k as an axis through a quadrant to coincide in direction with j and comparing the length of bj with ai , may be represented by the equation

$$\frac{b}{a}k = \frac{bj}{ai}. \quad (2)$$

Developing these equations according to the laws involved in the respective propositions, and introducing from time to time suitable notation for expressing certain operations, we may build up the entire system of Quaternions. And, in order to make it practical, it will only be necessary to interpret the results in accordance with the fundamental principles and the signification of the notation employed. This method was recognized as a correct logical process by Hamilton;* but as it is very arbitrary, we prefer to show in what sense the lemmas may be true.

4. Let the reader imagine that he travels three miles due south, thence northwesterly five miles reaching a point four miles due west of the starting point. Let S represent one mile in a due southerly direction ($N. W.$), one mile in the northwesterly direction, and W one mile due west. Then the operation of traveling may be expressed thus,

$$3S + 5(N.W.),$$

the result of which—in regard to the initial and terminal positions—is the same as if one traveled due west four miles, or $4W$. Hence, in regard to these positions, we may write

$$4W = 3S + 5(N.W.).$$

This equation is not algebraic, though its form appears to be such; for the letters W , &c., represent both length and direction. The entire distance traveled in one case is eight miles, and in the other four miles; which distances are unequal. Still, there is equality in the sense that each member represents the *result* of two independent operations. The addition in the second member is not algebraic, since, in algebra, two quantities having incongruous units cannot be added, but addition here implies a succeeding similar operation—one followed by another step.

*Lectures, p. 38.

5. In a similar manner, if the length and direction of AB be represented by γ , which direction may be parallel to some fixed line in space, and need have no reference to the points of the compass; and similarly β represent CA and α , CB ; then may the position of B in reference to C be represented in the same manner by the equation

$$\alpha = \beta + \gamma.*$$

An extension of this principle gives, for the relation between two adjacent angles of a polygon, the equation

$$\alpha = \beta + \gamma + \delta + \&c.$$

The signs $=$ and $+$ as here used have a broader meaning than in algebra, but they include their full signification as used in that science.

6. A VECTOR literally implies a transference of a point a given distance in a given direction.† Vectors which have the same direction are parallel.‡ Parallel vectors whose lengths are equal are said to be equal; and if their lengths are unequal they are multiples of each other. Thus one of the parallel sides of a trapezoid is a multiple vector of the other. If the length of a vector be unity, it is called a *unit vector*. A vector being positive in one direction, will be negative in the opposite direction.** The positive sign of a vector may be fixed arbitrarily, but being fixed it must retain that sign in that direction throughout any particular discussion. Co-initial vectors are such as radiate from one point.

If, in the triangle ABC , the transference be from B completely round the triangle and positive right-handed, we have

$$\gamma + \beta + \alpha = 0.††$$

If positive left-handed, we have

$$\alpha + \beta + \gamma = 0,$$

or, passing around right-handed,

$$-\gamma - \beta - \alpha = 0;$$

which is the same as the preceding with the signs changed.

7. THE COMMUTATIVE principle consists in interchanging the terms. Thus in the triangle ABC —vectors positive left-handed—we have, beginning at A ,

$$\gamma + \alpha + \beta = 0,$$

and beginning at C ,

$$\beta + \gamma + \alpha = 0.$$

Hence the terms may be interchanged as in algebra.



*Lectures, pp. 29, 30, 44.

†Ibid. p. 15.

**Ibid. p. 51.

††Ibid. p. 144.

‡Some authors of elementary geometry define parallel lines as those which have the same direction, and though it is strictly correct, and possibly the best definition ever given, yet, experience with beginners shows that it is not the most elementary.

8. THE ASSOCIATIVE principle consists in combining the terms in different groups, thus

$$(a + \beta) + \gamma = a + (\beta + \gamma) = (a + \gamma) + \beta,$$

which is the same as in algebra.*

9. A TENSOR is a numerical factor by which a vector is multiplied.† Thus, in the expressions ba , $-c\beta$, 3γ , the factors b , $-c$, and 3 , are tensors. A tensor is sometimes denoted by the letter T , thus, in the preceding expressions $Ta=b$, $T\beta=-c$, $T\gamma=3$. If a be a unit vector, and $a_1=ba$ be the entire vector then we have

$$a_1 = Ta(a).$$

10. The law of signs in relation to the transposition of terms is the same as in algebra. For in the triangle ABC , if we have, as given in eq'n (1), the position of B in reference to C ,

$$a = \beta + \gamma,$$

then for A in reference to C , we have

$$\beta = a - \gamma,$$

or, of A in reference to B ,

$$-\gamma = -a + \beta; \ddagger$$

which results may be obtained from (1) by the transposition of terms.

11. Co-planar vectors are such as are in one plane. Any three co-planar vectors which give the relation

$$aa + b\beta + c\gamma = 0,$$

will form a closed triangle. For, constructing a plane triangle the lengths of whose sides are the lengths respectively of aa , $b\beta$, $c\gamma$ (or if a , β , γ , be unit-vectors, then the lengths will be a , b , c , respectively) the triangle may be placed in such a position that its sides will be parallel to the respective vectors.

12. If as the result of analysis we have

$$\Sigma a + \Sigma \beta = 0,$$

then must

$$\Sigma a = 0, \text{ and } \Sigma \beta = 0.$$

For Σa is typical of a set of parallel vectors, and in a developed form may be represented by $ma+na+la+\&c.$; and similarly $\Sigma \beta = p\beta + q\beta + \&c.$, another set of parallel vectors; and since two vectors cannot enclose a space, or, more generally, since two vectors not parallel have no common unit, each must be zero in order that their sum shall be zero.

This may be illustrated by a familiar example. Thus, if one travels north and south, different distances at different times; also east and west in

*Lectures, pp. 102, 103. † Ibid. pp. 57, 87, 88, 114. ‡ Ibid. pp. 30, 101, 102.

a similar manner, he will not end his journey *at the starting point* unless the sum of the north and south distances be zero, and the sum of the east and west distances, also zero; and if these separately are zero his terminal position will coincide with the initial, and the resultant of all the distances will be zero.

13. The principles now established are sufficient for the solution of a large class of problems. Operations under equation (1) are called the Addition and Subtraction of vectors.

APPLICATIONS.

1.—If the corresponding sides of two triangles are proportional, the triangles are similar.

Let ABC and FEC be two triangles in which we have

$$\frac{AB}{FE} = \frac{BC}{EC} = \frac{AC}{FC},$$

then will the corresponding angles be equal and hence the triangles will be similar. One side CE of one triangle may be made to coincide in direction with the side CB of the other. Let the lengths of the sides be $CE = l$, $EF = m$, $FC = n$, $CB = x$, $BA = y$, $AC = z$; and the unit-vectors of the corresponding sides $\alpha, \beta, \gamma, \alpha, \delta, \epsilon$; observing that the unit-vector for CB is the same as for CE .



The triangle CEF gives

$$l\alpha + m\beta + n\gamma = 0,$$

and CBA ,

$$x\alpha + y\delta + z\epsilon = 0.$$

Eliminating α gives

$$xm\beta - yld + xn\gamma - z\epsilon = 0.$$

But the sides being proportional, we have

$$\frac{l}{x} = \frac{m}{y} = \frac{n}{z}, \text{ or } xm = yl, \text{ and } xn = zl;$$

which reduces the former equation to

$$y(\beta - \delta) + z(\gamma - \epsilon) = 0.$$

The relation of y to z is fixed by the condition of the problem, and β and γ are known; hence the equation can be satisfied only by making $\delta = \beta$ and $\epsilon = \gamma$; hence BA must be parallel to EF and CA to CF , and hence $\angle CEF = \angle CBA$, and $\angle CFE = \angle CAB$. In the figure CA coincides with CF , but, according to the analysis, it is only necessary that the sides be parallel.

2.—To find the centre of gravity of four equal particles situated anywhere in space.

Let A, B, E, D , represent the position of the particles. The centre of gravity of those at A and B will be at M , the middle point of the line AB . Similarly, the centre of gravity of D and E will be at N , the middle point of DE . Joining M and N , the centre of gravity of all the particles will be at C , the middle point of MN .



To find the position of this point, take any point O as the origin of vectors, and draw the vectors $OA = a$, $OB = \beta$, $OE = \gamma$, $OD = \delta$, and $OC = \mu$. Then as vectors $OA + AB = OB$,

or

$$a + AB = \beta;$$

$$\therefore AB = \beta - a,$$

and

$$AM = \frac{1}{2}(\beta - a).$$

Similarly,

$$DN = \frac{1}{2}(\gamma - \delta).$$

Also

$$ON = OD + DN,$$

or, substituting for DN ,

$$= \delta + \frac{1}{2}(\gamma - \delta),$$

reducing,

$$= \frac{1}{2}(\gamma + \delta).$$

Similarly,

$$OM = \frac{1}{2}(a + \beta).$$

Also,

$$ON + NM = OM,$$

or

$$NM = OM - ON;$$

substituting,

$$= \frac{1}{2}(a + \beta) - \frac{1}{2}(\gamma + \delta)$$

$$= \frac{1}{2}(a + \beta - \gamma - \delta);$$

hence

$$NC = \frac{1}{2}(a + \beta - \gamma - \delta).$$

Finally,

$$OC = ON + NC,$$

or

$$\mu = \frac{1}{2}(\gamma + \delta) + \frac{1}{2}(a + \beta - \gamma - \delta)$$

$$= \frac{1}{2}(a + \beta + \gamma + \delta);$$

that is, the vector to the centre of gravity is the mean of all the vectors. This point is also called the mean point of the polygon formed by joining the several particles.* It will be observed that the mode of solution consists in reaching a point by two independent routes. The particles, are not confined to one plane, and the quadrilateral $ABEF$ formed by joining the particles, may be a warped surface. An interpretation of the last result shows that—If a polygon be formed, beginning at the origin, whose successive sides are equal and parallel to the respective vectors and the closing side be divided into the same number of equal parts as there are particles, the centre of gravity will be at the first point of division from the origin.

(To be continued.)

*Hamilton's Lectures, p. 458.

ON SOME PROPERTIES OF POLYNOMIALS.

BY E. L. DE FOREST, M. A., WATERTOWN, CONN.

It was shown by me in the ANALYST for Sept., 1879, that if a single series of equidistant terms, unlimited in length, such as

$$\dots u_{-2}, u_{-1}, u_0, u_1, u_2, u_3, \dots,$$

is adjusted by an unsymmetrical formula, for example

$$u'_0 = l_0 u_0 + l_1 u_1 + l_2 u_2 + l_3 u_3 + l_4 u_4 + l_{-1} u_{-1} + l_{-2} u_{-2} + l_{-3} u_{-3} + l_{-4} u_{-4}, \quad (1)$$

thus affording the adjusted terms

$$\dots u'_{-2}, u'_{-1}, u'_0, u'_1, u'_2, u'_3, \dots,$$

in such manner that

$$\begin{aligned} u'_1 &= l_0 u_1 + l_1 u_2 + l_2 u_3 + l_3 u_4 + l_4 u_5 \\ &\quad + l_{-1} u_0 + l_{-2} u_{-1} + l_{-3} u_{-2} + l_{-4} u_{-3}, \\ u'_2 &= l_0 u_2 + l_1 u_3 + l_2 u_4 + l_3 u_5 + l_4 u_6 \\ &\quad + l_{-1} u_1 + l_{-2} u_0 + l_{-3} u_{-1} + l_{-4} u_{-2}, \\ &\quad \&c., \qquad \&c., \qquad \&c., \end{aligned}$$

and if this new series is adjusted again by another unsymmetrical formula, for example

$$u''_0 = L_0 u'_0 + L_1 u'_1 + L_2 u'_2 + L_{-1} u'_{-1} + L_{-2} u'_{-2}, \quad (2)$$

the repeated adjustment thus obtained is equivalent to one adjustment of the original series by a resultant formula in which the adjusted term u'' is expressed in linear terms of $u_0, u_1, u_2, \&c.$, the coefficients of these last being sums of products of L and l . For instance, the term u_2 enters the resultant formula with the coefficient

$$L_{-2} l_4 + L_{-1} l_3 + L_0 l_2 + L_1 l_1 + L_2 l_0. \quad (3)$$

Let us look a little more closely at the way in which this happens. In $u'_{-2}, u'_{-1}, u'_0, u'_1, u'_2, \&c.$, the term u_2 enters with the coefficients $l_4, l_3, l_2, l_1, l_0, \&c.$, respectively. In u''_0 the terms $u'_{-2}, u'_{-1}, u'_0, u'_1, u'_2$, enter with the coefficients $L_{-2}, L_{-1}, L_0, L_1, L_2$, respectively. Hence, u_2 enters into u''_0 through $u'_{-2}, u'_{-1}, u'_0, u'_1$, and u'_2 with the coefficients $L_{-2} l_4, L_{-1} l_3, L_0 l_2, L_1 l_1$ and $L_2 l_0$ respectively, the sum of the sub-indices of the L and l in each of these coefficients being 2, the sub-index of u_2 . The total influence of u_2 upon u''_0 is proportioned to the sum of all these coefficients as seen in (3).

Thus the resultant of two adjustments by any given component formulas similar to (1) and (2), no matter how many terms they may contain, will be a

formula in which any term u_r of the original series enters with a coefficient made up of the sum of all the products which can be formed by multiplying the L into the l in such manner that the sum of their sub-indices shall be r . And since it makes no difference whether we multiply an L into an l , or an l into an L , the resultant formula will be unchanged if we suppose the order of the operations to be reversed, so that the series is adjusted first by (2) and then by (1).

This mode of combining the coefficients L and l is precisely the same as that which occurs when we multiply together the two polynomials

$$\begin{aligned} l_4 z^4 + l_3 z^3 + l_2 z^2 + l_1 z^{-1} + l_0 z^0 + l_1 z + l_2 z^2 + l_3 z^3 + l_4 z^4, \\ L_{-2} z^{-2} + L_{-1} z^{-1} + L_0 z^0 + L_1 z + L_2 z^2, \end{aligned}$$

where the indices of z are the same as the sub-indices of l and L , and the same also as those of u in (1) and (2). In the product, any power of z , for example z^2 , will have for its coefficient the sum of all those products of L and l which have the sum of their sub-indices equal to 2, the index of the power of z under consideration. So too if we increase all the indices of z in the first factor by any number, for instance 4, and likewise increase all those of the second factor by any number, for instance 2, the product will be unchanged, except that all the indices of z in it will be increased by $4+2=6$. The coefficient of z^{2+6} , for instance, will be formed by those same combinations of L with l which formed the coefficient of z^2 in the previous case. Thus we establish the proposition, that any two component formulas of adjustment

$$\left. \begin{aligned} u'_0 &= l_0 u_0 + l_1 u_1 + l_2 u_2 + \dots + l_m u_m \\ &\quad + l_{-1} u_{-1} + l_{-2} u_{-2} + \dots + l_{-m} u_{-m}, \\ u'_0 &= L_0 u_0 + L_1 u_1 + L_2 u_2 + \dots + L_n u_n \\ &\quad + L_{-1} u_{-1} + L_{-2} u_{-2} + \dots + L_{-n} u_{-n}, \end{aligned} \right\} \quad (4)$$

no matter which is used first, are equivalent to a single resultant formula, whose coefficients are the same and in the same order as those of the powers of z in the product of the polynomials

$$\left. \begin{aligned} l_{-m} + l_{-m+1} z + l_{-m+2} z^2 + \dots + l_0 z^m + l_1 z^{m+1} + \dots + l_m z^{2m}, \\ L_{-n} + L_{-n+1} z + L_{-n+2} z^2 + \dots + L_0 z^n + L_1 z^{n+1} + \dots + L_n z^{2n}. \end{aligned} \right\} \quad (5)$$

The resultant coefficients of u_0, u_1, u_2 , &c., are the same as those of $z^{m+n}, z^{m+n+1}, z^{m+n+2}$, &c., respectively. The whole number of terms in the resultant is $2(m+n)+1$.

Combinations analogous to these are found to arise in the repeated adjustment of a double series, or table of values of a function of two variables. The terms of the series are supposed to be arranged in rectangular form, as in (6), and each of them has two sub-indices, the first denoting its rank in the horizontal direction, reckoned from an initial term $u_{0,0}$, and the second

$u_{-1, 2}$	$u_{0, 2}$	$u_{1, 2}$	$u_{2, 2}$
$u_{-1, 1}$	$u_{0, 1}$	$u_{1, 1}$	$u_{2, 1}$
$u_{-1, 0}$	$u_{0, 0}$	$u_{1, 0}$	$u_{2, 0}$
$u_{-1, -1}$	$u_{0, -1}$	$u_{1, -1}$	$u_{2, -1}$

(6)

its rank in the vertical direction. The series may be extended without limit in both directions. For an investigation of some simple symmetrical formulas for the adjustment of such series, see the *Smithsonian Reports* of 1871 and 1873, pp. 321 and 332; an error being corrected in the *ANALYST*, May, 1877, p. 84. We will now consider the adjustment formulas in the most general way, as unsymmetrical, the terms u having coefficients l which may be any numbers, subject only to the condition that their algebraic sum is unity. Each l will have the same two sub-indices as the u to which it originally belongs.

For the sake of simplicity, suppose that a first adjustment is made by the nine term formula

$$u'_{0, 0} = \left\{ \begin{array}{l} l_{-1, 1} u_{-1, 1} + l_{0, 1} u_{0, 1} + l_{1, 1} u_{1, 1} \\ + l_{-1, 0} u_{-1, 0} + l_{0, 0} u_{0, 0} + l_{1, 0} u_{1, 0} \\ + l_{-1, -1} u_{-1, -1} + l_{0, -1} u_{0, -1} + l_{1, -1} u_{1, -1} \end{array} \right\}.$$

To make the notation more compact, we will designate any such rectangular formula by means of the terms in its four angles only, thus;

$$u'_{0, 0} = \frac{l_{-1, 1} u_{-1, 1}}{l_{-1, -1} u_{-1, -1}} \bigg| \frac{l_{1, 1} u_{1, 1}}{l_{1, -1} u_{1, -1}}. \quad (7)$$

It gives an unlimited number of adjusted terms,

$$u'_{1, 0} = \frac{l_{-1, 1} u_{-1, 1}}{l_{-1, -1} u_{-1, -1}} \bigg| \frac{l_{1, 1} u_{1, 1}}{l_{1, -1} u_{1, -1}}, \quad u'_{1, 1} = \frac{l_{-1, 1} u_{-1, 1}}{l_{-1, -1} u_{-1, -1}} \bigg| \frac{l_{1, 1} u_{1, 1}}{l_{1, -1} u_{1, -1}}, \text{ \&c., \&c.}$$

Let these be adjusted again by another formula, for instance

$$u''_{0, 0} = \frac{L_{-2, 1} u'_{-2, 1}}{L_{-2, -1} u'_{-2, -1}} \bigg| \frac{L_{2, 1} u'_{2, 1}}{L_{2, -1} u'_{2, -1}}, \quad (8)$$

which includes 15 terms in three horizontal rows or five columns. A particular term in the original series (6), as $u_{2, 1}$ for instance, can only enter into formula (8) through the adjusted values of the nine terms which have $u_{2, 1}$ in their middle, and which we will denote by their corner terms only,

$$\begin{array}{ccc} u'_{1, 2} & \vdots & u'_{2, 2} \\ u'_{1, 0} & \vdots & u'_{2, 0} \end{array}$$

It would enter through these nine terms respectively with the nine coefficients

$$\begin{array}{c} L_{1,2} l_{1,-1} \\ L_{1,0} l_{1,1} \end{array} \vdots \begin{array}{c} L_{3,2} l_{-1,-1} \\ L_{3,0} l_{-1,1} \end{array}, \quad (9)$$

but for the fact that the first or horizontal sub-index of L in (8) nowhere exceeds 2, and the second or vertical one nowhere exceeds 1. Thus the five coefficients which contain $L_{1,2}$, $L_{2,2}$, $L_{3,2}$, $L_{3,1}$ and $L_{3,0}$ are all zero, and $u_{2,1}$ really enters (8) with only the four remaining coefficients from (9), its total influence upon $u''_{0,0}$ being proportioned to their sum

$$L_{1,1} l_{1,0} + L_{2,1} l_{0,0} + L_{1,0} l_{1,1} + L_{2,0} l_{0,1}.$$

In each of these products Ll the sum of the two horizontal sub-indices is 2, and the sum of the two vertical ones is 1, being the same as in $u_{2,1}$. Thus it appears that the resultant of any two component formulas such as (7) and (8), no matter how many terms they may contain, will be a formula in which any term $u_{r,s}$ of the original series enters with a coefficient made up of the sum of all the products which can be formed by multiplying the L into the l , in such manner that in each product the sum of the two horizontal sub-indices shall be r , and the sum of the two vertical ones shall be s . And as in the case of a single series, so here, the result will be unchanged if we reverse the order of operation, and adjust the double series first by (8) and then by (7).

Now let each u be replaced by the product of two variables, x and y , giving to the x an index or exponent equal to the horizontal sub-index of u , and to the y an index equal to the vertical sub-index of u . Then any two adjustment formulas, such as (7) and (8) for instance, will be respectively converted into the polynomials

$$\frac{l_{-1,1} x^{-1}y}{l_{-1,-1} x^{-1}y^{-1}} \mid \frac{l_{1,1} xy}{l_{1,-1} xy^{-1}}, \quad \frac{L_{-2,1} x^{-2}y}{L_{-2,-1} x^{-2}y^{-1}} \mid \frac{L_{2,1} x^2y}{L_{2,-1} x^2y^{-1}}. \quad (10)$$

If these two are multiplied together, the coefficient of a term in the product, that of x^2y for example, will be just the same as that of the corresponding term $u_{2,1}$ in the resultant formula of adjustment, for it will be the sum of all those products of the L and l which can be formed so as to make the sum of the two horizontal or x indices equal to 2, and the sum of the two vertical or y indices equal to 1. If we increase all the indices of x and y in the first polynomial by any two numbers, for instance by 1 and 1 respectively, and increase those of the second polynomial by any two numbers, for instance 2 and 1 respectively, the product will be unchanged, except that all the indices of x will be increased by $1+2=3$, and all those of y by $1+1=2$. This proposition then is established, that any two component formulas of adjustment,

$$u_{0,0} = \frac{l_{-m,n} u_{-m,n}}{l_{-m,-n} u_{-m,-n}} \bigg| \frac{l_{m,n} u_{m,n}}{l_{m,-n} u_{m,-n}}, \quad u'_{0,0} = \frac{L_{-p,q} u_{-p,q}}{L_{-p,-q} u_{-p,-q}} \bigg| \frac{L_{p,q} u_{p,q}}{L_{p,-q} u_{p,-q}}, \quad (11)$$

latter which is used first, are equivalent to a single resultant formula the coefficients are the same and in the same order as those of the product of the powers of x and y in the product of the two polynomials

$$\frac{l_{-m,n} x^0 y^{2n}}{l_{-m,-n} x^0 y^0} \bigg| \frac{l_{m,n} x^{2m} y^{2n}}{l_{m,-n} x^{2m} y^0}, \quad \frac{L_{-p,q} x^0 y^{2q}}{L_{-p,-q} x^0 y^0} \bigg| \frac{L_{p,q} x^{2p} y^{2q}}{L_{p,-q} x^{2p} y^0}. \quad (12)$$

the resultant coefficients of $u_{0,0}$, $u_{1,0}$, $u_{1,1}$, &c., are respectively the same as those of

$$x^{m+p} y^{n+q}, \quad x^{m+p+1} y^{n+q}, \quad x^{m+p+1} y^{n+q+1}, \quad \&c.$$

whole number of terms in the component formulas being respectively $(2n+1)$ and $(2p+1)(2q+1)$, the whole number in the resultant is

$$[2(m+p)+1][2(n+q)+1]. \quad (13)$$

it follows from the foregoing that if a double series is adjusted k times by a single formula

$$u'_{0,0} = \frac{\lambda_{-m,n} u_{-m,n}}{\lambda_{-m,-n} u_{-m,-n}} \bigg| \frac{\lambda_{m,n} u_{m,n}}{\lambda_{m,-n} u_{m,-n}}, \quad (14)$$

if the equivalent or resultant formula is denoted by

$$u^{(k)}_{0,0} = \frac{l_{-km,kn} u_{-km,kn}}{l_{-km,-kn} u_{-km,-kn}} \bigg| \frac{l_{km,kn} u_{km,kn}}{l_{km,-kn} u_{km,-kn}}, \quad (15)$$

the coefficients l will be the same as the coefficients in the expansion of the binomial

$$\left\{ \frac{\lambda_{-m,n} x^0 y^{2n}}{\lambda_{-m,-n} x^0 y^0} \bigg| \frac{\lambda_{m,n} x^{2m} y^{2n}}{\lambda_{m,-n} x^{2m} y^0} \right\}^k. \quad (16)$$

number of terms in the expansion is $(2km+1)(2kn+1)$.

As was shown in the ANALYST for Jan., 1880, p. 2, that if the coefficients of a polynomial of one variable,

$$a_0 + a_1 z + a_2 z^2 + \dots + a_n z^n,$$

be regarded as a system of equidistant parallel forces acting at right angles to the axis of X taken as a mathematical lever, and if h_1 denotes the distance from the centre of parallel forces to the point of application of a_0 , or, in other words, the lever arm of the system about the place of a_0 as a fulcrum, then in the k power of this polynomial the lever arm of the system about the place of the first one as a fulcrum will be kh_1 . This may be proved in a manner rather more simple and general, and without introducing the condition

$$a_0 + a_1 + a_2 + \dots + a_n = 1,$$

which holds when a_0, a_1 , &c., represent coefficients in a formula of adjustment. Let any two polynomials in z be

$$\left. \begin{aligned} a_0 + a_1 z + a_2 z^2 + \dots + a_m z^m, \\ c_0 + c_1 z + c_2 z^2 + \dots + c_n z^n, \end{aligned} \right\} \quad (17)$$

and let the sums of their coefficients be

$$\left. \begin{aligned} S_1 &= a_0 + a_1 + a_2 + \dots + a_m, \\ S_2 &= c_0 + c_1 + c_2 + \dots + c_n. \end{aligned} \right\} \quad (18)$$

Regarding these as systems of equidistant parallel forces, let the lever arms of the two systems about the points of application of a_0 and c_0 be h_1 and h_2 , respectively, and in each system let Δx be the interval between the consecutive points of application of the forces. The moments of the two systems about the places of a_0 and c_0 will be

$$\left. \begin{aligned} S_1 h_1 &= (a_1 + 2a_2 + 3a_3 + \dots + m a_m) \Delta x, \\ S_2 h_2 &= (c_1 + 2c_2 + 3c_3 + \dots + n c_n) \Delta x. \end{aligned} \right\} \quad (19)$$

When the two polynomials 17 are multiplied together, the moment of the coefficients of that part of the product which is due to c_0 , about the place of a_0 , will be $c_0 S_1 h_1$. The moment of those of the part which is due to $c_1 z$ will be

$$c_1 S_1 (h_1 + \Delta x),$$

for the part due to $c_2 z^2$ it will be

$$c_2 S_1 (h_1 + 2\Delta x),$$

and so on. Denoting by H the lever arm of the whole system of coefficients in the product, we shall have for the moment of this system about the place of the first term as a fulcrum,

$$\begin{aligned} S_1 S_2 H &= c_0 S_1 h_1 + c_1 S_1 (h_1 + \Delta x) + c_2 S_1 (h_1 + 2\Delta x) + \dots + c_n S_1 (h_1 + n\Delta x) \\ &= S_1 h_1 (c_0 + c_1 + \dots + c_n) + S_1 (c_1 + 2c_2 + \dots + n c_n) \Delta x. \end{aligned}$$

By (18) and (19) this reduces to

$$\begin{aligned} S_1 S_2 H &= S_1 S_2 h_1 + S_1 S_2 h_2, \\ \therefore H &= h_1 + h_2, \end{aligned} \quad (20)$$

so that the lever arm of the coefficients in the product is equal to the sum of the lever arms of those in the two factors. From this it follows that if any number of polynomials are multiplied together, the lever arm of the coefficients in the final product, about its first term as a fulcrum, is equal to the sum of the lever arms in all the factors, about their first terms as fulcrums. Hence, if the first polynomial in (17) is to be raised to the k power, the lever arm of the coefficients in the expansion of

$$(a_0 + a_1 z + a_2 z^2 + \dots + a_m z^m)^k$$

will be equal to kh_1 . Now the whole length of the polynomial is $m\Delta x$, and the whole length of its expansion to the k power is $km\Delta x$. Thus it is proved that as k increases from one integer to another, the lever arm of the product increases in the same ratio as the whole length of the product does,

so that the extremity of the lever arm, or the centre of parallel forces, divides the length of the expanded polynomial in a constant ratio. If the coefficients are all positive, they may be regarded as masses in a system of material points, and the centre of forces becomes the centre of gravity.

Analogous properties are found to belong to polynomials of two variables, whose terms are supposed to be arranged thus,

$$\begin{array}{cccc}
 + \dots & + \dots & + \dots & + \dots \\
 + a_{0,2} y^2 & + a_{1,2} xy^2 & + a_{2,2} x^2 y^2 + \dots & \\
 + a_{0,1} y & + a_{1,1} xy & + a_{2,1} x^2 y + \dots & \\
 + a_{0,0} & + a_{1,0} x & + a_{2,0} x^2 + \dots &
 \end{array} \quad (21)$$

so that the indices of x and y in any term correspond to the rank of that term reckoned from the left hand column and the lower row respectively. Let any two such polynomials be denoted for brevity by

$$\frac{a_{0,n} x^0 y^n}{a_{0,0} x^0 y^0} \bigg| \frac{a_{m,n} x^m y^n}{a_{m,0} x^m y^0}, \quad \frac{c_{0,p} x^0 y^p}{c_{0,0} x^0 y^0} \bigg| \frac{c_{p,q} x^p y^q}{c_{p,0} x^p y^0}, \quad (22)$$

so that the first one has $m+1$ and $n+1$ terms in its rows and columns, respectively, while the second one has $p+1$ and $q+1$. Let the sums of their coefficients be

$$S_1 = \frac{a_{0,n}}{a_{0,0}} \bigg| \frac{a_{m,n}}{a_{m,0}}, \quad S_2 = \frac{c_{0,p}}{c_{0,0}} \bigg| \frac{c_{p,q}}{c_{p,0}}. \quad (23)$$

Regarding these coefficients as systems of equidistant parallel forces acting upon the plane of XY at right angles to it, let the lever arms of the two systems about the lower row or axis of X be h_1 and h_2 , and let those about the left hand column or axis of Y be k_1 and k_2 . Also in each system let Δx and Δy be the intervals between consecutive points of application of the forces, in the directions of X and Y . In the second system, let the sum of the coefficients in the first or lower row be s''_0 , and let the sums of those in the other successive rows be s''_1, s''_2, s''_3 , &c., wherefore

$$S_2 = s''_0 + s''_1 + \dots + s''_q. \quad (24)$$

The moment of the second system about the axis of X is

$$S_2 h_2 = (s''_1 + 2s''_2 + \dots + q s''_q) \Delta y. \quad (25)$$

When the two polynomials (22) are multiplied together, the moment of the coefficients of that part of the product due to the first row in the second polynomial, about the axis of X or place of the first row in the product, will be $s''_0 S_1 h_1$. The moment for the part due to the second row will be

$$s''_1 S_1 (h_1 + \Delta y).$$

In the part due to the third row the moment will be

$$s''_2 S_1 (h_1 + 2\Delta y),$$

and so on. Denoting by H and K the lever arms of the whole system of

coefficients in the product about the axes of X and Y , we have for the total moment of this system about the line of its lower row as a fulcrum,

$$\begin{aligned} S_1 S_2 H &= s''_0 S_1 h_1 + s''_1 S_1 (h_1 + \Delta y) + s''_2 S_1 (h_1 + 2\Delta y) + \dots + s''_q S_1 (h_1 + q\Delta y) \\ &= S_1 h_1 (s''_0 + s''_1 + \dots + s''_q) + S_1 (s''_1 + 2s''_2 + \dots + qs''_q) \Delta y. \end{aligned}$$

By (24) and (25) this reduces to

$$\begin{aligned} S_1 S_2 H &= S_1 S_2 h_1 + S_1 S_2 h_2, \\ \therefore H &= h_1 + h_2, \end{aligned} \tag{26}$$

showing that the lever arm of the coefficients in the product, about the line of its lower row, is equal to the sum of the lever arms of those in the two factors, about the lines of their lower rows. From the similarity of situation of the axes of X and Y with respect to the rectangular system of coefficients, it follows that we shall have in like manner

$$K = k_1 + k_2. \tag{27}$$

Hence, if any number of such polynomials in x and y are multiplied together, the lever arm of the system of coefficients in the final product, about the line of its lower row, is equal to the sum of the lever arms of the systems in all the factors, about the lines of their lower rows; and likewise the lever arm for the product, about the line of its left hand column, is equal to the sum of the lever arms for all the factors, about the lines of their left hand columns. Similar properties would evidently exist if the rotations were supposed to be about the line of the upper row or of the right hand column, for the coefficients in the factors being arbitrary, their positions may be reversed, and those of the product will be reversed also. Hence too, if a single polynomial, the first one in (22) for instance, is to be raised to the k power, the lever arms H and K of the coefficients in the expansion of

$$\left\{ \frac{a_{0..n} x^0 y^n}{a_{0..0} x^0 y^0} \middle| \frac{a_{m..n} x^m y^n}{a_{m..0} x^m y^0} \right\}^k \tag{28}$$

will be equal to kh_1 and kk_1 respectively. But the length and breadth of the given polynomial are $m\Delta x$ and $n\Delta y$, and the length and breadth of its expansion to the k power will be $km\Delta x$ and $kn\Delta y$. Thus it is proved that as k increases from one integer to another, the lever arms of the product increase in the same ratio as the dimensions of the product do. The centre of parallel forces in each case will be that point in the plane of XY whose distances from the axes of X and Y are h_1 and k_1 for the given polynomial, and kh_1 and kk_1 for its expansion to the k power, so that as k is increased, the position of this centre of forces, relatively to the four sides of the rectangle formed by the polynomial, will remain unchanged. If the coefficients are all positive, we may regard them as masses, and the centre of forces is their centre of gravity.

(To be continued.)

See p. 53

ON THE QUADRATURE OF CURVES AND THE CUBATURE OF SOLIDS BY MEANS OF INFINITE SERIES.

BY PROF. D. TROWBRIDGE, A. M., WATERBUGH, N. Y.

WE shall need the sum of the series

$$1^p + 2^p + 3^p + \dots + n^p = A_0 n^{p+1} + \dots \text{suppose.}$$

To find A_0 —for the coefficients of the remaining terms of the sum we shall not need,—let $n = n+1$, then we shall have

$$1^p + 2^p + \dots + n^p + (n+1)^p = A_0(n+1)^{p+1} + \dots = (n+1)^p + A_0 n^{p+1} + \dots$$

$$\therefore n^p + p n^{p-1} + \dots = A_0[(n+1)^{p+1} - n^{p+1}] + \dots = (p+1)A_0 n^p + \dots$$

Equating the coefficients of like powers of n , we find

$$A_0 = \frac{1}{p+1}. \text{ Hence } 1^p + 2^p + \dots + n^p = \frac{n^{p+1}}{p+1} + \dots \quad (1)$$

Wherever the first member of (1) occurs in what follows, it is sufficient to put in its place the term given in the second member of (1); for we shall always have n^{p+1} multiplied by x'^{p+1} , n being infinitely great and x' infinitely small, and $nx' =$ some finite quantity. Since the remaining terms in the second member of (1) will contain the powers n^p , n^{p-1} , &c., and these will be multiplied by x'^{p+1} , it will be seen that

$$n^p \cdot x'^{p+1} = x' \times \text{finite quantity} = 0, \text{ and so of the other terms.}$$

I. THE CIRCLE.

Let the origin be at the centre of the circle, and let ordinates be drawn perpendicular to the radius CD dividing the quadrant ACD into rectangles of infinitely small width, each equal to x' . Let $CB = x = nx'$, and call the area of $ACBP = A$, the radius of the circle being unity. We know from Elementary Geometry that the area of the quadrant ACD equals $\frac{1}{4}\pi$, $\pi = 3.14159\dots$ It will now be seen that

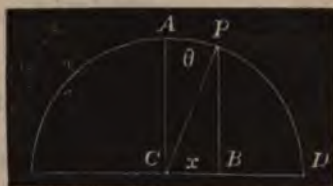
$$A = x'[y_1 + y_2 + \dots y_n] = x'[\sqrt{1-x'^2} + \sqrt{1-4x'^2} + \dots + \sqrt{1-n^2x'^2}]\quad (2)$$

since $y_n^2 = 1 - n^2x'^2$, n being an integer.

If we expand the radicals in the third member of (2) and arrange the terms, we shall have $(1+1+\dots=n)$

$$A = x'[n - \frac{1}{2}x'^2(1^2+2^2+\dots+n^2) - \frac{1}{2 \cdot 4}x'^4(1^4+2^4+\dots+n^4) - \frac{3}{2 \cdot 4 \cdot 8}x'^6(1^6+2^6+\dots+n^6) \dots]$$

$$= x - \frac{x^3}{2 \cdot 3} - \frac{x^5}{2 \cdot 4 \cdot 5} - \frac{3x^7}{2 \cdot 4 \cdot 6 \cdot 7} - \frac{3 \cdot 5 x^9}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 9} - \dots \quad (3)$$



If we make $x = 1 = CD$, we shall have

$$\frac{1}{2}\pi = 1 - \frac{1}{2.3} - \frac{1}{2.4.5} - \frac{3}{2.4.6.7} - \dots \quad (4)$$

Now make the angle $ACP = \theta$, and we find

$$\begin{aligned} A &= \frac{1}{2}\theta + \frac{1}{2}\sin \theta \cos \theta = \frac{1}{2}\theta + \frac{1}{2}x\sqrt{1-x^2}. \quad \text{Whence, by (3),} \\ \theta &= 2 \left[x - \frac{x^3}{2.3} - \frac{x^5}{2.4.5} - \frac{3x^7}{2.4.6.7} - \dots - \frac{1}{2}x\sqrt{1-x^2} \right] = 2 \left[(1-\frac{1}{2})x - \frac{1}{2}(\frac{1}{2}-\frac{1}{2})x^3 - \dots \right] \\ &= x + \frac{x^3}{2.3} + \frac{3x^5}{2.4.5} + \frac{3.5x^7}{2.4.6.7} + \dots \end{aligned} \quad (5)$$

If this series be reverted we shall have

$$x = \sin \theta = \theta - \frac{\theta^3}{1.2.3} + \frac{\theta^5}{1.2.3.4.5} - \dots \quad (6)$$

If we divide equation (6) by θ , and then make $\theta = 0$, we see that

$$\frac{\sin 0}{0} = 1. \quad (7)$$

If we make $\sin \theta = \theta z$, $z = 1 - \frac{1}{1.2.3}\theta^2 + \dots$, we have $\cos \theta = \sqrt{1-\theta^2 z^2} = 1 - \frac{1}{2}\theta^2 z^2 - \dots$, $\cos \theta - 1 = -\frac{1}{2}\theta^2 z^2 - \dots$, and $(\cos \theta - 1) + \theta = -\frac{1}{2}\theta^2 z^2$.

$$\text{If } \theta = 0, \quad \frac{\cos 0 - 1}{0} = \frac{1 - 1}{0} = \frac{0}{0} = 0. \quad (8)$$

If in (6) we make $\theta = \theta + h$, we shall have

$$\sin(\theta + h) = \sin \theta \cos h + \cos \theta \sin h = \theta + h - \frac{(\theta + h)^3}{1.2.3} + \frac{(\theta + h)^5}{1.2.3.4.5} - \dots$$

If we now subtract (6) from this equation, divide by h and then make $h = 0$, we shall have

$$\sin \theta \left(\frac{\cos 0 - 1}{0} \right) + \cos \theta \frac{\sin 0}{0} = 1 - \frac{\theta^2}{1.2} + \frac{\theta^4}{1.2.3.4} - \dots;$$

or by (7) and (8),

$$\cos \theta = 1 - \frac{\theta^2}{1.2} + \frac{\theta^4}{1.2.3.4} - \dots \quad (9)$$

If $t = \tan \theta$, then $\sin \theta = x = t / \sqrt{1+t^2}$. If this value of x be put in (5), and the second member be reduced, we shall find

$$\theta = t - \frac{1}{3}t^3 + \frac{1}{5}t^5 - \dots \quad (10)$$

The reversion of this will give t in terms of θ .

II. THE ELLIPSE.

The equation of the ellipse is

$$ay = b \sqrt{a^2 - x^2} = ab \sqrt{1 - x_1^2}, \quad x = ax_1. \quad (11)$$

$$\begin{aligned} \text{Area} &= ax'_1[y_1 + y_2 + \dots y_n] = abx'_1[\sqrt{1-x_1^2} + \sqrt{1-4x_1^2} + \dots \sqrt{1-n^2x_1^2}] \\ &= abA \text{ by (3). When } x = 1, \text{ Eq. (4), } \frac{1}{2} \text{ area of ellipse} = \frac{1}{2}\pi ab. \end{aligned} \quad (12)$$

III. THE PARABOLA.

Let the axis of the parabola be vertical, and also be the axis of y , and let h be the greatest value of y , a being the greatest value of x . The equation of the curve will be

$$x^2 = 2p(h-y), \text{ or } y = h - \frac{x^2}{2p}. \quad (13)$$

Since $nx' = a$, the area of the curve is

$$\begin{aligned} A &= 2x'[y_1 + y_2 + \dots y_n] = 2x'[nh - \frac{x'^2}{2p}(1+2^2+\dots n^2)] \\ &= 2ah - \frac{a^3}{3p} = \frac{2}{3}ah = \frac{2}{3} \text{ circumscribing rectangle.} \end{aligned} \quad (14)$$

IV. THE HYPERBOLA.

Let SAE be one branch of a hyperbola, CAX the axis, CN the asymptote. If $CD = x$, ED parallel to CN , $ED' = y$, perpendicular to CN ; if a and b are the semi-axes, then we have

$$2xy = ab \quad (15)$$

The area of $ABDE = x'(y_1 + y_2 + \dots y_n)$

$$\begin{aligned} &= \frac{1}{2}abx' \left[\frac{1}{x_0+x'} + \dots + \frac{1}{x_0+nx'} \right] \\ &= \frac{abx'}{2x_0} \left[\frac{1}{1+\frac{x'}{x_0}} + \frac{1}{1+\frac{2x'}{x_0}} + \dots + \frac{1}{1+\frac{nx'}{x_0}} \right] \\ &= \frac{abx'}{2x_0} \left[n - \frac{x'}{x_0}(1+2+\dots n) + \frac{x'^2}{x_0^2}(1+2^2+\dots) \right] \\ &= \frac{ab}{2x_0} \left[nx' - \frac{x'^2 n^2}{2} + \frac{x'^3 n^3}{3} - \dots \right] = \frac{ab}{2} \left[\frac{x-x_0}{x_0} - \frac{(x-x_0)^2}{2x_0^2} + \frac{(x-x_0)^3}{3x_0^3} - \dots \right] \\ &= \frac{ab}{2} \log \left(1 + \frac{x-x_0}{x_0} \right) = \frac{ab}{2} \log \frac{x}{x_0}, \end{aligned} \quad (16)$$

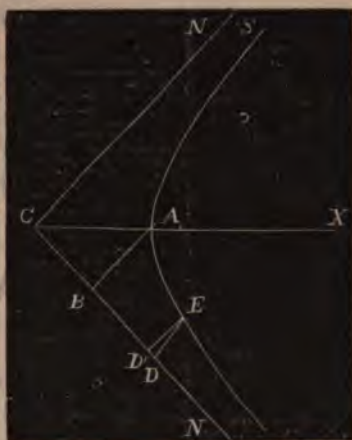
where $x_0 = CB$. It is easy now to find the area of the hyperbola.

V. THE SPHERE.

To find the surface of the sphere conceive it to be divided by planes perpendicular to a diameter into elementary solids (frusta of cones) the thickness, or height of each being x' , and let the slant height of these be $s_1, s_2, \&c.$ Then the surface of a hemi-sphere will be, y being an ordinate,

$$S = 2\pi[s_1y_1 + s_2y_2 + \dots + s_ny_n]. \quad (17)$$

Let $ABaP$ be one of the elementary frustums,



$AB = ab = x'$, $aP = s$, $BP = y$, $CP = r =$ radius of the sphere. If ab is perpendicular to Aa the two triangles abP and aAC will be similar and give $sy = rx'$. Hence $s_1y_1 = s_2y_2 = \dots = s_ny_n = rx'$. This in (17) gives

$$S = 2\pi rnx' = 2\pi rx, \quad (18)$$

where $nx' = x = CA$. If $x = r$, this gives for the surface of the hemisphere $2\pi r^2$, and of the whole sphere, $4\pi r^2$.

VI. THE PROLATE SPHEROID.

To find the surface of the prolate spheroid, conceive it divided, as in the case of the sphere, by planes perpendicular to the transverse axis. Then the surface of a hemi-sphere (or a part of it) will be

$$S = 2\pi[s_1y_1 + s_2y_2 + \dots + s_ny_n]. \quad (19)$$

If in the last fig. we consider PC a normal, N , we shall have $sy = Nx'$, and

$$S = 2\pi x'[N_1 + N_2 + \dots + N_n]. \quad (20)$$

But $N = b\sqrt{1 - (e^2x'^2/a^2)}$. Hence

$$S = 2\pi bx' \left[\sqrt{1 - \frac{e^2x'^2}{a^2}} + \sqrt{1 + \frac{4e^2x'^2}{a^2}} + \dots + \sqrt{1 - \frac{n^2e^2x'^2}{a^2}} \right]. \quad (21)$$

Suppose in the first figure that $CB = ex + a = nex' \div a = \sin \theta$; then, as in Prob. II,

$$\begin{aligned} \frac{ex'}{a} \left[\sqrt{1 - \frac{e^2x'^2}{a^2}} + \dots + \sqrt{1 - \frac{n^2e^2x'^2}{a^2}} \right] &= \frac{1}{2}\theta + \frac{1}{2}\sin \theta \cos \theta \\ &= \frac{1}{2}\sin^{-1} \frac{ex}{a} + \frac{ex}{2a} \sqrt{1 - \frac{e^2x^2}{a^2}}. \end{aligned}$$

We therefore have for that portion of the surface of the prolate spheroid, which is included between two planes drawn perpendicular to the transverse axis of the generating ellipse, one through the centre and the other at the distance x from it,

$$S = \frac{\pi ab}{e} \left[\sin^{-1} \frac{ex}{a} + \frac{ex}{a} \sqrt{1 - \frac{e^2x^2}{a^2}} \right]. \quad (22)$$

If $x = a$, surface of whole spheroid equals

$$2S' = 2\pi ab \left[\frac{1}{e} \sin^{-1} e + \sqrt{1 - e^2} \right]. \quad (23)$$

VII. THE OBLATE SPHEROID.

Equation (20) will give the surface of the oblate spheroid if we make $N = a\sqrt{1 + (a^2e^2x'^2/b^4)}$, and $b^2z = aex$, so that

$$S = \frac{2\pi b^2z'}{ae} \left[\sqrt{1 + z'^2} + \sqrt{1 + 4z'^2} + \dots + \sqrt{1 + n^2z'^2} \right]. \quad (24)$$

If the terms of (24) be expanded and the result treated as in Prob. II, we shall find, since $nz' = z$,

$$S = \frac{2\pi b^2}{ae} \left[z + \frac{z^3}{2.3} - \frac{z^5}{2.4.5} + \frac{z^7}{2.4.6.7} - \dots \right]. \quad (25)$$

VIII. THE LOGARITHMIC CURVE.

If we have $y = a^x$, (26)
 area of curve = $A = x'[y_1 + y_2 + \dots + y_n] = x'[a^{x_1} + a^{x_2} + \dots + a^{x_n}]$
 $= x' \left[n + \lambda x'(1 + 2 + \dots + n) + \frac{\lambda^2 x'^2}{1.2} (1 + 4 + \dots + n^2) + \dots \right]$
 $= x + \frac{\lambda x^2}{1.2} + \frac{\lambda^2 x^3}{1.2.3} + \dots = \frac{1}{\lambda} (a^x - 1), \quad (\lambda = \log a.)$ (27)

IX. THE ELLIPSOID.

To find the solid contents of the ellipsoid, suppose sections made by planes perpendicular to the axis of z , the distance between the planes being z' . Each section will be an ellipse whose axes we shall call $2x$ and $2y$. The area of a section will then be πxy . The solid contents of a portion of the ellipsoid will now be

$$S = \pi z' [x_1 y_1 + x_2 y_2 + \dots + x_n y_n]. \quad (28)$$

Now let two sections be made, one passing through the axes of x and z , and the other through y and z , each section will be an ellipse, giving

$$cx = a\sqrt{c^2 - z^2}, \quad cy = b\sqrt{c^2 - z^2} \quad (29)$$

The product of these will be

$$c^2 xy = ab(c^2 - z^2). \quad (30)$$

We now have, since $nz' = z$,

$$S = \frac{\pi ab z'}{c^2} \left[nc^2 - z'^2 (1 + 4 + \dots + n^2) \right] = \frac{\pi ab}{c^2} \left(c^2 z - \frac{z^3}{3} \right). \quad (31)$$

If $z = c$, $2S = \frac{4}{3}\pi abc$, (32). This solution includes the sphere and the spheroids.

It is easy to extend this method to the paraboloid, hyperboloid of revolution, elliptical paraboloid, and a great variety of other cases, but the process is so easy that we need not work out these cases here. We see here how the Integral Calculus sorts out the terms of series which are needed in the final result.

The surface of the hyperboloid of revolution can be found without difficulty. For this case, N in Eq. (20) is equal to $b\sqrt{(e^2 x^2 \div a^2) - 1} = b\sqrt{(z^2 - 1)}$, $az = ex$. Hence

$$S = \frac{2\pi ab z'}{e} \left[\sqrt{(z'^2 - 1)} + \sqrt{(4z'^2 - 1)} + \dots + \sqrt{(n^2 z'^2 - 1)} \right].$$

By using Prob. IV, we can find this equal to

$$S = \pi b \left[\frac{x}{a} \sqrt{(e^2 x^2 - a^2)} - b + \frac{a}{e} \log \left(\frac{ex + \sqrt{(e^2 x^2 - a^2)}}{ae + b} \right) \right].$$

We can also easily find the surface of the paraboloid of revolution.

ORIGIN OF THE NAME "QUATERNIONS".

BY PROF. W. W. JOHNSON, ANNAPOLIS, MD.

PROF. Wood appears to have fallen inadvertently into error in referring the name of Quaternions to the four quantities; Vector, Tensor, Versor, Scalar: See page 11. It appears from the Preface to the "Lectures" (see page 47), that the four units are the heterogeneous quantities 1, i , j and k , of which the first is the real or scalar unit, and the others correspond to three mutually rectangular directions in space. Hamilton had in fact found that expressing directed lines in the triplex form $ix + jy + kz$, x , y and z being real coefficients, he was able to interpret their product as of the form

$$w + ix + jy + kz,$$

a quadruplex quantity, viz.—the sum of a number and line, the latter being perpendicular to the plane of the factor lines.

It seems to have been a favorite conception of Hamilton's to refer the real quantities of simple Algebra to the uni-dimensional quantity time, as in his Essay on Algebra as the Science of Pure Time; thus in the combination of line and number to which his investigations led him, he saw united the one of time and the three of space, as expressed in the sonnet below, which was published after his death in an Article upon the life and genius of Hamilton, in the North British Review.

SONNET BY THE INVENTOR OF QUATERNIONS.

(Supposed to have been addressed to Sir John Herschel.)

THE TETRACTYS,
Or high Matheis, with her charm severe
Of line and number, was our theme; and we
Sought to behold her unborn progeny,
And thrones reserved in Truth's celestial sphere:
While views, before attained, became more clear;
And how the One of Time, of Space the Three,
Might, in the chain of Symbol, girdled be:
And when my eager and reverted ear
Caught some faint echoes of an ancient strain,
Some shadowy outlines of old thoughts sublime,
Gently he smiled to see, revived again,
In later age, and occidental clime,
A dimly traced Pythagorean lore,
A westward floating, mystic dream of FOUR.

NOTE ON SCHULENBURG'S SOLUTION OF THE GENERAL EQUATION OF THE FIFTH DEGREE.

BY W. E. HEAL, WHEELING, INDIANA.

SCHULENBURG makes the solution of the quintic

$$x^5 + p_2x^2 - p_3x^2 + p_4x - p_5 = 0,$$

depend upon that of the sextic

$$y^6 - \frac{1}{2}q_2y^4 + \frac{1}{16}q_4y^2 - q_5y + \frac{1}{64}q_6 = 0.$$

(See ANALYST, Vol. IV, page 50, Eq. V.)

If from (1) and (4) of Eq. VIII (Vol. IV, pp. 51 and 52) we eliminate $\frac{1}{3}(a-20d)$ we find

"IX. (1) Function $a\beta\gamma\delta = 0$; thus

(2) Function $A. B. C. D = N$ having been formed, N must $= 0$."

I use the letters q_2, q_4, q_6, q_5 instead of A, B, C, D and I write "IX. (2)" thus;

$$F(q_2, q_4, q_6, q_5) = N.$$

If we write (1) and (4) of Eq. "VIII" in the form

$$a_1m^4 + b_1m^3 + c_1m^2 + d_1m + e_1 = 0,$$

$$a_2m^4 + b_2m^3 + c_2m^2 + d_2m + e_2 = 0,$$

and determine the eliminant $F(p_2, p_4, p_6, p_5)$, (see Salmon's Algebra, p. 396), it is readily seen that the term $a_1^4e_2^4$ does not vanish and therefore $F(p_2, p_4, p_6, p_5)$ cannot $= 0$. Or, in other words, the equation $N = 0$, which is assumed to be true by Schulenburg, is *not* true.

NOTE BY PROF. HYDE.—In my solution of Prob. 283, in the ANALYST for Jan. 1880, by inadvertence, a V was omitted from all the equations after the first which makes the results untrue. It should have been

$$Va\rho + V\beta\rho \quad V(a+\beta)\rho,$$

which contains the proof, since

$$UVa\rho = UV\beta\rho = UV(a+\beta)\rho,$$

(because a, β and ρ are coplanar), and therefore

$$TVa\rho + TV\beta\rho = TV(a+\beta)\rho,$$

when P is outside the parallelogram; and

$$UVa\rho = -UV\beta\rho = \pm UV(a+\beta)\rho, \text{ and}$$

$$\therefore TVa\rho - TV\beta\rho = \pm TV(a+\beta)\rho,$$

when P is inside. These tensors are the product required.

FORMULA FOR FORCE OF MORTALITY.

Editor Analyst:

An essay of mine on "Methods of Interpolation applicable to the graduation of irregular series, such as tables of mortality, &c.", part of which appeared in the *Smithsonian Report* of 1873, contained among other things, at page 343 of that report, a demonstration of the formula

$$\mu_3 = \frac{8(y_2 - y_4) - (y_1 - y_5)}{12y_3},$$

where y_1, y_2 , &c., are the "numbers living" at any five consecutive birthdays, out of a certain number of persons born, and μ_3 is the "force of mortality" at the middle birthday. The same formula has since been published, with a different demonstration and notation, in an article by Mr. W. S. B. Woolhouse, in the *London Journal of the Institute of Actuaries and Assurance Magazine* for April, 1878, page 64. I wrote to the editor in the October following, stating my priority, and asking to have it mentioned in the *Journal*. This request has not been complied with up to July, 1879, the date of the latest number I have seen. The January 1879 number contains, at page 285, a laudatory reference to the formula, and the name of Mr. Woolhouse only is mentioned as its author. I ask leave therefore to put the facts on record, and remark further, that a copy of the *Smithsonian Report* referred to was sent to the editor of the *Journal* in 1875, so that he had no need to be in ignorance—if he was ignorant—of the existence of my work, in April, 1878.

E. L. DE FOREST.

Watertown, Conn., Jan. 8, 1880.

CORRESPONDENCE RELATIVE TO PROBLEM 286.

Editor Analyst:

Problem 286 was proposed some years ago, in the *Educational Times*, and discussed thoroughly. Mr. Woolhouse made a solution the same as that by Mr. Adcock. Stephen Watson afterward made a solution obtaining the result $\frac{2}{3}$, which Mr. Woolhouse admitted was correct, and I believe was finally so admitted by all. (See *Educational Times* for Jan. 1870, June '70 and Sept. '71. The solution of Prob. 193, ANALYST, also solves this.

WALTER SIVERLY.

Oil City, Pa., Dec. 29, 1879.

[Because, in the denominator of the fraction which represents the req'd probability, the points through which the circles are supposed to be drawn

are evenly distributed over the area of the given circle, while in the numerator, of the published solution, they are represented by circumferences of circles that intersect one another, and therefore cannot be evenly distributed over the given circle; the terms, therefore, of the fraction which represents the required probability in the published solution, are non-homogeneous and hence do not necessarily represent the required result.

Instead of publishing Mr. Seitz's solution, as we promised in No. 1, we refer the reader to the solution of Problem 193 (Vol. V, p. 61), which can easily be modified so as to meet the requirements of this problem.—Ed.]

NOTE ON THE COMPUTATION OF π . By PROF. J. SCHEFFER.—Perhaps the most convenient way of calculating π may be derived from the development of $\sin^{-1}x \div \sqrt{1-x^2}$ into a series. Multiplying the well known series of $\sin^{-1}x$ and $1 \div \sqrt{1-x^2}$ together, the law of the formation of the coefficients of the ascending powers of x would not easily become apparent, but this difficulty will be obviated by the following method.

$$\text{Putting } (\sin^{-1}x)^2 = Ax^2 + Bx^4 + Cx^6 + Dx^8 + Ex^{10} + \dots \quad (1)$$

and differentiating, we obtain:

$$2\sin^{-1}x \div \sqrt{1-x^2} = 2Ax + 4Bx^3 + 6Cx^5 + 8Dx^7 + 10Ex^9 + \dots \quad (2)$$

Multiplying (2) by $\sqrt{1-x^2}$, and again differentiating,

$$2 \div \sqrt{1-x^2} = (2A + 3.4Bx^2 + 5.6Cx^4 + 7.8Dx^6 + 9.10Ex^8 + \dots) \sqrt{1-x^2} \\ - (2Ax + 4Bx^3 + 6Cx^5 + 8Dx^7 + 10Ex^9 + \dots) x \div \sqrt{1-x^2}.$$

Multiplying this equation by the radical, and comparing the coefficients of the same powers of x , we find for the coefficients the following values:

$$A = 1, B = \frac{2^2}{3.4}, C = \frac{4^2}{5.6}, D = \frac{2^2.4^2}{3.4.5.6}; E = \frac{6^2}{7.8}, C = \frac{2^2.4^2.6^2}{3.4...8}, \text{ etc.}$$

We may present these coefficients in another form, viz.;

$$A = 1, B = \frac{2}{3} \cdot \frac{1}{2}, C = \frac{2.4}{3.5} \cdot \frac{1}{3}, D = \frac{2.4.6}{3.5.7} \cdot \frac{1}{4}, E = \frac{2.4.6.8}{3.5.7.9} \cdot \frac{1}{5}, \text{ etc.}$$

Substituting in (2) we get

$$\frac{\sin^{-1}x}{\sqrt{1-x^2}} = x + \frac{2}{3}x^3 + \frac{2.4}{3.5}x^5 + \frac{2.4.6}{3.5.7}x^7 + \frac{2.4.6.8}{3.5.7.9}x^9 + \dots \quad (3)$$

Put $x = y \div \sqrt{1+y^2}$, then is $\sin^{-1}x = \sin^{-1}[y \div \sqrt{1+y^2}]$, and (3) becomes

$$\tan^{-1}y = \frac{y}{1+y^2} \left[1 + \frac{2}{3} \left(\frac{y^2}{1+y^2} \right) + \frac{2.4}{3.5} \left(\frac{y^2}{1+y^2} \right)^2 + \frac{2.4.6}{3.5.7} \left(\frac{y^2}{1+y^2} \right)^3 + \dots \right].$$

Since $\frac{1}{2}\pi = \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3}$, we have

$$\frac{\pi}{4} = \frac{4}{10} \left[1 + \frac{2}{3} \left(\frac{2}{10} \right) + \frac{2.4}{3.5} \left(\frac{2}{10} \right)^2 + \frac{2.4.6}{3.5.7} \left(\frac{2}{10} \right)^3 + \dots \right] \\ + \frac{3}{10} \left[1 + \frac{2}{3} \left(\frac{1}{10} \right) + \frac{2.4}{3.5} \left(\frac{1}{10} \right)^2 + \frac{2.4.6}{3.5.7} \left(\frac{1}{10} \right)^3 + \dots \right].$$

SOLUTION OF PROBLEM 120. (SEE VOL. III, P. 132.)

Editor Analyst:

As you requested me to examine Prob. 120 under the supposition that it should have been written

$$\text{Given } \left\{ \begin{array}{l} [x^2 + xy + y^2] \frac{1}{x+y} = m, \\ [x^2 - xy + y^2] \frac{1}{x-y} = n, \end{array} \right\} \text{ To find } x \text{ and } y,$$

I submit the following solution:

Clearing the equations of fractions and multiplying the first by $x-y$ and the second by $x+y$, and reducing, we get

$$(n-m)x^3 = (n+m)y^3;$$

whence we find

$$x = \left(\frac{n+m}{n-m} \right)^{\frac{1}{3}} \cdot y, \quad y = \left(\frac{n-m}{n+m} \right)^{\frac{1}{3}} \cdot x.$$

Substituting for y and x , respectively, in the first and second eq's, we get

$$x = m \cdot \frac{(n^2-m^2)^{\frac{1}{2}} + (n+m)^{\frac{3}{2}}}{(n+m)^{\frac{5}{2}} + (n^2-m^2)^{\frac{1}{2}} + (n-m)^{\frac{3}{2}}},$$

$$y = m \cdot \frac{(n^2-m^2)^{\frac{1}{2}} + (n-m)^{\frac{3}{2}}}{(n+m)^{\frac{5}{2}} + (n^2-m^2)^{\frac{1}{2}} + (n-m)^{\frac{3}{2}}}.$$

GEORGE H. HARVILL.

Bonner, La., Jan. 24, 1880.

PROB. 281. (SEE P. 190, VOL. VI.) SOLUTION BY PROF. DE VOLSON WOOD.—Assuming that the attraction of the sun is constantly parallel to the line of centres of the earth and sun, and of constant intensity F , then will $F \sin \varphi$ be the force which causes the ball to roll, φ being the angular distance on the earth of the ball from the line of centres. Let θ be the angle through which the ball has rolled when its angular dist. from the line of centres is φ and t the corresponding time, m the mass of the ball, v the present velocity of the earth in its orbit, D the distance between the earth and sun, R the radius of the earth, r the radius of the ball and k its radius of gyration and ω the angular velocity of the earth. Then we have



$$F = m \frac{v^2}{D}, \quad (1)$$

$$\frac{d^2 \theta}{dt^2} = \frac{F r \sin \varphi}{m k^2}, \quad (2)$$

$$R \omega t = R \varphi + r \theta. \quad (3)$$

Differentiating (3), $R\omega dt = R d\varphi + r d\theta$. (4)

Differentiating again, $0 = R d^2\varphi + r d^2\theta$;

$$\therefore d^2\varphi = -\frac{r}{R} d^2\theta = -n d^2\theta. \quad (5)$$

From (4) $dt = \frac{d\varphi + n d\theta}{\omega}$. (6)

Equations (5) and (6) reduce (2) to

$$-\frac{m\omega^2 k^2}{nrF} \cdot \frac{1}{\sin\varphi} \cdot \frac{d^2\varphi}{d\theta^2} = \frac{d\varphi^2}{d\theta^2} + 2n \frac{d\varphi}{d\theta} + n^2.$$

The integral of this (which I have not yet found in finite terms) gives

$$\varphi = f(\theta),$$

and this in equation (3) gives θ in terms of t . As t is known (12 hours) θ becomes known and hence φ is known; the latter of which makes known the distance of the ball from the line of centres. The relative velocities will be $r d\theta \div R \omega dt$.

The values of the fixed constants are, approximately, $D = 92,500,000$ miles, $v = 68,000$ miles per hour, $R = 25,000 \div 2\pi$ miles, $r = 2\frac{1}{2}$ miles, $k^2 = \frac{2}{3}r^2$, $m = (r^2 + R^2)$ times the mass of the earth, $\omega = 2\pi \div (12 \text{ hours})$. In using these the corresponding quantities must all be reduced to the same unit. The value of F is about 1,300,000,000 gross tons.

[This problem, as published, in the absence of Mr Schneider's diagram that accompanied his manuscript, leaves the direction of the sun from the point A , indeterminate. It appears, however, from Mr. Schneider's figure that he contemplated the attractive force as being in a direction from A , *opposite* to that indicated by the arrow in the foregoing figure; hence equation (3) should be written $R\omega t = R\varphi - r\theta$.

If we multiply (2) by n , then substitute in (2) for $n d^2\theta$ its value $d^2\varphi$ from (5), (2) will become

$$\frac{d^2\varphi}{dt^2} = \frac{r^2 F}{mk^2 R} \sin\varphi.$$

Multiplying this eq. by $2d\varphi \div dt$, integrating once, introducing the correction and extracting the square root of both members, we get

$$\frac{d\varphi}{dt} = \frac{r}{k} \sqrt{\left(\frac{2F}{mR}\right) \cdot \sqrt{(a - \cos\varphi)}}. \quad (7)$$

Substituting for dt in (7) its value from (6), reducing and integrating again, we get

$$\theta = \frac{R}{r} \varphi - \frac{\omega k R}{r^2} \sqrt{\left(\frac{mR}{2F}\right)} \int \frac{d\varphi}{\sqrt{(a - \cos\varphi)}}$$

which is an elliptic function of the first order.—Ed.]

288. [Prof. J. H. Kershner, E. J. Edmunds, Geo. Eastwood and Dr. H. Eggers, each sent a solution of this question, but they all lead to complicated equations of a high degree, and are too long for the space that remains at our disposal in this No. We give below the result obtained by Mr. Eastwood.]

Putting r and R for the radii of the two circles, whose centers are O_1 and O_2 , O_1 being the origin and (x_1, y_1) , (x_2, y_2) coordinates of M and N respectively; also putting $AC = a$ and $AB = \beta$, the equation of line MN is

$$y - y_1 = \frac{y_1 - y_2}{x_1 - x_2}(x - x_1). \quad (7)$$

Substituting for x_1, y_1, x_2, y_2 from equations (1), (2), (5) and (6) we get

$$[y(a^2 + r^2) - r(a^2 - r^2)] \cdot [(2R^2r^2a - \beta a^2 - \beta r^2)(\beta - a)^2 + R^2(\beta - 2a)a^2 + \beta R^2r^2] \\ - [2r(R^2 + \beta - a)a^2 - 2R(a^2 + r^2)(\beta - a)^2] \cdot [x(a^2 + r^2) - 2R^2a] = 0. \quad (8)$$

The differential coefficient of (8), with respect to a , equated to zero gives an equation of the 5th degree in a , any real root of which substituted in (8) will give the envelop of MN .

If P denote the middle point of MN the equation of the line CP will be

$$y - r = \frac{r - \frac{1}{2}(y_1 - y_2)}{a - \frac{1}{2}(x_1 + x_2)}(x - a).$$

Combining this equation with (7) we get for the eqn. of the locus of P

$$2y - (r + y_1) = \frac{y_1 - y_2}{x_1 - x_2}(x - x_1) + \frac{r - \frac{1}{2}(y_1 + y_2)}{a - \frac{1}{2}(x_1 + x_2)}(x - a). \quad (10)$$

SOLUTIONS OF PROBLEMS IN NUMBER ONE.

SOLUTIONS of problems in No. 1 have been received as follows:

From R. J. Adcock, 289, 292, 295; Prof. R. C. Arendt, 295; Prof. W. W. Beman, 295; Marcus Baker, 289; Alex. S. Christie, 291; Dr. H. Eggers, 295; Prof. A. B. Evens, 295; George Eastwood, 295; Prof. A. Hall, 295; Prof. W. W. Johnson, 289, 294, 295; Prof. J. H. Kershner, 289, 291, 295; Chas. H. Kummell, 289, 291, 292, 295; Prof. H. T. J. Ludwick, 295; Prof. J. Scheffer, 289, 295; Prof. E. B. Seitz, 289, 291, 295; Walter Siverly, 289; Geo. Lilley, 289; Prof. W. P. Casey, 289, 295.

289. "In any spherical triangle, show that

$$\tan c = \frac{\cot A \cot a + \cot B \cot b}{\cot a \cot b - \cos A \cos B},$$

SOLUTION BY GEORGE LILLEY, A. M., CORNING, IOWA.

Multiplying together the two well known formulæ of spherical trigonometry,

$$\left. \begin{aligned} \cos a &= \cos b \cos c + \sin b \sin c \cos A \\ \sin a \cos B &= \sin c \cos b - \cos c \sin b \cos A \end{aligned} \right\}, \text{ we readily obtain}$$

$$\tan c = \frac{\cot a \cos A + \cot b \cos B}{\cot a \cot b - \cos A \cos B}.$$

Hence the proposition is incorrectly stated.

[All our correspondents, who are credited with solutions of this problem, have obtained the same value for $\tan c$ as is here given by Mr. Lilley.]

290. No solution received.

291. "If $ABC, A'B'C'$ are two trirectangular triangles on the surface of a sphere (the letters being arranged in the same order of rotation); show that $\cos AA' = \cos BB' \cos CC' - \cos B'C \cos BC'.$ "

SOLUTION BY ALEX. S. CHRISTIE, U. S. COAST SURVEY, WASH., D. C.

Adopting the usual quaternion notation, and putting in all the steps of the process:

$$\begin{aligned} \cos AA' &= -Saa' = -S.V\beta\gamma V\beta'\gamma' \\ &= S\gamma V\beta V\beta'\gamma' = S\gamma(\gamma'S\beta\beta' - \beta'S\beta\gamma') \\ &= S\beta\beta'S\gamma\gamma' - S\beta'\gamma S\beta\gamma' \\ &= \cos BB' \cos CC' - \cos B'C \cos BC' \end{aligned}$$

when $a = V\beta\gamma$ and $a' = V\beta'\gamma'$, or the triangles lettered in the *same* order of rotation; but $= \cos BC' \cos B'C - \cos BB' \cos CC'$

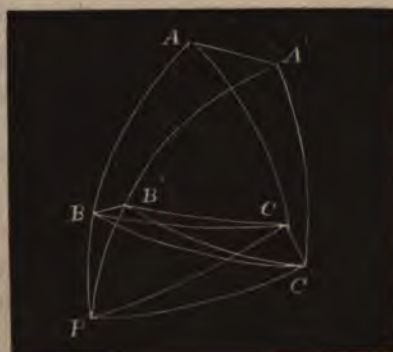
when $a = V\beta\gamma$ and $a' = V\gamma'\beta'$, or the triangles lettered in the *opposite* order.

SOLUTION BY PROF. E. B. SEITZ, KIRKSVILLE, MO.

Produce AB and $A'B'$ to P , and draw the arcs PC and PC' , each of which is a quadrant.

Since BPC and $B'PC'$ are right angles, the angles BPB' and CPC' are equal; hence

$$\begin{aligned} \cos BPB' &= \cos CPC' = \cos CC', \\ \cos B'PC &= \sin CPC' = \sin CC', \\ \text{and } \cos BPC &= -\sin CPC' \\ &= -\sin CC' \end{aligned}$$



In the triangles PBB' and PAA' we have

$$\cos BB' = \cos PB \cos PB' + \sin PB \sin PB' \cos CC', \quad (1)$$

$$\begin{aligned} \cos AA' &= \cos PA \cos PA' + \sin PA \sin PA' \cos CC' \\ &= \sin PB \sin PB' + \cos PB \cos PB' \cos CC'. \end{aligned} \quad (2)$$

In the triangles $PB'C$ and PBC' we have

$$\cos B'C = \sin PB' \cos B'PC = \sin PB' \sin CC' \quad (3)$$

$$\cos BC' = \sin PB \cos BPC' = -\sin PB \sin CC'. \quad (4)$$

Substituting the values of $\sin PB'$ and $\sin PB$ from (3) and (4) in (1) and (2), and reducing, we find

$$\cos BB' \sin^2 CC' = \cos PB \cos PB' \sin^2 CC' - \cos B'C \cos BC' \cos CC', \quad (5)$$

$$\cos AA' \sin^2 CC' = \cos PB \cos PB' \sin^2 CC' \cos CC' - \cos B'C \cos BC'. \quad (6)$$

Subtracting (5) multiplied by $\cos CC'$ from (6), and dividing by $\sin^2 CC'$, we have

$$\cos AA' = \cos BB' \cos CC' - \cos B'C \cos BC'.$$

292. "Two surveyors measure a plane quadrangular field, one measuring the four sides a, b, c, d with a chain and the other, the angles $(ab), (bc), (cd), (da)$ with a theodolite. From former experience it is known that the first is liable to a probable error of m inches per chain and the other to a probable error of n'' per angle. Required the weights of the linear and angular measurements, also conditions to be fulfilled by the measured quantities in approximate linear form and the analytical formation of the normal equations for determining the most probable corrections to the measured quantities."

SOLUTION BY R. J. ADCOCK, ROSEVILLE, ILL.

Let the true or required values of the measures a, b, c, d , be x, y, z, u , and the measured angles $(ab), (bc), (cd), (da)$, denote by A_1, B_1, C_1, D_1 , and the required values of these angles by A, B, C, D . Since the angles are separate independent measures and have a condition independent of the sides, they are to be considered separately. Then, by the principle of Least Squares announced at page 22,

$$(A - A_1)^2 + (B - B_1)^2 + (C - C_1)^2 + (D - D_1)^2 = \text{a minimum.} \quad (1)$$

Eliminating D by the equation

$$A + B + C + D = 2\pi, \quad (2)$$

and differentiating with respect to each of the other variables A, B, C , and adding the resulting equations, gives

$$A + B + C = \frac{3}{2}\pi + A_1 + B_1 + C_1.$$

Hence by (2) $D = \frac{1}{2}\pi + \frac{1}{4}D_1 - \frac{1}{4}(A_1 + B_1 + C_1)$, $C = \frac{1}{2}\pi + \frac{1}{4}C_1 - \frac{1}{4}(A_1 + B_1 + D_1)$, $B = \frac{1}{2}\pi + \frac{1}{4}B_1 - \frac{1}{4}(A_1 + C_1 + D_1)$, $A = \frac{1}{2}\pi + \frac{1}{4}A_1 - \frac{1}{4}(B_1 + C_1 + D_1)$.

For the sides, by the principle above alluded to,

$$(x-a)^2 + (y-b)^2 + (z-c)^2 + (u-d)^2 = \text{a minimum}; \quad (3)$$

from which two variables may be eliminated by the trigonometrical relations

$$z = \frac{u \sin D - y \sin A}{\sin(C+D)} = nu - ry, \quad x = \frac{u \sin C - y \sin D}{\sin(C+D)} = n_1 u - r_1 y, \quad (4)$$

giving $(n_1 u - r_1 y - a)^2 + (y - b)^2 + (nu - ry - c)^2 + (u - d)^2 = \text{a minimum}$.

Differentiating and reducing,

$$y = \frac{(n_1 a + nc + d)(nr + n_1 r_1) - (r_1 a + rc - b)(n^2 + n_1^2 + 1)}{(r^2 + r_1^2)(n^2 + n_1^2 + 1) - (nr + n_1 r_1)^2}.$$

$$\begin{aligned} 2 \text{ area} &= ab \sin A_1 + cd \sin C_1 = bc \sin B_1 + ad \sin D_1 \\ &= \frac{1}{2}(ab \sin A_1 + bc \sin B_1 + cd \sin C_1 + ad \sin D_1), \text{ nearly.} \end{aligned}$$

Differentiating, and using the probable errors m inches and n'' for the differentials, gives

$$\frac{n''}{m} = 2 \frac{bc \sin B_1 + ad \sin D_1 - ab \sin A_1 - cd \sin C_1}{ab \cos A_1 + cd \cos C_1 - bc \cos B_1 - ad \cos D_1},$$

the square of which gives the relative weights required.

[Mr. Kummell has sent us an extended general solution of this problem which is omitted here for want of room but will be inserted in a future No. He also sent a solution of 282 with condition (3) modified, which will appear in some future number.]

293. No solution received.

294. "One curve rolls upon another; prove that a series of carried parallel curves envelope a series of parallel curves, or, involutes of the same evolute envelope involutes of the same evolute."

SOLUTION BY. PROF. JOHNSON.

Parallel curves, or involutes of a common evolute, have common normals, and intercept upon these normals a constant length. Consider the normal (or normals) of the carried curve which at any instant pass through the instantaneous centre (point of contact of rolling curves).

Since every carried point is moving in a direction perpendicular to the line which joins it with the instantaneous centre, the points where the normal cuts the parallel carried curves are moving in the directions of the tangents to these curves, and will therefore be points of contact with their envelopes. These envelopes have therefor common normals, and are parallel curves.

295. "Given the common astronomical equations

$$\text{tang } (\lambda - \mathcal{Q}) = \cos i \text{ tang } u,$$

$$\sin \beta = \sin i \sin u,$$

eliminate u , and show in this manner that

$$\text{tang } \beta = \text{tang } i \sin (\lambda - \mathcal{Q})."$$

SOLUTION BY CHAS. H. KUMMELL, U. S. LAKE SURVEY, DETROIT, MICH.

1. The first two relations are consistent with Napier's rules of circular parts as exhibited in the figure. The third is then true by the same rules.



$$\begin{aligned} 2. \quad \tan \beta &= \frac{\sin \beta}{\sqrt{1 - \sin^2 \beta}} = \frac{\sin i \sin u}{\sqrt{1 - \sin^2 i \sin^2 u}} \\ &= \frac{\sin i \tan u}{\sqrt{1 + \cos^2 i \tan^2 u}} = \frac{\tan i \tan(\lambda - \mathcal{Q})}{\sqrt{1 + \tan^2(\lambda - \mathcal{Q})}} \\ &= \tan i \sin (\lambda - \mathcal{Q}). \end{aligned}$$

SOLUTION BY PROF. A. B. EVENS, LOCKPORT, N. Y.

For the sake of brevity put x for $(\lambda - \mathcal{Q})$. The first and second equations give, respectively,

$$\tan u = \frac{\tan x}{\cos i}, \quad \sin^2 \beta = \frac{\tan^2 \beta}{1 + \tan^2 \beta} = \frac{\sin^2 i \tan^2 u}{1 + \tan^2 u}.$$

Eliminating $\tan^2 u$ and solving for $\tan^2 \beta$, we find

$$\tan^2 \beta = \frac{\tan^2 i \tan^2 x}{1 + \tan^2 x} = \tan^2 i \sin^2 x;$$

and therefore, restoring the value of x

$$\tan \beta = \tan i \sin (\lambda - \mathcal{Q}).$$

SOLUTION BY PROF. ASAPH HALL.

From the first equation

$$\frac{\sin^2(\lambda - \mathcal{Q})}{\cos^2(\lambda - \mathcal{Q})} = \frac{\cos^2 i \sin^2 u}{\cos^2 u}.$$

Adding unity to each side and putting $D = \sqrt{(\cos^2 u + \cos^2 i \sin^2 u)}$, we find

$$\sin(\lambda - \mathcal{Q}) = \frac{\cos i \sin u}{D} : \quad \cos(\lambda - \mathcal{Q}) = \frac{\cos u}{D}.$$

From the second equation, $1 - \sin^2 \beta = \cos^2 \beta = \cos^2 u + \cos^2 i \sin^2 u$ or we have

$$\cos \beta = D.$$

$$\therefore \frac{\sin(\lambda - \mathcal{Q})}{\sin \beta} = \frac{\cos i}{\cos \beta \sin i},$$

$$\text{or} \quad \text{tang } \beta = \text{tang } i \sin (\lambda - \mathcal{Q}).$$

[Solutions equally brief and elegant were sent by Professors Beman, Seitz and Johnson.]

PROBLEMS.

296. (Selected) By *L. G. Barbour, Richmond, Ky.*—Given

$$x^6 + y^6 = \frac{1}{2}\sqrt{2}, \quad x^4 + y^4 = 1;$$

to find x and y by quadratics.

297. By *Dr. H. Eggers, Milwaukee, Wis.*—Given three circles and their centers; to find the points of contact with any fourth circle with the ruler alone.

298. By *T. P. Stowell, Rochester, N. Y.*—On the diameter AB produced, of a given circle, is a given point P . A chord CD is parallel to the diameter AB . Find the position of CD , by a geometrical construction, such that the angle CPD shall be the greatest possible.

299. By *E. P. Thompson, Elizabeth, New Jersey.*—Prove that the number of terms in an expanded polynomial of t terms is equal to the number of combinations of $(n+t-1)$ quantities taken $t-1$ at a time, where n is the exponent of the polynomial.

300. By *Marcus Baker, U. S. Coast Survey, Wash., D. C.*—In a plane triangle ABC are given, the perimeter $2s$ and the radii of the inscribed and circumscribed circles, r and R , to determine the triangle.

301. Determine the angles of a plane triangle from the following relations

$$(I) \begin{cases} \tan^2 A + \tan^2 B + \tan^2 C = m, & (1) \\ \tan^4 A + \tan^4 B + \tan^4 C = n. & (2) \end{cases}$$

$$(II) \begin{cases} \tan^{\frac{2}{3}} A + \tan^{\frac{2}{3}} B + \tan^{\frac{2}{3}} C = m, & (1) \\ \tan^{\frac{4}{3}} A + \tan^{\frac{4}{3}} B + \tan^{\frac{4}{3}} C = n. & (2) \end{cases}$$

$$(III) \begin{cases} \tan \frac{1}{2} A + \tan \frac{1}{2} B + \tan \frac{1}{2} C = m, & (1) \\ \tan \frac{1}{2} A \times \tan \frac{1}{2} B \times \tan \frac{1}{2} C = n. & (2) \end{cases}$$

302. By *Alex. S. Christie, U. S. Coast Survey, Wash., D. C.*—Given n points in space, to construct a polygon of n sides, having its sides bisected by the n given points, and determine the number of solutions.

303. By *Benjamin Headley, Dillsborough, Ind.*—Required the length of one branch of a curtate cycloid; the generating circle having a radius of 4 inches and the generating point a radius of 8 inches.

304. *By Prof. W. W. Beman, Ann Arbor, Mich.*—Find the limit of error in the following method of trisecting an arc approximately: Trisect the chord of the arc, also the diameter parallel to it, and the corresponding semicircumference. Join the corresponding points of trisection of the semicircumference and the diameter, producing the lines till they intersect. From the point thus found pass lines through the points of trisection of the chord until they meet the arc.

305. *By Prof. J. Scheffer.*—Find the eq. of the curve, rect. co-ord. (x, y) , in which the length $s = \frac{ny}{x} = \int_0^x dx \sqrt{\left[1 + \left(\frac{dy}{dx}\right)^2\right]}$, n being a constant.

306. *By Prof. W. W. Johnson.*—If N is an odd number and m the number of odd numbers less than N and prime to it (including 1), then N is a divisor of one of the two numbers $2^m \pm 1$.

307. *By Prof. E. B. Seitz, Kirksville, Mo.*—Three points are taken at random in the surface of the northern hemisphere; find the chance that the small circle through the points lies wholly north of the equator, supposing the earth a perfect sphere.

308. *By Chas. H. Kummell, Assist. U. S. Lake Survey.*—Evaluate $\int_0^{\frac{\pi}{2}} dx \sin [x(a-x)]$ and $\int_0^{\frac{\pi}{2}} dx \cos [x(a-x)]$ by rapidly converging series.

PUBLICATIONS RECEIVED.

The Cosmogony of Laplace (pamphlet). By DANIEL KIRKWOOD, LL. D. (Read before the American Philosophical Society, Sept. 19th, 1879.)

Micrometrical Measurement of 1054 Double Stars, Observed with the 11 inch Refractor from Jan. 1, 1878 to September 1, 1879, Under the Superintendence of ORMOND STONE, A. M., Director. 180 pp., 8vo. Cincinnati. 1879.

The National Quarterly Review, Edited by DAVID A. GORTON, M. D. and CHAS. H. WOODMAN. New York. January, 1849.

The Jan. No. of this very able *Review* contains, among other valuable articles, an article by Dr. Gorton, the senior editor, on "The Hygiene of Water" which is especially interesting and a historical and critical review of the Nebular Hypothesis, by Prof. David Trowbridge, in which an effort is made to reconcile the hypothesis with the anomaly presented by the inner satellite of Mars.

ERRATA.

On page 31, line 9, for "inverse order of rotation", read same order of rotation.

" " 52, last line, for "mystic", read mistic.

THE ANALYST.

OL. VII.

MAY, 1880.

No. 3.

QUATERNIONS.

BY PROF. DE VOLSON WOOD, HOBOKEN, NEW JERSEY.

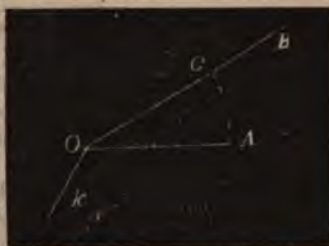
(Continued from page 38.)

MULTIPLICATION AND DIVISION OF RECTANGULAR VECTORS.

14. THE addition and subtraction of vectors, the principles of which were given in the preceding Art., are useful in determining the relation between points. This includes a large and important field in geometry, some of the problems of which may be classed under the general heads of Symmetry, Transversals, Anharmonic Ratio, Mean Point, &c.* We next proceed to compare one vector with another.

15. Conceive two vectors in space, not parallel, and of unequal lengths. Through some point draw two lines parallel and equal in length respectively to the two lines, then will these lines, as vectors, be equal respectively to the former ones. Let $OA = a$ and $OB = b$. To compare these vectors, it is required to compare OB with OA , the comparison involving both their relative directions and relative lengths. Proceeding in a practical way, we first revolve the line OA about the end O , or, more properly, about an axis through O perpendicular to the plane of AB , to coincide in direction with OB . Suppose that A falls at C , then will their relative length be expressed by

$$\frac{\text{length } OB}{\text{length } OA} = \frac{OB}{OC}.$$



*Hamilton devoted about 30 pages to this part of the subject in his Lectures, and about 10 pages in his Elements of Quaternions—a posthumous work of 762 pages, pub. 1866: Also Hail and Kelland Introduction to Quaternions about 25 pp., and about 30 pp. in the writer's co-ordinate Geometry.

The problem before us may be stated thus, to find a value for the expression

$$\frac{\text{vector } OB}{\text{vector } OA} = \frac{\beta}{\alpha}.$$

The notation of ordinary algebra not being applicable to this case, a notation and system had to be invented.

16. *Positive rotation* will be considered left-handed,* or in a direction opposite to that of the motion of the hands of a watch, the face of the watch being towards the eye as the observer views the rotation. The UNIT-ANGLE will be taken as a quadrant—or 90° —and *rotation* through a quadrant will be A UNIT-OPERATION in regard to rotation; hence a rotation of a part of a quadrant will be a part of a unit operation, and more than a quadrant, a multiple unit.

17. Beginning with the simplest case, we take three mutually perpendicular lines† of equal length passing through a point, and let their common length be called a unit; then will each line serve as an axis about which either of the others may be revolved. Let $OA = i$, be positive to the right, $OB = j$, positive upwards, and $OC = k$, positive in front or the plane of AOB ; then will i, j, k constitute a system of three mutually perpendicular unit-vectors. OA being positive to the right, AO will be negative, and may be represented by OD , drawn to the left of O on AO prolonged; and, making its length unity, it will be represented by $-i$. Similarly for $-j$ and $-k$. Let the line OA be revolved positively



*Hamilton considered right-handed rotation as *positive*, but truly remarked that either direction may be assumed as such. I choose the former because it agrees with the conventional method of generating an angle in Trigonometry, Analytical Geometry and Mechanics.

†It is impossible to have more than three mutually perpendicular lines in a set having a common point, and hence all problems involving four or more rectangular dimensions of space are necessarily hypothetical. But such problems may properly be the subject of analysis when laws are assigned to the hypothetical conditions.

Thus, if the radius be unity, it is impossible to have a cosine equal to 2, but by assuming that the laws of trigonometry apply to hypothetical (or impossible) cases in that science, an expression may be found for each of the trigonometrical functions when $\cos z = 2$. For we will have $\cos^2 z + \sin^2 z = 1$; $\therefore \sin z = \sqrt{-3}$, and similarly for other values. This result shows that the assumption involves an impossible condition—one contrary to the limitations of the science. The principle is the same as that contained in the old arithmetical problem,

"If one-third of six be three, What will a fourth of twenty be?"

Hamilton, at one time, sought to connect calculation with *Geometry*, through some undiscovered extension, to space of three dimensions. Lectures, preface, p. (31).

‡This is Hamilton's notation, Lectures, p. 59. The letters i, j, k , are, by common consent, appropriated for this purpose. The letter i had long been used by mathematicians to represent the even root of negative unity.

(that is, left-handed) through a unit angle, about $+k$ as an axis, to coincide with $+j$. It is required to devise an expression which will involve simultaneously the unit angle and the three vectors, and which will call to mind the nature of the operation. Conceive that k is the agent producing the rotation and that $+k'$, or simply k , represents the fact that it *produces a unit operation*, and writing " $=$ " for *produces*, we have

$$k = \frac{j}{i}. \quad (3)$$

This expression is read ' k equals j divided by i , or i divided into j ' but not ' k equals i into j ', for this last expression has been reserved for multiplication.* This expression, though called an *equation*, is not algebraic in the ordinary sense of that science.† It is a vector equation and must be multiplied as such. Every notation carries whatever is assigned to it; hence every equation between three vectors of the form of (3) expresses the fact that—the left member represents a vector perpendicular to the plane of the other two, about which as an axis the vector represented by the denominator of the right member is turned in a positive direction through a quadrant to coincide with the vector of the numerator.

In this manner each of the following equations may be interpreted.

$$k = \frac{-j}{-i}, \quad k = \frac{i}{-j}, \quad k = \frac{-i}{j}, \quad (4)$$

$$i = \frac{k}{j}, \quad i = \frac{-j}{k}, \text{ \&c.} \quad j = \frac{i}{k}, \quad j = \frac{-k}{i}, \text{ \&c.}$$

If the vector axis be $-k$, positive rotation will be from j towards i &c.; hence

$$-k = \frac{-j}{i}, \quad -k = \frac{i}{j}, \text{ \&c.}, \quad (5)$$

which results are the same as a negative rotation about $+k$.

A comparison of equations (3) and (5) shows that a reciprocal of the fraction *changes the sign* of the vector axis, instead of producing its reciprocal.

*Hamilton sought to preserve not only the symbols of algebra—as letters and signs—but also, in a measure, its language in the use of terms—as addition, subtraction, multiplication, division, raising to powers, and extracting roots. The enlarged and modified use of a symbol, as occasion demands, is a process constantly going on in mathematics, and is classed as "The principle of extension by the removal of restrictions." This was recognized by Hamilton—See *Lectures* p. (15), foot note.

This enlargement of the signification of terms and symbols is much more desirable than the introduction of new terms and signs to represent every new shade of meaning; for it not only avoids the necessity of taxing the memory in using them but when the analysis degenerates into the more ordinary operations, the necessity of transforming a new into an ordinary notation is avoided.

†It really involves four units, one of length and three of relative direction.

18. Suppose that k operates upon i turning it in one operation through two unit angles, it will bring i into the position $-i$. Then will the right member of the equation become $i + -i = -1$, but what shall be the form of the left member? It must be $2k, 3k, k, k^2$, or some other form. Instead of making an arbitrary choice we observe that multiplying (3) and (4) together, member by member, will give minus 1 (as it should, since it is simply combining two successive unit operations) and we have

$$k.k = \frac{j}{i} \cdot \frac{-i}{j} = \frac{-i}{i} = -1.$$

The left member is of the form of the multiplication of two equal factors in algebra; hence borrowing (or extending) the notation of that science for this case,* we have

$$k^2 = -1, \quad (6)$$

which is read in the usual way ' k square equals minus 1'—or more fully, 'the square of a unit vector produces minus unity'. In other words, it reverses the *direction* of a line.

19. The index (or exponent) expresses the number of units through which a line is revolved, *the number of quadrants being equal to the units in the index*. Hence k^2 implies a rotation through three quadrants about k as an axis; k^4 , four quadrants; $k^{\frac{1}{2}}$, one half a quadrant or 45° ; $k^{\frac{1}{3}}$, a rotation of 30° ; k^{-1} , a rotation in a *negative* direction through a quadrant, $-k^{\frac{1}{2}}$, a *positive* rotation of 60° about a *negative* vector axis; $-k^{-\frac{1}{2}}$, a *negative* rotation of 135° about a *negative* vector axis; and generally ' k^t ' represents a rotation through t quadrants, or parts of a quadrant, about a vector axis; where t may be positive or negative, entire or fractional, and k a unit vector, positive or negative. If θ be an angle measured by the subtended arc, we have $t = \theta + \frac{1}{2}\pi$, and

$$k^t = k^{\frac{\theta}{\pi}}. \quad (7)$$

20. A **VERSOR**† is a representative of rotation. The axis about which rotation takes place is taken as a representative of the agent producing the rotation, and the angle as the amount of rotation produced by the agent; and the axis and rotation combined is called *the versor*, and may be represented as in equation (7) above. If the rotation be through a quadrant, it is called a *quadrantal versor*, hence i, j, k , or any other unit vector with unity for an exponent, is a quadrantal versor. The letter U ‡ placed before

*One of the peculiarities of Hamilton's style may be noticed by carefully reading Articles 75, 81, and 85, of his Lectures. It will be observed that he uses $\sqrt{-1}$ before he introduces the exp. 2; and when he introduces the latter (p. 81) he simply says "or using the EXP. 2" &c.

†From a Latin word signifying that which turns about.

‡Lectures, pp. 88, 118.

an expression implies that the rotation, or the act of version, is to be considered. Thus, $U(\beta + \alpha)$ implies the turning of vector α to coincide in direction with vector β , hence if θ be the angle between α and β we have

$$U \frac{\beta}{\alpha} = k^{2\theta + \pi*}, \quad (8)$$

where θ may be positive or negative and k also positive or negative. The letter U may also be used to denote a unit vector; thus $U\beta$ is a unit vector parallel to vector β . As a versor it implies that β has been turned from some arbitrary direction into the given one. Hence we may write

$$\beta = T\beta \cdot U\beta.$$

The versor of a number is merely a sign, $+$ or $-$, but as multiplying by unity does not change the product, it may be written $+1$ or -1 ; thus

$$U.3b = +1, \quad U(-\sqrt{2}) = -1.$$

From equation (6) we have

$$k = -\frac{1}{k} = -(k^{-1}); \dagger \quad (9)$$

that is, a vector equals minus the reciprocal of the same vector.

21. From equation (6), by taking one-half the exponents of both sides and using the radical sign as in algebra, we have

$$k = (-1)^{\frac{1}{2}} = \sqrt{-1}. \quad (10)$$

This is the form of the simplest imaginary of algebra, but it is not to be interpreted algebraically. The right member is simply a particular expression for the left member,** and hence implies that a line is rotated through a quadrant. The same expression may not only be found for i and j ,†† but the line to be rotated will also disappear from the expression. Hence, Hamilton called the expression $\sqrt{-1}$, when separated from i , j , k , or any other axis, "an INDETERMINATE VECTOR-UNIT, or a unit vector with indeterminate direction."‡‡ In this system it is perfectly real,† but in algebra it is impossible and is properly called imaginary†—it does not represent any real line.† On account of its indeterminate character it is rarely used in this science.

The expression $k^2 = -1$ has an infinite number of real roots, since it is applicable to an infinite number of pairs of rectangular vectors in a plane perpendicular to vector k . The corresponding case in algebra is furnished by the equation of the sphere

$$x^2 = x^2 + y^2 + z^2,$$

*Lectures, p. 88. †Ibid., p. 114. ‡Ibid., pp. 122, 123, 126. **It simply denotes a certain vector unit, Ibid., p. 397. ††"Every unit vector is regarded as one of the square roots of negative unity." Preface, p. (61), foot note. ‡‡Ibid., p. 178. †Ibid., p. 180. †'Ibid., pp. 635-36. †''Ibid., p. 638.

which, being true and real for all values of x , y and z less than r , gives an infinite number of values of r , all equal in length, but different in position. The equation $k^2 = -1$, when k has a fixed position, is the equation of a circumference whose radius is unity; but as the position of k is indeterminate, it is the equation of a sphere whose radius is unity.*

Although we have

$$i^2 = j^2 = k^2 = -1,$$

it does not follow that

$$i = j = k,$$

and it is evident from the figure and the definition of a vector that they cannot be equal, for they necessarily differ in *direction*. The case is similar to one in algebra, where we have

$$x^2 = y^2 = z^2 = 4^2 = 1,$$

from which it does not follow that

$$x = y = z = 4 = 1;$$

neither does it follow from the equation

$$0 \cdot x = 0 \cdot y \text{ that } x = y.$$

These are cases where the general values of the quantities differ, but which reduce to an identity under particular forms of expression.

22. The preceding operations are called The Division of Vectors; we now proceed to consider their multiplication. If k operate upon i producing j , as before described, and then the lines be multiplied together, we may express the operation thus

$$ki = j, \quad (11)$$

which is read ' k into i equals j ' or ' i multiplied by k equals j '; but not ' i into k ' nor ' k multiplied by i '.† This compared with equation (3) shows that the expression for division may be changed to multiplication by clearing the former of fractions as in algebra, being particular however to place the versor as the *first* factor, for reasons which will soon appear. We thus see that the multiplication and division of rectangular unit vectors are intrinsically the same.‡ An examination of the figure shows that, in the last equation, j may be considered as the versor turning k into i .

If k turns i through two unit angles, the expression becomes

$$k^2i = -i; \therefore k^2 = -1, \text{ and } k = \sqrt{-1},$$

as before. If the length of the vector be a we have

$$(ak)(ak) = a^2k^2 = -a^2,$$

that is, the square of a vector equals minus the square of a line of equal length.†' This furnishes one mode of passing from a vector to a line.

*Lectures, p. 181. †Ibid., p. 37. Hamilton interprets this eq'n thus; —a quadrantal rotation about i from k to $-j$ followed by a quadrantal rotation about $-k$ from $-j$ to i is the same as a single quadrantal rotation about j from k to i . ‡Ibid., pp. 90, 91. †Ibid., p. 81.

23. If i operate upon k positively, we have (see the figure)

$$ik = -j, \quad (12)$$

which compared with (11) shows that ki does not equal ik , but that we have

$$ki = -ik.$$

This is called the *non-commutative* principle of the multiplication of vectors—by which is meant that the product is not the same when the factors are interchanged.* This property draws a sharp line of distinction between this science and ordinary algebra.†

The signs $+$ and $-$ are commutative; for an exam. of the fig. shows that

$$(+i)(-j) = -k = (-i)(+j) = -ij.$$

If both sides of the equation $ij = k$, be multiplied into k , we have

$$ijk = k^2 = -1, \text{ also } i^2 = j^2. \ddagger$$

If the cyclical order be preserved the result will remain the same, but if that order be deranged the sign of the result will be changed, and we will have

$$ijk = jki = kij = -kij = -ijk = -jki = -1. \S$$

24. The *associative principle* consists in the grouping of the factors in different ways. It may be shown that this principle is the same in this system as in algebra,** so that we have

$$ai.jk = a.ij.k = aijk. \P$$

25. The *distributive principle* consists in making the product of a factor into the sum of two or more quantities equal to the sum of the products of the factors into each of the quantities separately. This principle is also the same in this system as in algebra; hence we have

$$a(2\beta + b\gamma) = 2a\beta + ba\gamma.$$

(To be continued.)

*Hamilton labored to retain the *commutative* principle but was compelled to abandon it in order to establish his system. Lectures, preface, pp. (43)-(46). ††Ibid., pp. 208, 237, 333.

†There are however many instances in arithmetical and algebraic analysis in which a change of place of the characters or symbols produces a change in the result, as will be seen by comparing the following pairs of expressions;— $\log \cos x$, $\cos \log x$; $\sin \tan^{-1} x$, $\tan^{-1} \sin x$; $\log \sec^{-1} x$, $\sec^{-1} \log x$; a^x , x^a ; $\frac{1}{3}$, $\frac{1}{4}$; 45, 54.

‡The establishment of this equation was considered as the foundation of the syst. p. (46)

**The associative principle is not a necessity. Thus, in the octinomials of Caley and Graves, of the form $a+ib+jc+kd+le+mf+ng+oh$, where i, j , &c., are each $\sqrt{-1}$, we have $i^2 = j^2 = k^2 = l^2 = m^2 = n^2 = o^2 = ijk = ilm = ino = jln = jmo = klo = knm$.

But $i.jl = in = -0 = -kl = -ij.l$. Phil. Mag., May, 1845, p. 210. Trans. R. Irish A. Vol. XXI, part II, pp. 338, 339. ††Lectures, pp. 199, 289, 300.

DETERMINATION OF THE AREA OF AN ELLIPSE GEOMETRICALLY, THAT OF THE CIRCLE BEING KNOWN.

BY OCTAVIAN L. MATHIOT, BALTIMORE, MD.

LET $ABDE$ represent any upright circular cylinder whose diameter is AB and perpendicular height BD . Let the cylinder be cut by a plane through D and A , the cutting edge of which is parallel to the plane of the base; then will the section $AFDH$ be an ellipse whose transverse axis is AD and whose conjugate axis $FH = AB$, the diameter of the cylinder; and it is evident from the figure that this ellipse divides the cylinder into two equal parts.

Prolong the cylinder to IK so that $DI = DB$, and through IE pass a plane parallel to the plane through DA , then will the section $EMIL$ be an ellipse similar and equal to the ellipse $AFDH$.

Let the diameter of the base, $AB = b$ and the height $BD = h$, then is $AD = \sqrt{b^2 + h^2} = a$, say; hence a and b are the axes of the ellipse.

Produce AD to meet a perpendicular from I in O , then will the triangles ABD and IOD be similar, and we shall have $AD : AB :: ID : IO$, or, $a : b :: h : p$, where $p = IO$ the perpendicular distance between the sections through AD and IE ; hence

$$p = \frac{bh}{a}; \therefore A \frac{bh}{a} = \frac{1}{2}\pi b^2 h, \quad (1)$$

where A = the area of the ellipse;

$$\therefore A = \frac{1}{2}\pi ab. \quad (2)$$

The equality expressed by (1) is obvious from the construction of the Fig., and the result, in (2) is the recognized expression for the area of an ellipse, but, so far as I know, it has hitherto been obtained from a higher analysis than the foregoing.

In giving the above formula for the mensuration of the ellipse, a well-known author says; "this rule and the one for the following problem (the parabola) cannot be demonstrated without the aid of principles not yet considered" &c. (See Davies' Legendre, for the use of schools, p. 284.)



ON SOME PROPERTIES OF POLYNOMIALS.

BY E. L. DE FOREST.

(Continued from page 46.)

WE pass now to the consideration of that other property of polynomials of one variable, which was demonstrated in the *ANALYST* for Jan., 1880, pp. 5 to 7. It can be treated in a more simple and general way as follows.

Let the coefficients in the two polynomials, (17) of the present paper, be regarded as magnitudes placed at equal distances Δx along a straight line which is taken as the axis of X . If these two systems of coefficients were parallel forces acting as was supposed, they would have each a centre of parallel forces, located in the axis of X . As before, let the distances of these centres from the places of a_0 and c_0 be h_1 and h_2 , and let the sums of the coefficients in the two systems be S_1 and S_2 . Also let r_1 and r_2 be distances such that $S_1 r_1^2$ in the first system is equal to the sum of the products formed by multiplying each coefficient of that system into the square of its distance from the centre of parallel forces, while $S_2 r_2^2$ has a similar significance in the second system. These distances, r_1 and r_2 would be the radii of gyration, if the coefficients in each system were the masses of material points attached to the imponderable axis of X , and rotating in the plane of XY about the centre of parallel forces. They cannot however be completely represented as such because masses are always positive, while the coefficients in (17) may be either positive or negative. Still, for convenience of nomenclature, we will call r_1 and r_2 radii of gyration.

Now according to the definition,

$$\left. \begin{aligned} S_1 r_1^2 &= a_0 h_1^2 + a_1 (h_1 - \Delta x)^2 + a_2 (h_1 - 2\Delta x)^2 + \dots + a_m (h_1 - m\Delta x)^2, \\ S_2 r_2^2 &= c_0 h_2^2 + c_1 (h_2 - \Delta x)^2 + c_2 (h_2 - 2\Delta x)^2 + \dots + c_n (h_2 - n\Delta x)^2. \end{aligned} \right\} \quad (29)$$

Expanding the binomials and arranging the terms,

$$\left. \begin{aligned} S_1 r_1^2 &= h_1^2 (a_0 + a_1 + \dots + a_m) - 2h_1 \Delta x (a_1 + 2a_2 + \dots + ma_m) \\ &\quad + (\Delta x)^2 (1^2 a_1 + 2^2 a_2 + \dots + m^2 a_m), \\ S_2 r_2^2 &= h_2^2 (c_0 + c_1 + \dots + c_n) - 2h_2 \Delta x (c_1 + 2c_2 + \dots + nc_n) \\ &\quad + (\Delta x)^2 (1^2 c_1 + 2^2 c_2 + \dots + n^2 c_n). \end{aligned} \right\} \quad (30)$$

Let g_1 and g_2 be the radii of gyration of the two systems about the places of their first terms, in the same sense as r_1 and r_2 are radii about the centres of parallel forces. Then by the definition,

$$\left. \begin{aligned} S_1 g_1^2 &= (1^2 a_1 + 2^2 a_2 + \dots + m^2 a_m) (\Delta x)^2, \\ S_2 g_2^2 &= (1^2 c_1 + 2^2 c_2 + \dots + n^2 c_n) (\Delta x)^2. \end{aligned} \right\} \quad (31)$$

Substituting in (30), we find with the help of (18) and (19),

$$r_1^2 = g_1^2 - h_1^2, \quad r_2^2 = g_2^2 - h_2^2. \quad (32)$$

Now when the two polynomials (17) are multiplied together, let R and G be the radii of gyration of the coefficients in the product, in the same sense as before, about the centre of parallel forces and the first term respectively. As in (32) for the factors, so here for the product, we shall have

$$R^2 = G^2 - H^2,$$

and taking the value of H from (20),

$$R^2 = G^2 - (h_1 + h_2)^2. \quad (33)$$

The sum of all the coefficients in the product is $S_1 S_2$, and by the definition of what we here call the radius of gyration, $S_1 S_2 G^2$ is equal to the sum of all the products formed by multiplying each coefficient into the square of its distance from the first term. Hence, supposing the first polynomial in (17) to be multiplied successively by the terms of the second one, we get

$$\begin{aligned} S_1 S_2 G^2 = & c_0(1^2 a_1 + 2^2 a_2 + \dots + m^2 a_m)(\Delta x)^2 \\ & + c_1[1^2 a_0 + 2^2 a_1 + \dots + (m+1)^2 a_m](\Delta x)^2 \\ & + c_2[2^2 a_0 + 3^2 a_1 + \dots + (m+2)^2 a_m](\Delta x)^2 \\ & + \dots \dots \dots \\ & + c_n[n^2 a_0 + (n+1)^2 a_1 + \dots + (n+m)^2 a_m](\Delta x)^2. \end{aligned}$$

In the last or general term, the coefficient of $c_n(\Delta x)^2$, by expansion of the binomials, and by virtue of (18), (19) and (31), reduces to

$$S_1 \left[n^2 + 2n \left(\frac{h_1}{\Delta x} \right) + \left(\frac{g_1}{\Delta x} \right)^2 \right].$$

Assigning to n the values 0, 1, 2, &c., in succession, we get expressions for the coefficients of $c_0(\Delta x)^2$, $c_1(\Delta x)^2$, &c., and thus find

$$\begin{aligned} S_1 S_2 G^2 = & c_0 S_1 g_1^2 + c_1 S_1 [1^2 (\Delta x)^2 + 2.1 h_1 \Delta x + g_1^2] \\ & + c_2 S_1 [2^2 (\Delta x)^2 + 2.2 h_1 \Delta x + g_1^2] \\ & + \dots \dots + c_n S_1 [n^2 (\Delta x)^2 + 2 n h_1 \Delta x + g_1^2], \end{aligned}$$

which we put in the form

$$\begin{aligned} S_2 G^2 = & g_1^2 (c_0 + c_1 + \dots + c_n) + 2 h_1 \Delta x (c_1 + 2 c_2 + \dots + n c_n) \\ & + (\Delta x)^2 (1^2 c_1 + 2^2 c_2 + \dots + n^2 c_n). \end{aligned}$$

By means of (18), (19) and (31) this is reduced to

$$G^2 = g_1^2 + g_2^2 + 2 h_1 h_2,$$

which being substituted in (33), gives

$$R^2 = g_1^2 - h_1^2 + g_2^2 - h_2^2,$$

so that by (32) we have finally

$$R^2 = r_1^2 + r_2^2. \quad (34)$$

Thus it is proved that when two entire polynomials are multiplied

together, the square of the radius of gyration in the product is equal to the sum of the squares of the radii in the two factors, the axis of rotation being at the centre of parallel forces in each case. If this product is multiplied by a third factor, the square of the radius for the new product is equal to the sum of the squares of the radii for the three factors, and so on. We thus establish the general theorem, that if any number of polynomial factors having coefficients either positive or negative, are multiplied together, the square of the radius of gyration for the product is equal to the sum of the squares of the radii for all the factors, the centres of parallel forces being the axes of rotation. As a case under this, we have the property demonstrated in my previous article, that the square of the radius of gyration in the k power of a polynomial is k times the square of the radius for the polynomial itself. In the notation of that article, $b'_2(\Delta x)^2$ and $b_2^{(k)}(\Delta x)^2$ correspond to r_1^2 and R^2 in the notation used here, and since the sum of the coefficients in each polynomial was supposed to be unity in the former demonstration, $b'_2(\Delta x)^2$ there corresponds also to the product $S_1 r_1^2$ here. By the mechanical analogy employed, this product answers to the moment of inertia of the system of coefficients whose sum is S_1 .

Particularly interesting results are those already given at pp. 3, 8 and 22 of my former article, where p and q are probabilities whose sum is unity, and the binomial $p+q$ or $p+qx$ is seen to have its centre of parallel forces at the distance $h_1 = qx$ from the place of its first term p , while the square of its radius of gyration about this centre is $r_1^2 = pq(\Delta x)^2$. Then in the expansion of $(p+q)^m$ our theorems prove that the lever arm or distance of the centre of parallel forces from the first term is $H = qm\Delta x$, and the square of the radius of gyration is $R^2 = pqm(\Delta x)^2$. Since the coefficients are all positive, they may be regarded as masses, and the centre of parallel forces becomes their centre of gravity. Whenever qm is a whole number, it is known to be the rank of the greatest term in the expansion, and our proof that the place of this term is then the centre of gravity, shows that in a series of observations whose probabilities or frequencies are represented by the terms of such an expansion, the most probable value of the observed quantity is the arithmetical mean of all the observed values.* Even if qm is not a whole number, the place of the arithmetical mean cannot deviate from the place of the greatest term so far as to reach the place of the next term on either side of the greatest. The propriety of adopting the arithmetical mean as the most probable value, is thus shown without

*Quetelet has stated, in effect, that the arithmetical mean divides the length of the series in the ratio of p to q . (*Lettres sur la The'orie des Probabilite's*, p. 181.) I am not aware that any writer has hitherto given a demonstration of it.

the aid of the assumption, usually made, that positive and negative errors of equal amount are equally probable. Again, R^2 is obtained by taking the sum of the products formed by multiplying each term of the expansion into the square of its distance from the centre of gravity, and dividing this by the sum of the terms, which is unity. Hence if qm is an integer, R is simply the quadratic (mean) error, and whether qm be an integer or not, R is the quadratic error in the sense that it is the square root of the mean of the squares of the deviations of the typical set of observations from their arithmetical mean. When m is made an infinity of the second order, and Δx becomes dx , the difference, if any, between the places of the greatest term and of the arithmetical mean vanishes, and we have

$$R = dx \sqrt{pqm}, \quad (35)$$

a well known expression for the quadratic mean error. (ANALYST, May, 1879, p. 69.)

Polynomials of two variables are found to have radii of gyration with properties analogous to those in the case of one variable. Using the same notation as in (22) and (23), let a horizontal and a vertical straight line be drawn through the centre of parallel forces, in the plane of XY , and regard these lines as axes of rotation for each of the two polynomials (22). First, let r_1 and r_2 be their radii of gyration about the horizontal axis, in the sense that $S_1 r_1^2$ is the sum of the products formed by multiplying each coefficient of the first polynomial into the square of its distance from this axis, and $S_2 r_2^2$ in like manner for the second polynomial. Let $s'_0, s'_1, \&c.$, be the sums of the coefficients in the first, second, &c. rows from the bottom, in the first polynomial, and $s''_0, s''_1, \&c.$, in like manner for the second polynomial. Then, denoting the lever arms by h_1 and h_2 as before, we have

$$\left. \begin{aligned} S_1 r_1^2 &= s'_0 h_1^2 + s'_1 (h_1 - \Delta y)^2 + s'_2 (h_1 - 2\Delta y)^2 + \dots + s'_n (h_1 - n\Delta y)^2, \\ S_2 r_2^2 &= s''_0 h_2^2 + s''_1 (h_2 - \Delta y)^2 + s''_2 (h_2 - 2\Delta y)^2 + \dots + s''_q (h_2 - q\Delta y)^2. \end{aligned} \right\} \quad (36)$$

Expanding binomials and arranging the terms,

$$\left. \begin{aligned} S_1 r_1^2 &= h_1^2 (s'_0 + s'_1 + \dots + s'_n) - 2h_1 \Delta y (s'_1 + 2s'_2 + \dots + ns'_n) \\ &\quad + (\Delta y)^2 (1^2 s'_1 + 2^2 s'_2 + \dots + n^2 s'_n), \\ S_2 r_2^2 &= h_2^2 (s''_0 + s''_1 + \dots + s''_q) - 2h_2 \Delta y (s''_1 + 2s''_2 + \dots + qs''_q) \\ &\quad + (\Delta y)^2 (1^2 s''_1 + 2^2 s''_2 + \dots + q^2 s''_q). \end{aligned} \right\} \quad (37)$$

The statical moments of the coefficients of the polynomials about their lower rows or axes of X will be as in (25),

$$\left. \begin{aligned} S_1 h_1 &= (s'_1 + 2s'_2 + \dots + ns'_n) \Delta y, \\ S_2 h_2 &= (s''_1 + 2s''_2 + \dots + qs''_q) \Delta y. \end{aligned} \right\} \quad (38)$$

Let g_1 and g_2 be the radii of gyration of the two systems of coefficients about their lower rows, in the same sense as r_1 and r_2 are radii about the axis of rotation. Then by the definition,

and by (40) we have finally

$$R^2 = r_1^2 + r_2^2. \quad (42)$$

Thus it appears that when any two entire polynomials in x and y are multiplied together, and the horizontal line through the centre of parallel forces is taken as an axis, the square of the radius of gyration of the coefficients about this axis in the product is equal to the sum of the squares of the corresponding radii in the two factors. From the similar situation of the polynomials in the horizontal and the vertical directions, it follows also that if the vertical line through the centre of parallel forces is taken as an axis, denoting by v_1 and v_2 the radii of gyration about it for the two factors, and by V that for their product, we shall have in like manner

$$V^2 = v_1^2 + v_2^2. \quad (43)$$

The sum of the squares of the distances of any coefficient from the horizontal and vertical axes of rotation, is equal to the square of its distance from the centre of parallel forces. Hence, if we suppose the system of coefficients to rotate in the plane of XY , about the centre of parallel forces, denoting by ρ_1 and ρ_2 the radii of gyration for the two factors, and by ρ that for their product, we shall have, by the definition,

$$S_1 \rho_1^2 = S_1 r_1^2 + S_1 v_1^2, \quad S_2 \rho_2^2 = S_2 r_2^2 + S_2 v_2^2, \quad S_1 S_2 \rho^2 = S_1 S_2 R^2 + S_1 S_2 V^2$$

$$\rho_1^2 = r_1^2 + v_1^2, \quad \rho_2^2 = r_2^2 + v_2^2, \quad \rho^2 = R^2 + V^2, \quad (44)$$

whence by virtue of (42) and (43),

$$\rho^2 = \rho_1^2 + \rho_2^2. \quad (45)$$

This shows that the square of the radius of gyration for the product, in the plane of XY and about the centre of parallel forces, is equal to the sum of the squares of the corresponding radii for the two factors.

The theorems here proved for the product of two factors may evidently be extended to that of three, four, or any number of factors, so that always, whether the axis of rotation is horizontal through the centre of parallel forces, or vertical through that centre, or at right angles to the intersection of these two and to the plane of XY , the square of the radius of gyration for the product will be equal to the sum of the squares of the corresponding radii for all the factors. If the factors are equal, so that the first polynomial in (22) is to be raised to the k power as in (28), then, denoting the three radii of gyration for the polynomial by r_1 , v_1 and ρ_1 , and those of its expansion to the k power by R , V and ρ , we shall have

$$R^2 = k r_1^2, \quad V^2 = k v_1^2, \quad \rho^2 = k \rho_1^2. \quad (46)$$

As an illustration of the foregoing properties, we will take two factors, the sums of their terms being $S_1 = 12$ and $S_2 = 8$, and whose coefficients, arranged as in (21), are

— 2	2	4	—2	3
1	—3	5	2	—1
2	1	—1	0	1

2	—1	1
0	3	—2
—1	4	2

(47)

The lever arms of the two systems about their lower rows and their left-hand columns are found as in (25), taking for convenience Δx and Δy as the units of x and y ;

$$\begin{aligned}
 12h_1 &= 1 \times 4 + 2 \times 5 = 14, & \therefore h_1 &= \frac{7}{6}, \\
 12k_1 &= 1 \times 0 + 2 \times 8 + 3 \times 0 + 4 \times 3 = 28, & \therefore k_1 &= \frac{7}{3}, \\
 8h_2 &= 1 \times 1 + 2 \times 2 = 5, & \therefore h_2 &= \frac{5}{8}, \\
 8k_2 &= 1 \times 6 + 2 \times 1 = 8, & \therefore k_2 &= 1,
 \end{aligned}$$

and these give the positions of the centres of parallel forces. The squares of the radii of gyration about horizontal and vertical axes through the centre of parallel forces are found as in (36).

$$\begin{aligned}
 12r_1^2 &= 3(\frac{7}{6})^2 + 4(\frac{1}{3})^2 + 5(\frac{5}{6})^2, & \therefore r_1^2 &= \frac{23}{3}, \\
 12v_1^2 &= 1(\frac{7}{3})^2 + 0(\frac{4}{3})^2 + 8(\frac{1}{3})^2 + 0(\frac{2}{3})^2 + 3(\frac{5}{3})^2, & \therefore v_1^2 &= \frac{11}{3}, \\
 8r_2^2 &= 5(\frac{5}{8})^2 + 1(\frac{3}{8})^2 + 2(\frac{11}{8})^2, & \therefore r_2^2 &= \frac{47}{4}, \\
 8v_2^2 &= 1(1)^2 + 6(0)^2 + 1(1)^2, & \therefore v_2^2 &= \frac{1}{2}.
 \end{aligned}$$

Now when the polynomials (47) are multiplied together, the coefficients of their product are as in (48), and the sum of them all is $S_1 S_2 = 96$.

— 4	6	4	— 6	12	— 5	3
2	—13	24	4	—13	16	— 7
6	— 7	—12	45	— 6	0	9
— 1	13	—16	7	21	3	— 4
— 2	7	9	— 2	— 3	4	2

(48)

To get the lever arms of this product we proceed as before:

$$96H = 1 \times 23 + 2 \times 35 + 3 \times 13 + 4 \times 10, \quad \therefore H = \frac{43}{8},$$

$$96K = 1 \times 6 + 2 \times 9 + 3 \times 48 + 4 \times 11 + 5 \times 18 + 6 \times 3, \quad \therefore K = \frac{19}{8},$$

and these give the position of the centre of parallel forces.

For the squares of the radii of gyration about horizontal and vertical axes through this centre, we have

$$96R^2 = 15(\frac{43}{8})^2 + 23(\frac{19}{8})^2 + 35(\frac{5}{4})^2 + 13(\frac{7}{4})^2 + 10(\frac{11}{4})^2, \quad \therefore R^2 = \frac{791}{8},$$

$$96V^2 = 1(\frac{19}{8})^2 + 6(\frac{7}{8})^2 + 9(\frac{1}{4})^2 + 48(\frac{1}{8})^2 + 11(\frac{3}{8})^2 + 18(\frac{5}{8})^2 + 3(\frac{9}{8})^2, \quad \therefore V^2 = \frac{53}{8}.$$

Now comparing the results for the two factors with those for their product, we see that

$$\frac{7}{8} + \frac{1}{8} = \frac{4\frac{3}{4}}{4}, \quad \text{and} \quad \frac{7}{8} + 1 = \frac{15}{8},$$

which are in agreement with (26) and (27); also that

$$\frac{3\frac{3}{8}}{4} + \frac{4\frac{7}{8}}{4} = \frac{7\frac{7}{8}}{4}, \quad \text{and} \quad \frac{1}{8} + \frac{1}{4} = \frac{3}{8},$$

which agree with (42) and (43); so that the calculation fully accords with the theory.

Properties analogous to the foregoing can also be shown to belong to polynomials of three variables, x , y , and z . Suppose two such polynomials, the sums of whose coefficients are S_1 and S_2 , these coefficients being arranged in each case in the form of a rectangular parallelepiped or block, three adjacent sides of which are taken as the coordinate planes of ZX , ZY and XY . Of course any such polynomial can be put in this form, for the absent terms, if any, can be supposed to enter with coefficients zero.

Let the exponents of the highest powers of x , y and z be m , n and v in the first polynomial, and p , q and w in the second. Then in the first one the number of terms lying along the edges in the three directions of X , Y and Z are $m+1$, $n+1$, $v+1$ respectively, and in the second, $p+1$, $q+1$, $w+1$ respectively. The total numbers of terms in the two are therefore

$$(m+1)(n+1)(v+1), \quad \text{and} \quad (p+1)(q+1)(w+1).$$

When the two are multiplied together, their product is a polynomial which, being similarly arranged, will have $m+p+1$, $n+q+1$, and $v+w+1$ terms along its edges, the total number of terms in it will be

$$(m+p+1)(n+q+1)(v+w+1),$$

and the sum of all its coefficients will be $S_1 S_2$. Let each of the two polynomial factors be divided into sections or slices parallel to one of the coordinate planes, for instance that of ZX , so that the exponent of y in the terms of the first, second, third, &c., sections shall be 0, 1, 2, &c., respectively. Also let the sums of the coefficients in these sections be $s'_0, s'_1, s'_2, \&c.$, in the first polynomial, and $s''_0, s''_1, s''_2, \&c.$, in the second one. For convenience of expression, let the product of any coefficient into its perpendicular distance from a plane be called its statical moment with respect to that plane, let its product into the square of that distance be called its moment of inertia with respect to the plane, and let the lever arm and the radius of gyration be defined in accordance with this, so that h_1, h_2 and H are the lever arms of the coefficients in the two polynomial factors and their product, with respect to the plane of ZX , while g_1, g_2 and G are their radii of gyration with respect to that plane. Then, denoting by $\Delta x, \Delta y$ and Δz the intervals between the places of successive coefficients in the three co-ordinate directions, formulas (24) and (25) will hold good for the new meaning of the

same symbols, and when we suppose the first polynomial to be successively multiplied by the first, second, &c., sections of the second one, we shall find by the same reasoning and formulas as before, that (26) is true in its new sense, that is to say, the lever arm of the product, with respect to its side in the plane of ZX , is equal to the sum of the lever arms of the two factors with respect to their sides lying in the same plane. The same is evidently true with respect to any other corresponding sides of the two given factors and their product. Hence if any number of polynomials in x , y and z are multiplied together, the lever arm of the coefficients in the final product, with respect to the plane of one of its sides, is equal to the sum of the lever arms for all the factors with respect to their corresponding sides. So too if a single polynomial, whose lever arms with respect to the three coordinate planes are h_1 , k_1 and l_1 , is to be raised to the k power, the corresponding lever arms H , K and L of the expansion will be equal to kh_1 , kk_1 and kl_1 respectively. If the polynomial is intersected by three planes parallel to the coordinate planes and at intervals from them equal to the corresponding lever arms, their point of common intersection may be called the centre of forces, and if the polynomial is raised to any power or powers, the centre of forces keeps always the same position, relatively to the sides of the block which the expansion occupies. When all the coefficients are positive, they may be regarded as masses, and the centre of forces becomes their centre of gravity.

Passing now to the radius of gyration, the symbols retaining their new significance, we will also denote by r_1 , r_2 and R the radii for the two polynomial factors and their product, with respect to a plane parallel to the plane of ZX and passing through the centre of forces. We shall find that all the formulas from (36) to (42) hold good in their new meaning, so that R^2 is equal to the sum of r_1^2 and r_2^2 . The same would evidently be true if the radii were taken with respect to planes through the centre of forces and parallel to either ZY or XY . Hence if any number of polynomials in x , y and z are multiplied together, the square of the radius of gyration in the product, taken with respect to a plane through the centre of forces and parallel to a side of the block which the product occupies, is equal to the sum of the squares of the corresponding radii for all the factors. If a polynomial with radius r_1 is raised to the k power, the square of the corresponding radius in the expansion will be kr_1^2 .

Again, let us define the moment of inertia of a coefficient with respect to a point, as the product of that coefficient into the square of its distance from the point, and define the radius of gyration of a system of coefficients with respect to the point, in accordance with this. Let the radii of two poly-

nomial factors in x, y, z , with respect to the three coordinate planes through the centre of forces in each, be r_1, v_1 and u_1 for the first and r_2, v_2 and u_2 for the second, and let R, V and U be the radii for their product. Also let the radii for the two factors and their product, with respect to the centre of forces in each, be ρ_1, ρ_2 and ρ . The square of the distance of any coefficient from the centre of forces, is equal to the sum of the squares of its distances from three coordinate planes through that centre. We have then, by reasoning similar to that by which (44) was obtained,

$$\rho_1^2 = r_1^2 + v_1^2 + u_1^2, \quad \rho_2^2 = r_2^2 + v_2^2 + u_2^2, \quad \rho^2 = R^2 + V^2 + U^2. \quad (49)$$

But we have already found

$$R^2 = r_1^2 + r_2^2, \quad V^2 = v_1^2 + v_2^2, \quad U^2 = u_1^2 + u_2^2, \quad (50)$$

whence finally,

$$\rho^2 = \rho_1^2 + \rho_2^2. \quad (51)$$

Thus, in the product of two or any number of polynomials in x, y and z , the square of what we call the radius of gyration with respect to the centre of forces, is equal to the sum of the squares of the radii for all the factors with respect to their centres of forces.

(To be concluded in No. 4.) *See p. 105*

PARALLEL CHORDS IN AN ELLIPSE.

BY PROF. ASAPH HALL.

LET φ be the eccentric angle of a point on the ellipse, or its eccentric anomaly. The rectangular coordinates of this point are

$$x = a \cos \varphi : \quad y = b \sin \varphi.$$

Let the equation of a right line cutting the ellipse be

$$y = a \cdot x + \beta.$$

Denote by φ_1 the eccentric angle where the right line cuts the ellipse, and by φ_{10} the similar angle for the opposite point in which this line meets the ellipse. For a second right line we use φ_2 and φ_{20} . Hence we have

$$b \cos \frac{1}{2}(\varphi_{10} + \varphi_1) = -a \cdot a \sin \frac{1}{2}(\varphi_{10} + \varphi_1). \quad \text{etc.}$$

If Δ_1 be the length of the chord from φ_1 to φ_{10} ,

$$\Delta_1^2 = (x_{10} - x_1)^2 + (y_{10} - y_1)^2,$$

or

$$\Delta_1^2 = 4a^2 \cdot (1 + a^2) \cdot \sin^2 \frac{1}{2}(\varphi_{10} + \varphi_1) \cdot \sin^2 \frac{1}{2}(\varphi_{10} - \varphi_1)^2.$$

Similarly

$$\Delta_2^2 = 4a^2 \cdot (1 + a^2) \cdot \sin^2 \frac{1}{2}(\varphi_{20} + \varphi_2) \cdot \sin^2 \frac{1}{2}(\varphi_{20} - \varphi_2)^2,$$

$$D_3^2 = 4a^2 \cdot (1 + \alpha^2) \cdot \sin \frac{1}{2}(\varphi_{30} + \varphi_3)^2 \cdot \sin \frac{1}{2}(\varphi_{30} - \varphi_3)^2, \\ \text{etc.}$$

If we substitute in the equation of the right line the coordinates for the points $\varphi_{10}, \varphi_1, \varphi_{20}, \varphi_2$, and eliminate α and β , we have for the equation of the chords,

$$\frac{\cos \frac{1}{2}(\varphi_{10} + \varphi_1)}{a} \cdot x + \frac{\sin \frac{1}{2}(\varphi_{10} + \varphi_1)}{b} \cdot y - \cos \frac{1}{2}(\varphi_{10} - \varphi_1) = 0.$$

$$\frac{\cos \frac{1}{2}(\varphi_{20} + \varphi_2)}{a} \cdot x + \frac{\sin \frac{1}{2}(\varphi_{20} + \varphi_2)}{b} \cdot y - \cos \frac{1}{2}(\varphi_{20} - \varphi_2) = 0.$$

The condition that these chords are parallel is

$$\varphi_{10} + \varphi_1 = \varphi_{20} + \varphi_2.$$

The length of the perpendicular on the first chord is

$$p_1 = \frac{-b \cos \frac{1}{2}(\varphi_{10} - \varphi_1)}{\sqrt{(1 + \alpha^2) \cdot \sin \frac{1}{2}(\varphi_{10} + \varphi_1)}}.$$

If we consider four parallel chords and denote the six differences of the perpendiculars by (1.2), (1.3), (1.4), (2.3), (2.4), (3.4), we shall have

$$(1.2) = \frac{b}{\sqrt{(1 + \alpha^2) \cdot \sin \frac{1}{2}(\varphi_{10} + \varphi_1)}} \cdot \left\{ \cos \frac{1}{2}(\varphi_{10} - \varphi_1) - \cos \frac{1}{2}(\varphi_{20} - \varphi_2) \right\} \\ = A \cdot (c - d),$$

$$\text{and } (1.3) = A \cdot (c - e); (1.4) = A \cdot (c - f); (2.3) = A \cdot (d - e);$$

$$(2.4) = A \cdot (d - f); (3.4) = A \cdot (e - f),$$

in which A is a factor constant for parallel chords. It is plain that the values of D^2 may be written

$$D_1^2 = B \cdot (1 - c^2); D_2^2 = B \cdot (1 - d^2); \text{etc.},$$

where B is also constant for parallel chords.

If now we form the expression

$$+ (2.3) (2.4) (3.4) \cdot D_1^2 - (1.3) (1.4) (3.4) \cdot D_2^2$$

$$+ (1.2) (1.4) (2.4) \cdot D_3^2 - (1.2) (1.3) (2.3) \cdot D_4^2,$$

we have, after substitution, and the omission of constant factors,

$$+ (d - e)(d - f)(e - f)(1 - c^2) - (c - e)(c - f)(e - f)(1 - d^2)$$

$$+ (c - d)(c - f)(d - f)(1 - e^2) - (c - d)(c - e)(d - e)(1 - f^2),$$

and we shall find that this symmetrical function reduces to zero. That is, if we form the product of the square of the chord by the mutual distances of the three other chords, the sum of the products of the first and third is equal to the sum of the products of the second and fourth.

This theorem was announced by Savary (Conn. des Tems, 1830, p. 65), in a Memoir on the orbits of double stars. It will be seen that this property holds when the intersecting line is inclined to the chords.

$$[pal]a_1 + [pbl]\beta_1 + \dots + [pl^2]\lambda_1 - p_1 l_1 = 0, \quad (5_{w,1})$$

$$[pal]a_2 + [pbl]\beta_2 + \dots + [pl^2]\lambda_2 - p_2 l_2 = 0, \quad (5_{w,2})$$

$$[pal]a_m + [pbl]\beta_m + \dots + [pl^2]\lambda_m - p_m l_m = 0, \quad (5_{w,m})$$

Summing products by $a_1, a_2, \dots, a_m$ then for each of the n sets we see that we must have

$$[aa] = 1; \quad [a\beta] = 0; \dots [a\lambda] = 0; \quad (6_x)$$

$$[ba] = 0; \quad [b\beta] = 1; \dots [b\lambda] = 0; \quad (6_y)$$

$$[la] = 0; \quad [l\beta] = 0; \dots [l\lambda] = 1. \quad (6_w)$$

We have then, using equations (5) for forming these aggregates,

$$[aa] = [pa^2][a^2 \div p] + [pab][a\beta \div p] + \dots + [pal][a\lambda \div p] = 1, \quad (7_{x,a})$$

$$[ba] = [pab][a^2 \div p] + [pb^2][a\beta \div p] + \dots + [pbl][a\lambda \div p] = 0, \quad (7_{y,a})$$

$$[la] = [pal][a^2 \div p] + [pbl][a\beta \div p] + \dots + [pl^2][a\lambda \div p] = 0, \quad (7_{w,a})$$

$$[a\beta] = [pa^2][a\beta \div p] + [pab][\beta^2 \div p] + \dots + [pal][\beta\lambda \div p] = 0, \quad (7_{x,\beta})$$

$$[b\beta] = [pab][a\beta \div p] + [pb^2][\beta^2 \div p] + \dots + [pbl][\beta\lambda \div p] = 1, \quad (7_{y,\beta})$$

$$[l\beta] = [pal][a\beta \div p] + [pbl][\beta^2 \div p] + \dots + [pl^2][\beta\lambda \div p] = 0, \quad (7_{w,\beta})$$

$$[a\lambda] = [pa^2][a\lambda \div p] + [pab][\beta\lambda \div p] + \dots + [pal][\lambda^2 \div p] = 0, \quad (7_{x,\lambda})$$

$$[b\lambda] = [pab][a\lambda \div p] + [pb^2][\beta\lambda \div p] + \dots + [pbl][\lambda^2 \div p] = 0, \quad (7_{y,\lambda})$$

$$[l\lambda] = [pal][a\lambda \div p] + [pbl][\beta\lambda \div p] + \dots + [pl^2][\lambda^2 \div p] = 1. \quad (7_{w,\lambda})$$

Multiplying the normal equations in their order by the quantities $[a^2 \div p]$, $[a\beta \div p]$, $\dots$, $[a\lambda \div p]$ which result from these equations and adding the products we obtain

$$x = [a\mu] = [a^2 \div p][pa\mu] + [a\beta \div p][pb\mu] + \dots + [a\lambda \div p][pl\mu], \quad (8_x)$$

Similarly we obtain

$$y = [\beta\mu] = [a\beta \div p][pa\mu] + [\beta^2 \div p][pb\mu] + \dots + [\beta\lambda \div p][pl\mu], \quad (8_y)$$

$$w = [\lambda\mu] = [a\lambda \div p][pa\mu] + [\beta\lambda \div p][pb\mu] + \dots + [\lambda^2 \div p][pl\mu]. \quad (8_w)$$

These equations would have resulted as normal equations from the following equations as observation equations:

$$\frac{a_1}{p_1} [pa\mu] + \frac{\beta_1}{p_1} [pb\mu] + \dots + \frac{\lambda_1}{p_1} [pl\mu] - \mu_1 = A_1, \text{ w't, } p_1, \quad (9_1)$$

$$\frac{a_2}{p_2} [pa\mu] + \frac{\beta_2}{p_2} [pb\mu] + \dots + \frac{\lambda_2}{p_2} [pl\mu] - \mu_2 = A_2, \text{ " } p_2, \quad (9_2)$$

$\frac{a_m}{p_m} [pa\mu] + \frac{\beta_m}{p_m} [pb\mu] + \dots + \frac{\lambda_m}{p_m} [pl\mu] - \mu_m = \Delta_m$, w't, p_m , (9.)
by making $[p\Delta^2] = \text{a minimum}$.

Resolving equations (8) we obtain the following formulæ analogous to (5);

$$a_1 = [a^2 \div p] p_1 a_1 + [a\beta \div p] p_1 b_1 + \dots + [a\lambda \div p] p_1 l_1, \quad (10_{a1})$$

$$\beta_1 = [a\beta \div p] p_1 a_1 + [\beta^2 \div p] p_1 b_1 + \dots + [\beta\lambda \div p] p_1 l_1, \quad (10_{\beta1})$$

$$\lambda_1 = [a\lambda \div p] p_1 a_1 + [\beta\lambda \div p] p_1 b_1 + \dots + [\lambda^2 \div p] p_1 l_1, \quad (10_{\lambda1})$$

Let r = probable error of an observation, weight 1, then $(r \div \sqrt{p})$ = the probable error of an observation, weight p , and applying the well known theorem [(43'), ANALYST, Vol. III, p. 140] we have the probable errors of the unknown quantities

$$r_a^2 = \left[\frac{a^2}{p} \right] r^2; r_\beta^2 = \left[\frac{\beta^2}{p} \right] r^2; \dots r_\lambda^2 = \left[\frac{\lambda^2}{p} \right] r^2, \quad (11)$$

and their reciprocal weights

$$\frac{1}{p_a} = \left[\frac{a^2}{p} \right]; \quad \frac{1}{p_\beta} = \left[\frac{\beta^2}{p} \right]; \dots \frac{1}{p_\lambda} = \left[\frac{\lambda^2}{p} \right]. \quad (12)$$

To obtain the probable error of $\mu_1 + \Delta_1, \mu_2 + \Delta_2, \dots$ viz., the observed quantities, corrected, we can use either (1) or (9). By (1) we have, using (4),

$$\begin{aligned} \mu_1 + \Delta_1 &= a_1[a\mu] + b_1[\beta\mu] + \dots + l_1[\lambda\mu] \\ &= (a_1 a_1 + b_1 \beta_1 + \dots + l_1 \lambda_1) \mu_1 \\ &\quad + (a_1 a_2 + b_1 \beta_2 + \dots + l_1 \lambda_2) \mu_2 \\ &\quad + \dots \\ &\quad + (a_1 a_m + b_1 \beta_m + \dots + l_1 \lambda_m) \mu_m. \end{aligned}$$

Denoting the probable error of $\mu_1 + \Delta_1$ by R_1 we have

$$\begin{aligned} R_1^2 &= \{ a_1^2 [a^2 \div p] + a_1 b_1 [a\beta \div p] + \dots + a_1 l_1 [a\lambda \div p] \\ &\quad + a_1 b_1 [a\beta \div p] + b_1^2 [\beta^2 \div p] + \dots + b_1 l_1 [\beta\lambda \div p] \\ &\quad + \dots \\ &\quad + a_1 l_1 [a\lambda \div p] + b_1 l_1 [\beta\lambda \div p] + \dots + l_1^2 [\lambda^2 \div p] \} r^2 \\ &= (a_1 a_1 + b_1 \beta_1 + \dots + l_1 \lambda_1) (r^2 \div p_1), \text{ by virtue of (10), } (13) \end{aligned}$$

and the reciprocal weight

$$\frac{1}{P_1} = \frac{1}{p_1} (a_1 a_1 + b_1 \beta_1 + \dots + l_1 \lambda_1). \quad (14)$$

Likewise we have the expression for the correction Δ_1 , by (1),

$$\begin{aligned} \Delta_1 &= -\mu_1 + a_1[a\mu] + b_1[\beta\mu] + \dots + l_1[\lambda\mu] \\ &= (-1 + a_1 a_1 + b_1 \beta_1 + \dots + l_1 \lambda_1) \mu_1 \\ &\quad + (a_1 a_2 + b_1 \beta_2 + \dots + l_1 \lambda_2) \mu_2 \\ &\quad + \dots \\ &\quad + (a_1 a_m + b_1 \beta_m + \dots + l_1 \lambda_m) \mu_m. \end{aligned}$$

noting the probable error of Δ_1 by ρ_1 we have

$$\rho_1^2 = [1 - 2(a_1\alpha_1 + b_1\beta_1 + \dots + l_1\lambda_1)](r^2 \div p_1) + R_1^2 \\ = (r^2 \div p_1) - R_1^2, \text{ by (13), } = r_1^2 - R_1^2, \quad (15)$$

$$r_1^2 = R_1^2 + \rho_1^2, \quad (16)$$

$$1 \div p_1 = 1 \div P_1 + 1 \div \pi_1, \text{ if } \pi_1 \text{ denotes the weight of } \Delta_1; \quad (17)$$

the remarkable theorem :

probable error of an observed quantity is the hypotenuse of a right triangle of which the probable error of the corrected quantity and that correction are the sides.

have the weighted sum of equations like (13)

$$[pR^2] = ([aa] + [bb] + \dots + [ll])r^2 = nr^2, \text{ by (6),} \quad (18)$$

$$[p \div P] = n. \quad (19)$$

∞ $[p \div p] = m$ we have the theorem :

the adjustment, the observed quantities are improved on the average in ratio of n to m.

$$\text{have also } [p\rho^2] = (m-n)r^2, \text{ (20); } [p \div \pi] = m-n. \quad (21)$$

can place $[p\epsilon^2] = [p\Delta^2]$ where $\epsilon_1, \epsilon_2, \dots \epsilon_m$ are the mean errors of corrections, but $[p\rho^2] = 2\rho^2[p\epsilon^2]$ where $\rho = 0.47694$, the probable error absolute precision, = 1. We have therefore from (20) the formula

$$r = \rho \sqrt{\left(\frac{2[p\Delta^2]}{m-n}\right)} = 0.6745 \sqrt{\left(\frac{[p\Delta^2]}{m-n}\right)}; \quad (22)$$

the average probable error for an observed quantity

$$r_0 = r \sqrt{(m \div [p])}, \quad (23)$$

$$\text{that of a corrected quantity } R_0 = r \sqrt{(n \div [p])}. \quad (24)$$

it is required to find the probable error of

$$U = M + Ax + By + \dots + Lw, \quad (25)$$

we, in terms of $\mu_1, \mu_2, \dots \mu_m$,

$$U = M + (Aa_1 + Bb_1 + \dots + Ll_1)\mu_1 \\ + (Aa_2 + Bb_2 + \dots + Ll_2)\mu_2 \\ + \dots \\ + (Aa_m + Bb_m + \dots + Ll_m)\mu_m;$$

$$\therefore r^2 = \{ A^2 [a^2 \div p] + AB[a\beta \div p] + \dots + AL[a\lambda \div p] \\ + AB[a\beta \div p] + B^2 [\beta^2 \div p] + \dots + BL[\beta\lambda \div p] \\ + \dots \\ + AL[a\lambda \div p] + BL[\beta\lambda \div p] + \dots + L^2 [\lambda^2 \div p] \} r^2. \quad (26)$$

U is not a linear function of $x, y, \dots w$ we replace them by $x_0 + \Delta x, y_0 + \Delta y, \dots w_0 + \Delta w$ and have, by Taylor's theorem,

$$= U_0 + \Delta U = f(x_0, y_0, \dots w_0) + \frac{df}{dx} \Delta x + \frac{df}{dy} \Delta y + \dots + \frac{df}{dw} \Delta w, \\ = f(x_0, y_0, \dots w_0) - \frac{df}{dx} x_0 - \frac{df}{dy} y_0 - \dots - \frac{df}{dw} w_0$$

$$+ \frac{df}{dx}x + \frac{df}{dy}y + \dots + \frac{df}{dw}w, \quad (25')$$

which is of the form (25) and we have

$$\begin{aligned} r^2 = & \left\{ \frac{df^2}{dx^2} \left[\frac{a^2}{p} \right] + \frac{df}{dx} \cdot \frac{df}{dy} \left[\frac{a\beta}{p} \right] + \dots + \frac{df}{dx} \cdot \frac{df}{dw} \left[\frac{a\lambda}{p} \right] \right. \\ & + \frac{df}{dx} \cdot \frac{df}{dy} \left[\frac{a\beta}{p} \right] + \frac{df^2}{dy^2} \left[\frac{\beta^2}{p} \right] + \dots + \frac{df}{dy} \cdot \frac{df}{dw} \left[\frac{\beta\lambda}{p} \right] \\ & + \frac{df}{dx} \cdot \frac{df}{dw} \left[\frac{a\lambda}{p} \right] + \frac{df}{dy} \cdot \frac{df}{dw} \left[\frac{\beta\lambda}{p} \right] + \dots + \frac{df^2}{dw^2} \left[\frac{\lambda^2}{p} \right] \left. \right\} r^2. \quad (26') \end{aligned}$$

• CONTINUATION OF THE NOTE AT P. 57 BY THE EDITOR.—Substituting for a its value $\frac{\omega^2 m R}{5F} + 1$, also substituting for F its value, $\frac{v^2}{D}$, from (3), and writing h for $\omega^2 + v^2$, we have

$$\theta = \frac{R}{r} \varphi - \frac{R}{r} \sqrt{\left(\frac{RDh}{5} \right)} \int \frac{d\varphi}{\sqrt{\left(\frac{1}{5} RDh + 1 - \cos\varphi \right)}};$$

or, substituting for $\cos\varphi$ its equivalent, $1 - 2\sin^2\frac{1}{2}\varphi$, dividing the quantity under the second radical by $\frac{1}{5}RDh$, placing this divisor in the denominator of the first radical and writing ψ for $\frac{1}{2}\varphi$ and c^2 for $10 + RDh$, we get

$$\theta = \frac{R}{r} \varphi - \frac{R}{r} \int \frac{2d\psi}{\sqrt{(1 + c^2 \sin^2\psi)}}.$$

From (3) we have

$$\theta = \frac{R}{r} \varphi - \frac{R}{r} \omega t,$$

$$\therefore t = \frac{2}{\omega} \int \frac{d\psi}{\sqrt{(1 + c^2 \sin^2\psi)}}. \quad (8)$$

Substituting for $\sin^2\psi$ its equivalent $1 - \cos^2\psi$, dividing the quantity under the radical by $1 + c^2$, and writing e^2 for $c^2 + (1 + c^2)$ we have

$$t = \frac{2}{\omega \sqrt{(1 + c^2)}} \int \frac{d\psi}{\sqrt{(1 - e^2 \cos^2\psi)}}. \quad (9)$$

The foregoing was sent to Mr. Charles H. Kummell, of the U. S. Lake Survey, with the request that he should calculate the numerical value of t ; and, in response, Mr. Kummell has put the integral in the form

$$t = \int_{\frac{1}{2}\pi - \phi}^{\frac{\pi}{2}} \frac{d\psi}{\sqrt{(a^2 \cos^2\psi + b^2 \sin^2\psi)}} \quad (10)$$

which is more convenient for numerical calculation, and from which he finds

$$t_n = 9^h 05^m 38^s.4.$$

Hence it appears that the ball will pass a point 180° from the initial point, $2^h 54^m 21^s.6$ before the point A will pass the same point in space.

CURVES OF PURSUIT.

BY JAS. M. INGALLS, 1ST LIEUT. 1ST ART'Y, U. S. A., FT. TRUMBULL, CONN.

Problem.—A point, *B*, moves with a uniform velocity along a given straight line: another point, *A*, moves continually toward *B* with a velocity also uniform. Required the locus of *A*.

Let *MO*, Fig. 1, be the given straight line, and *A* and *I* the positions of the points *A* and *B*, respectively, at any instant. Suppose that while *B* moves from *B* to *B'*, *A* moves from *A* to *A'*, describing the curve *AEA'*. Then from the nature of the problem the right lines *AB* and *A'B'* are tangents to the curve *AEA'* at *A* and *A'*, respectively.

Take *A* as the origin of rectangular co-ordinates, of which *AX* and *AY*, respectively perpendicular and parallel to *MO*, are the axes. Draw *A'D* perpendicular, and *A'C* parallel to *MO*; and make *AC* = *x*, *A'C* = *y*, *A'B'* = *t*, *A'D* = *z*, *A'B'M* = *φ*, *DA'B'* = *θ*, curve *AEA'* = *s*, *AB* = *T*, *AX* = *b*, and *ABM* = *β*. Of these quantities the last three relate to the relative initial positions of the points *A* and *B*, and may be considered constant. Further, if *m* = the ratio of the two given velocities, *BB'* = *ms*.

Since *B'D* + *DX* = *B'B* + *BX*, therefore

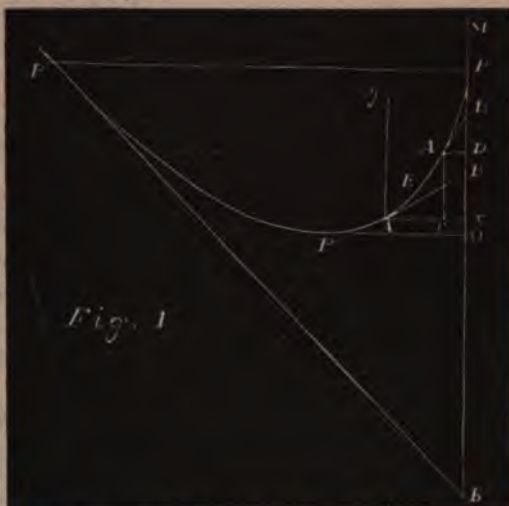
$$(b - x) \frac{dy}{dx} + y = ms + \text{constant.}$$

Or, differentiating,

$$(b - x) \frac{d^2y}{dx^2} = m \frac{ds}{dx}. \quad (1)$$

By substituting for $\frac{ds}{dx}$ its value, $\left(1 + \frac{dy^2}{dx^2}\right)^{\frac{1}{2}}$, the above equation may be written

$$\frac{d \frac{dy}{dx}}{\left(1 + \frac{dy^2}{dx^2}\right)^{\frac{1}{2}}} = \frac{m dx}{b - x},$$



the integration of which gives

$$\log \left[\frac{dy}{dx} + \left(1 + \frac{dy^2}{dx^2} \right)^{\frac{1}{2}} \right] = -\log (b-x)^m + \log C.$$

At the origin,

$$\frac{dy}{dx} = -\cot \beta; \text{ and } \left(1 + \frac{dy^2}{dx^2} \right)^{\frac{1}{2}} = \operatorname{cosec} \beta.$$

Therefore $C = b^m (\operatorname{cosec} \beta - \cot \beta) = b^m \tan \frac{1}{2} \beta$; and consequently,

$$\frac{dy}{dx} + \left(1 + \frac{dy^2}{dx^2} \right)^{\frac{1}{2}} = \frac{b^m \tan \frac{1}{2} \beta}{(b-x)^m}. \quad (2)$$

Hence we derive

$$\frac{dy}{dx} = \frac{b^m \tan \frac{1}{2} \beta}{2(b-x)^m} - \frac{(b-x)^m}{2b^m \tan \frac{1}{2} \beta}, \quad (3)$$

and

$$\frac{ds}{dx} = \frac{b^m \tan \frac{1}{2} \beta}{2(b-x)^m} + \frac{(b-x)^m}{2b^m \tan \frac{1}{2} \beta}. \quad (4)$$

In the above equations $b-x = s$ = the perpendicular from any point of the curve upon MO . Call that perpendicular which is tangent to the curve, a . ($P'O$, Fig. 1.) Then, from (3), we have, $a^m = b^m \tan \frac{1}{2} \beta$; or $b^m = a^m \times \cot \frac{1}{2} \beta$. Therefore, since the origin may be at any point of the curve, we have

$$(b-x)^m = s^m = a^m \cot \frac{1}{2} \beta. \quad (5)$$

The integration of equations (3) and (4) gives different forms according as $m < 1$, $m = 1$, or $m > 1$. We will discuss each case separately.

CASE 1. $m < 1$.

Using a^m instead of $b^m \tan \frac{1}{2} \beta$, and integrating equation (3), we have, since $b = T \sin \beta$,

$$y = \frac{(b-x)^{1+m}}{2a^m(1+m)} - \frac{a^m(b-x)^{1-m}}{2(1-m)} + \frac{T(m-\cos \beta)}{1-m^2}. \quad (6)$$

The integration of equation (4) gives,

$$s = \frac{T(1-m \cos \beta)}{1-m^2} - \frac{(b-x)^{1+m}}{2a^m(1+m)} - \frac{a^m(b-x)^{1-m}}{2(1-m)}. \quad (7)$$

When A overtakes B (at P , Fig. 1) we shall have $x = b$; and the above equation reduces to

$$y = PX = \frac{T(m-\cos \beta)}{1-m^2}; \text{ and } s = AEA'P = \frac{T(1-m \cos \beta)}{1-m^2}.$$

Equations (6) and (7) may be simplified by changing the origin. Thus if we change the origin to P' , we shall have $a = b = T$, and $\beta = \frac{1}{2}\pi$. Therefore equation (6) reduces to

$$y = \frac{(a-x)^{1+m}}{2a^m(1+m)} - \frac{a^m(a-x)^{1-m}}{2(1-m)} + \frac{am}{1-m^2} \quad (8)$$

and
$$s = \frac{a}{1-m^2} - \frac{(a-x)^{1+m}}{2a^m(1+m)} - \frac{a^m(a-x)^{1-m}}{2(1-m)} \quad (9)$$

Therefore $PO = \frac{am}{1-m^2}$; and $PEP' = \frac{a}{1-m^2}$; and consequently,

$$PEP' = (1+m)PO.$$

Since $m < 1$, we may make $\cos \beta = m$, and thus cause the constant term in equation (6) to disappear. For this particular value of β , we find from equation (5), $b = a\left(\frac{1+m}{1-m}\right)^{\frac{1}{2m}}$, which call c . Therefore equation (6) becomes, in terms of c ,

$$y = \frac{(c-x)^{1+m}}{2c^m(1-m^2)^{\frac{1}{2}}} - \frac{c^m(c-x)^{1-m}}{2(1-m^2)^{\frac{1}{2}}} \quad (10)$$

Since $y = 0$, in the above equation, when $x = 0$, and when $x = c$, it is evident that this new axis of X , is perpendicular to OM at P where A overtakes B ; and that the curve cuts the axis of X at P'' at an angle whose sine is m . From equation (7) we have

$$s = T - \frac{(c-x)^{1+m}}{2c^m(1-m^2)^{\frac{1}{2}}} - \frac{c^m(c-x)^{1-m}}{2(1-m^2)^{\frac{1}{2}}} \quad (11)$$

where $T = P''B''$. Therefore when $x = c$, we have $s = T$. That is, the curve $PP'P''$ is equal to the tangent $P''B''$.

An expression for t , that is, the distance between the points A and B at any instant, may easily be found. It is evident from the figure that

$$t = (b-x) \frac{ds}{dx} = z \frac{ds}{dx} = z \operatorname{cosec} \varphi = z \sec \theta. \quad (12)$$

But from equation (4)

$$\frac{ds}{dx} = \frac{a^{2m} + z^{2m}}{2a^m z^m} \therefore t = \frac{(a^{2m} + z^{2m})z^{1-m}}{2a^m}.$$

Another expression for t may be found as follows:— Differentiate the first of equations (12), and we have,

$$\frac{dt}{dx} = (b-x) \frac{d^2s}{dx^2} - \frac{ds}{dx};$$

but from equations (3) and (4) we easily deduce

$$(b-x) \frac{d^2s}{dx^2} = m \frac{dy}{dx}.$$

Therefore

$$\frac{dt}{dx} = m \frac{dy}{dx} - \frac{ds}{dx} \quad (13)$$

$$\therefore t = my - s + T. \quad (14)$$

When $t = 0$, $s = my + T$. That is (Fig. 1), $AEP = AB + m \times PX$.

RADIUS OF CURVATURE.

Designate the radius of curvature by R . Then, since generally,

$$R = \pm \frac{\frac{ds}{dx}}{\frac{d^2y}{dx^2}} \cdot \frac{ds^2}{dx^2};$$

if we use the plus sign and reduce by means of equation (1) we shall have,

$$R = \frac{b-x}{m} \cdot \frac{ds^2}{dx^2}. \quad (15)$$

Therefore by equation (12)

$$R = \frac{t^2}{mz} = \frac{t \sec \theta}{m} = \frac{a \cot \frac{1}{2}\varphi}{m \sin^2 \varphi}.$$

Substituting for $\frac{ds}{dx}$, in equation (15), its value already found, we have,

$$R = \frac{(a^{2m} + z^{2m})^2 z^{1-2m}}{4ma^{2m}} \text{ or } \frac{(a^{2m} + z^{2m})^2}{4ma^{2m} z^{2m-1}},$$

according as $m < \text{or} > \frac{1}{2}$. When $m = \frac{1}{2}$, we have the simple exp'n,

$$R = \frac{(a+z)^2}{2a}.$$

From the above expression it will be seen that the radius of curvature at P is zero when m is less than one half; $\frac{1}{2}a$ when $m = \text{one half}$; and infinite when m is greater than one half. The radius of curvature at P' is, in all cases, $\frac{a}{m}$; and at P'' , $R = \frac{a}{m(1-m^2)}$.

To determine whether R has a minimum, we differentiate equation (15), and obtain,

$$\frac{dR}{dx} = \left(2 \frac{dy}{dx} - \frac{1}{m} \cdot \frac{ds}{dx} \right) \frac{ds}{dx} = \frac{2m \sin \theta - 1}{m \cos^2 \theta}.$$

Therefore when $dR \div dx = 0$, we have $\sin \theta = (1+2m)$.

The second differential coefficient is

$$\frac{d^2R}{dx^2} = \frac{2m}{b-x} \left(\frac{dy^2}{dx^2} + \frac{ds^2}{dx^2} \right) - \frac{2}{b-x} \cdot \frac{dy}{dx} \cdot \frac{ds}{dx}, \text{ which reduces to } \frac{2m}{b-x}$$

on the supposition that $dR \div dx = 0$. As this is positive, R is a minimum

when $\sin \theta = \frac{1}{2m}$, or $z = a \left(\frac{2m-1}{2m+1} \right)^{\frac{1}{2m}}$.

If we designate by R_m the minimum value of R , we shall have

$$R_m = \frac{4am}{(2m+1)^{\frac{2m+1}{2m}} (2m-1)^{\frac{2m-1}{2m}}}.$$

We may also have by means of the preceding formulas, the following relations, viz.:—

$$R_m = \frac{4mz}{4m^2 - 1} = 2z \tan \theta \sec \theta = 2t \tan \theta = \frac{2t}{\sqrt{4m^2 - 1}}.$$

In these formulas, z , t and θ refer, of course, to the point of maximum curvature.

If we represent m by the proper fraction $p \div q$, where p and q are prime to each other, there will be three varieties of the curve, viz.:—when p is odd and q even; when p is even and q odd; and when p and q are both odd.

Before continuing the discussion it will be convenient to change the origin to P , at which point $b = T = 0$; and $\beta = \pi$. Therefore equations (6) and (7) become, changing the signs of x and s ,

$$y = \frac{a^m x^{1-m}}{2(1-m)} - \frac{x^{1+m}}{2a^m(1+m)}; \quad s = \frac{a^m x^{1-m}}{2(1-m)} + \frac{x^{1+m}}{2a^m(1+m)}.$$

Figure 2 represents the first variety, where $m = \frac{1}{2}$. We have

$$y = \pm \frac{(3a - x)\sqrt{x}}{3\sqrt{a}}; \quad \text{and } s = \frac{(3a + x)\sqrt{x}}{3\sqrt{a}}.$$

$$\text{Also } t = \pm \frac{a + x}{2} \left(\frac{x}{a} \right)^{\frac{1}{2}},$$

$$\text{and } R = \frac{(a+x)^2}{2a} = \frac{2t^2}{x} = 2t \sec \theta.$$

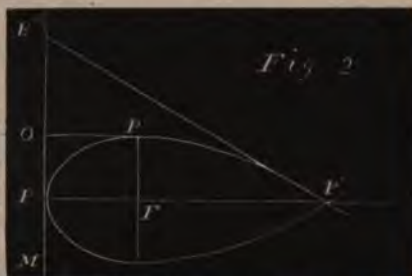
The curve extends from $x = 0$ to $x = +\infty$, and is symmetrical with respect to the axis of x .

There is a multiple point at P'' where the two branches intersect at an angle of 60° . The following values are easily deduced:—

Arc $PP'P'' = B''P'' = a\sqrt{12}$. Arc $PP' = 2 \times OP = \frac{4}{3}a$. $PP'' = 3 \times OP'$. The values of R at P , P' , and P'' , are respectively, $\frac{1}{2}a$, $2a$ and $8a$ —three quantities in geometrical progression. It is, perhaps worth mentioning that if the arc $PP'P''$ be revolved about PP'' as an axis, the volume generated will be $\frac{9}{16}$ of the circumscribing cylinder.

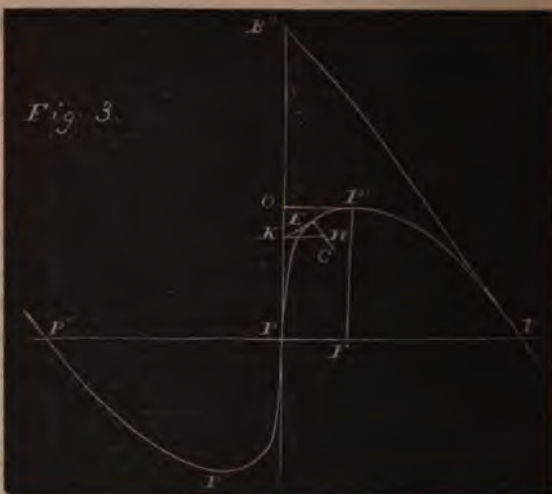
Figure 3 is a type of the second variety. In its construction, m was made equal to $\frac{2}{3}$. Its equation is

$$y = \frac{5x^{\frac{1}{3}}}{18a^{\frac{1}{3}}} \left[9a^{\frac{2}{3}} - x^{\frac{2}{3}} \right].$$



In all of these examples, by changing the sign of the second term within the brackets, of the equation of the curve, we have an expression for s ; i. e., the length of the curve measured from P to the point whose abscissa is x .

The curve extends from $x = +\infty$ to $x = -\infty$; and the parts on opposite sides of the axis of y are similar. The maximum curvature is at E , where



$$x = \left(\frac{3}{13}\right)^{\frac{2}{3}}a.$$

The minimum radius of curvature may be constructed as follows: At E draw the tangent EK and the normal EC . Draw KH perpendicular to the axis of y , and take $HC = EH$. Then is EC the radius of curvature at E . The radius of curvature at P is infinite, as it is in all these curves when $m > \frac{1}{2}$.

The third variety is illustrated by Fig. 4, where $m = \frac{1}{3}$. Its eq'n is

$$y = \frac{3x^{\frac{2}{3}}}{8a^{\frac{1}{3}}} \left[2a^{\frac{2}{3}} - x^{\frac{2}{3}} \right].$$

The curve consists of two similar branches extending to infinity on either side of the axis of y .

The radius of curvature at the origin is zero.



CASE 2. $m = 1$.

The integration of equations (3) and (4) gives, when $m = 1$,

$$y = \frac{a}{2} \log \left(\frac{b}{b-x} \right) + \frac{(b-x)^2}{4a} - \frac{b^2}{4a},$$

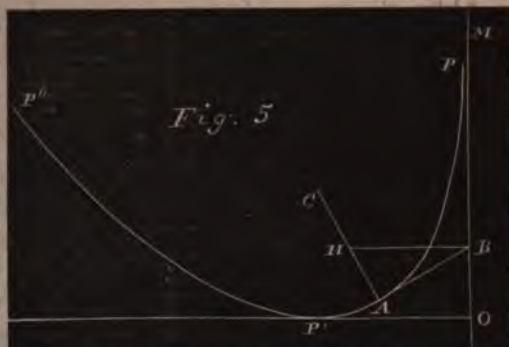
$$s = \frac{a}{2} \log \left(\frac{b}{b-x} \right) - \frac{(b-x)^2}{4a} + \frac{b^2}{4a}.$$

As before, the origin may be at any point of the curve determined by the value of b . If the origin be at P' , Fig. 5, we shall have $b = a$ and the above equations reduce to

$$y = \frac{a}{2} \log \left(\frac{a}{a-x} \right) + \frac{x^2}{4a} - \frac{x}{2} = \frac{a}{2} \log \left(\frac{a}{z} \right) + \frac{z^2}{4a} - \frac{a}{4},$$

$$s = \frac{a}{2} \log \left(\frac{a}{a-x} \right) - \frac{x^2}{4a} + \frac{x}{2} = \frac{a}{2} \log \left(\frac{a}{z} \right) - \frac{z^2}{4a} + \frac{a}{4}.$$

When $x = a$, y and s are both infinite; but from the above equations we have, when $x = a$, $s - y = \frac{1}{2}a$. That is, if the curved line $P'AP$, which meets the straight line OM at infinity were straightened or developed along OM , it would reach below O by a distance equal to $\frac{1}{2}a$.



The following values of t are easily deduced, viz.:—

$$t = z \sec \theta = \frac{a}{1 + \sin \theta} = \frac{dx}{d\theta} = a + y - s = \frac{z^2}{2a} + \frac{a}{2} = \sqrt{Rz}.$$

Therefore when $z = 0$, $t = \frac{1}{2}a$; which is the distance between the points A and B , at infinity.

The following are some of the expressions for R , viz.:—

$$R = \frac{t^2}{z} = t \sec \theta = \frac{(a^2 + z^2)^2}{4a^2z}.$$

When R is a minimum $\theta = 30^\circ$, $z = a \div \sqrt{3}$ and $t = 2a \div 3$; and the minimum value of R is $R_m = 4a \div 3 \sqrt{3}$. A , Fig. 5, is the point of maximum curvature, and $AC = 2 \times AH$ is the radius of curvature. The radius of curvature at any point of the curve whose coordinates are given may be easily constructed as follows: Let A be any point of the curve. Then by means of its coordinates, the tangent AB and normal AC may be drawn. From B draw BH parallel to OP' . Then is BH equal to the radius of curvature at the given point. At P' , $R = a$.

The area between the curve $P'P$ and its asymptote is $\frac{1}{3}a^2$; and the volume generated by revolving $P'P$ about $P'O$ is $\frac{1}{6}\pi a^3$; which is therefore the volume of a disc whose base is a circle of infinite radius and whose altitude is a .

If the curve PP' revolve about its asymptote, the surface generated by the curve $= \frac{4}{3}\pi a^2$, and the volume $= \frac{1}{8}\pi a^3$.

CASE 3. $m > 1$.

When $m > 1$, the integration of equations (3) and (4) gives, placing the origin at P' , Fig. 6.

$$y = \frac{a^m}{2(m-1)(a-x)^{m-1}} + \frac{(a-x)^{m+1}}{2a^m(m+1)} - \frac{am}{m^2-1},$$

$$s = \frac{a^m}{2(m-1)(a-x)^{m-1}} - \frac{(a-x)^{m+1}}{2a^m(m+1)} - \frac{a}{m^2-1}.$$

Although when m is finite, s and y are both infinite when $x = a$, yet their difference, as in case 2, is finite. The value of $s - y$ is, in all cases, equal to $a \div (m+1)$.—We have already found

$$t = \frac{a^{2m} + z^{2m}}{2a^m z^{m-1}},$$

which is infinite when $z = 0$. To determine whether t has a minimum value, we make the second member of equation (13) equal to zero, and find $\sin \theta = (1 \div m)$. This substituted in the second diff. coeff't reduces it to

$$\frac{m}{a} \left[(m+1)^{\frac{m+1}{2m}} (m-1)^{\frac{m-1}{2m}} \right];$$

and this is positive, so long as $m > 1$, for positive roots; and those are the only ones considered in this discussion. The preceding formulæ give, when $\sin \theta = (1 \div m)$,

$$z = a \left(\frac{m-1}{m+1} \right)^{\frac{1}{2m}};$$

and thence we deduce for t_m the following value, viz. :—

$$t_m = \frac{am}{(m+1)^{\frac{m+1}{2m}} (m-1)^{\frac{m-1}{2m}}}.$$

We have already found, when $\sin \theta = 1 \div (2m)$,

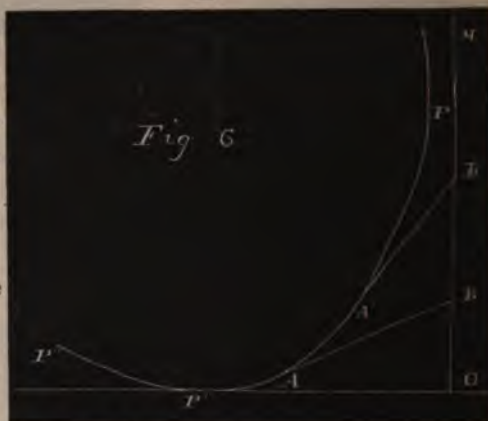
$$R_m = \frac{4am}{(2m+1)^{\frac{2m+1}{2m}} (2m-1)^{\frac{2m-1}{2m}}}$$

and

$$z = a \left(\frac{2m-1}{2m+1} \right)^{\frac{1}{2m}}.$$

In Fig. 6, where $m = \frac{5}{4}$, A is the point of maximum curvature, and $A'B'$ the minimum distance between the points.

For all values of $m > 1$, A lies between A' and P' ; and as m is increased A and A' approach still nearer to P' .



SOLUTIONS OF PROBLEMS IN NUMBER TWO.

SOLUTIONS of problems in No. 2 have been received as follows:

From R. J. Adcock, 299, 307, 308; Prof. L. G. Barbour, 296; Prof. W. P. Casey, 296, 299, 300, 301, 302; Alex. S. Christie, 301, 302; Prof. E. J. Edmunds, 296; George Eastwood, 298, 300; Dr. Wm. Hillhouse, 296, 300; W. E. Heal, 300, 302, 303, 305, 306; Prof. W. W. Johnson, 306; Chas. H. Kummell, 303, 308; Prof. J. H. Kershner, 296, 298, 299, 300, 301, 303; Prof. D. J. Mc Adam, 296; P. Richardson, 296, 300; T. P. Stowell, 298; Prof. E. B. Seitz, 296, 300, 307; Prof. J. Scheffer, 296, 298, 299, 300, 301, 303, 304; E. P. Thompson, 296.

296. "Given $x^6 + y^6 = \frac{1}{2}\sqrt{2}$, $x^4 + y^4 = 1$; to find x and y by quadratics."

SOLUTION BY PROF. L. G. BARBOUR, RICHMOND, KY.

Let $m+n = x^2$, $m-n = y^2$; then

$$m^3 + 3m^2n + 3mn^2 + n^3 + m^3 - 3m^2n + 3mn^2 - n^3 = x^6 + y^6 = \frac{1}{2}\sqrt{2}.$$

$$\therefore m^3 + 3mn^2 = \frac{1}{2} \cdot \frac{1}{2}\sqrt{2} = \frac{1}{4}\sqrt{2}. \quad (1)$$

Also $m^2 + 2mn + n^2 + m^2 - 2mn + n^2 = x^4 + y^4 = 1$; $\therefore m^2 + n^2 = \frac{1}{2}$.

$\therefore n^2 = \frac{1}{2} - m^2 =$ by (1), $(\frac{1}{4}\sqrt{2} - m^3) \div 3m$, or $\frac{3}{2}m - 3m^2 = \frac{1}{4}\sqrt{2} - m^3$.

$$\therefore 4m^3 - 3m = -\frac{1}{2}\sqrt{2}, \therefore 4m^4 - 3m^2 = -\frac{1}{2}\sqrt{2} \cdot m,$$

$$\text{or } 4m^4 - m^2 + \frac{1}{6} = 2m^2 - \frac{1}{2}\sqrt{2} \cdot m + \frac{1}{6};$$

$$\therefore 2m^2 - \frac{1}{4} = m\sqrt{2} - \frac{1}{4}; \therefore 2m = \sqrt{2}, \text{ or } m = \frac{1}{2}\sqrt{2}.$$

Therefore $n^2 = 0$, and $x^4 = m^2 = \frac{1}{2}$; $\therefore x = \frac{1}{2}\sqrt{2} = y$.

[This problem was also solved as above, with a slight difference in notation, by Professors Seitz and Mc Adam.]

297. No solution received.

298. "On the diameter AB produced, of a given circle, is a given point P . A chord CD is parallel to the diameter AB . Find the position of CD , by a geometrical construction, such that the angle CPD shall be the greatest possible."

SOLUTION BY PROF. J. H. KERSHNER, MERCERSBURG, PA.

Represent the point D by (x, y) , the origin being at the centre. The tan of $CPD = \tan(CPB - DPB) = (\tan CPB - \tan DPB) \div (1 + \tan CPB \times \tan DPB)$

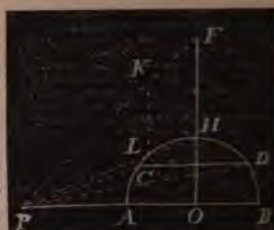
$$\begin{aligned}
 &= \left(\frac{y}{a-x} - \frac{y}{a+x} \right) \div \left(1 - \frac{y^2}{a^2-x^2} \right) \\
 &= \left(\frac{2xy}{a^2-x^2+y^2} \right) = \text{a maximum,}
 \end{aligned}$$

where $a = PO$.

Substituting $\sqrt{(r^2-x^2)}$ for y , we find by the Diff. Calculus, the expression has a maximum for

$$x = \frac{r\sqrt{(a^2-r^2)}}{a\sqrt{2}}.$$

To find x by geometrical construction, erect the perpendicular $OF=OP$, join PF and PH , lay off $FK=r$, draw KL , parallel to OF , meeting PH in L , and we shall have $HL=x$ = the abscissa of C or D .



299. "Prove that the number of terms in an *expanded* polynomial of t terms is equal to the number of combinations of $(n+t-1)$ quantities taken $t-1$ at a time, where n is the exponent of the polynomial."

SOLUTION BY PROF. W. P. CASEY, SAN FRANCISCO, CAL.

The number of terms in the expansion of any polynomial, the exponent n , being a positive integer, is

$$\frac{t(t+1)(t+2)\dots(t+n-1)}{n!},$$

where t is the number of terms and n the exponent; which is evidently = to the number of combinations of $(n+t-1)$ quantities taken $t-1$ at a time.

SOLUTION BY R. J. ADCOCK, ROSEVILLE, ILL.

The expansion of $(a+b)^n$ has $n+1$ terms; that of $(a+b+c)^n$ $= a^n + na^{n-1}(b+c) + \frac{1}{2}n(n-1)a^{n-2}(b+c)^2 + \dots + na(b+c)^{n-1} + (b+c)^n$, which has the sum of $n+1$ terms of the series $1+2+3+4+\dots+(n+1) = \frac{1}{2}(n+2)(n+1)$ terms. The expansion of $(a+b+c+d)^n$ has the sum of $n+1$ terms of the series $1+3+6+\dots + \frac{(n+1)(n+2)}{1.2} = \frac{(n+3)(n+2)(n+1)}{1.2.3}$

by differences. In like manner, the expansion of $(a+b+c+d+e)^n$ has $\frac{(n+4)(n+3)(n+2)(n+1)}{1.2.3.4}$ terms. Hence $\frac{(n+t-1)(n+t-2)\dots(n+1)}{1.2.3\dots(t-1)}$

which expresses the proposition that was to be proved.

300. "In a plane trian. ABC are given, the perimeter $2s$ and the radii of the inscribed and circumscribed circles, r and R , to determine the triangle."

SOLUTION BY P. RICARDSON, BROOKLYN, N. Y.

Let the sides opposite the angles A, B, C , be a, b, c , then we have given

$$a + b + c = 2s. \quad (1)$$

Double the area is $ab \sin C = 2rs$, $\sin C = c \div 2R$; $\therefore$ by substitution we get

$$abc = 4Rrs. \quad (2)$$

The area is $\sqrt{s(s-a)(s-b)(s-c)} = rs$, or, by squaring and performing operations indicated in left hand member, we get

$$s^4 - (a+b+c)s^3 + (ab+ac+bc)s^2 - abcs = r^2s^2,$$

and by substituting from (1) and (2) we get

$$[-s^4 + (ab+ac+bc)s^2 - 4Rrs^2] = r^2s^2,$$

whence

$$ab+ac+bc = r^2 + s^2 + 4Rr. \quad (3)$$

* Equations (1), (2) and (3) show that a, b, c , must be the roots of the cubic equation $x^3 - 2sx^2 + (r^2 + s^2 + 4Rr)x - 4Rrs = 0$.

301. "Determine the angles of a plane triangle from the following relations.

$$(I) \begin{cases} \tan^2 A + \tan^2 B + \tan^2 C = m, \\ \tan^4 A + \tan^4 B + \tan^4 C = n. \end{cases} \quad (1) \quad (2)$$

$$(II) \begin{cases} \tan^{\frac{2}{3}} A + \tan^{\frac{2}{3}} B + \tan^{\frac{2}{3}} C = m, \\ \tan^{\frac{4}{3}} A + \tan^{\frac{4}{3}} B + \tan^{\frac{4}{3}} C = n. \end{cases} \quad (1) \quad (2)$$

$$(III) \begin{cases} \tan^{\frac{1}{2}} A + \tan^{\frac{1}{2}} B + \tan^{\frac{1}{2}} C = m, \\ \tan^{\frac{1}{2}} A \times \tan^{\frac{1}{2}} B \times \tan^{\frac{1}{2}} C = n. \end{cases} \quad (1) \quad (2)$$

SOLUTION BY ALEX. S. CHRISTIE, U. S. C. SURV., WASH., D. C.

Let the sought tangents in each case be the roots a, b, c of the cubic

$$x^3 + px^2 + qx + r = 0.$$

Then by well known forms in the theory of equations and trigonometry (say Faa' de Bruno's Forms, Binaires, p. 27, and Chauvenet's Trig., p. 62) we have for determining p, q, r ,

$$(I) \begin{cases} m = a^2 + b^2 + c^2 = p^2 - 2q \\ n = a^4 + b^4 + c^4 = p^4 - 4p^2q + 4pr + 2q^2 \\ 0 = p - r \end{cases} \quad (1) \quad (2) \quad (3)$$

$$(II) \quad (1) \text{ and } (2) \text{ with } 0 = 1 - q$$

$$(III) \quad \begin{cases} m = -p \\ n = -r \\ 0 = 1 - q \end{cases}$$

Whence, for the three cases in order

where

$$x^3 + x^2 \sqrt{m+2q} + xq + \sqrt{m+2q} = 0,$$

$$q = \frac{1}{2} \{ 4 \pm \sqrt{16 - 2(n - 4m - m^2)} \};$$

$$x^3 + x^2 \sqrt{m+2} + x + \frac{n - m^2 + 2}{4\sqrt{m+2}} = 0,$$

$$x^3 - mx^2 + x - n = 0.$$

302. "Given n points in space, to construct a polygon of n sides, having its sides bisected by the n given points, and determine the No. of solutions."

SOLUTION BY ALEX. S. CHRISTIE.

With any point in space as origin, let $\rho_n, \rho_{n-1}, \dots, \rho_1$ be the quaternion vectors to the vertices of the required polygon, and $a_n, a_{n-1}, \dots, a_1$ the corresponding intermediate vectors to the given points.

Then

$$\begin{aligned} \rho_n &= a_n + a_n - \rho_{n-1} = 2a_n - \rho_{n-1} \\ &= 2a_n - 2a_{n-1} + \rho_{n-2} \\ &= 2a_n - 2a_{n-1} + a_{n-2} - \rho_{n-3} \\ &\dots \dots \dots \\ &= 2a_n - 2a_{n-1} + 2a_{n-2} - 2a_{n-3} + \dots + (-)^{n-1} 2a_1 + (-)^n \rho_1. \end{aligned}$$

When n is even $0 = a_n - a_{n-1} + a_{n-2} - a_{n-3} + \dots + a_2 - a_1. \quad (1)$

" n is odd $\rho_n = a_n - a_{n-1} + a_{n-2} - a_{n-3} + \dots - a_2 + a_1. \quad (2)$

Hence, when n is even, there are an infinite number of solutions, or none, according as the geometrical condition (1) is, or is not fulfilled; and when n is odd, (2) gives the construction for any vertex. It is obvious that there are, in this case, $\frac{1}{2}(n-1)!$ different solutions.

303. "Required the length of one branch of a curtate cycloid; the generating circle having a radius of 4 inches and the generating point a radius of 8 inches."

SOLUTION BY CHAS. H. KUMMELL, U. S. LAKE SURVEY, DETROIT, MICH.

Denote the radius of the generating circle by $r = 4$ inches and that of the generating point by $R = 8$ inches. Suppose the rolling of the generating circle begins when the radius of the generating point has a position C_0P_0 vertically downward. Let θ be the angle of turning from this position, then we have, taking A for origin (see Fig. on next page),

$$x = AM = AN - MN = r\theta - R \sin \theta, \quad (1)$$

$$y = PM = CN + PQ = r - R \cos \theta. \quad (2)$$

We have then $dx = (r - R \cos \theta)d\theta$; $dy = R \sin \theta d\theta$.

Therefore $ds = d\theta \sqrt{[R^2 - 2Rr \cos \theta + r^2]} = d\theta \sqrt{[(R+r)^2 - 4Rr \cos^2 \frac{1}{2}\theta]}. \quad (3)$

$2^2b_4 =$	119.47124	
$-2b_3 \div b_4 =$	-59.73537	$\log \text{Sum} = 1.2352268$
$-b_2 \div b_4 =$	-29.69116	$\log \pi = 0.4971499$
$-\frac{1}{2}b_1 \div b_4 =$	-12.85665	$\log (s_{2\pi} - s_0) = 1.7323767$
<u>Sum =</u>	17.18806	$s_{2\pi} - s_0 = 53.9976 \text{ inches.}$

304. "Find the limit of error in the following method of trisecting an arc approximately: Trisect the chord of the arc, also the diameter parallel to it, and the corresponding semicircumference. Join the corresponding points of trisection of the semicircumference and the diameter, producing the lines till they intersect. From the point thus found pass lines through the points of trisection of the chord until they meet the arc."

SOLUTION BY PROF. J. SCHEFFER, MERCERSBURG, PA.

Let E, E' represent the points of trisection of the diameter MN ; C, C' , those of the chord AB , and F, F' , those of the semicircle.

Draw the perpendicular PD , and join the centre O with F and G . Denote the angle AOD by α , the angle GOH by φ , and the radius by r .

We now easily find $OE = \frac{1}{3}r$ $CD = \frac{1}{3}\sin \alpha$.

In the triangle EOF , we have $EF^2 = EO^2 + OF^2 - 2EO \cdot OF \cos 60^\circ$; $\therefore EF = \frac{1}{3}r\sqrt{7}$; $\cos FEO = (FE^2 + EO^2 - OF^2) \div (2EF \cdot EO) = -\frac{1}{14}\sqrt{7}$; $\therefore OEP = \frac{1}{14}\sqrt{7}$, and $\sin OEP = \frac{3}{14} \times \sqrt{21}$; therefore $\tan OEP = 3\sqrt{3}$. But $PO = \frac{1}{3}r \times \tan OEP = r\sqrt{3}$. Since $OD = r \cos \alpha$, we have $PD = r(\cos \alpha + \sqrt{3})$.



We have $\tan GPO = CD \div PD = \frac{1}{3}[\sin \alpha \div (\cos \alpha + \sqrt{3})]$; therefore $\sin GPO = \frac{\sin \alpha}{\sqrt{[\sin^2 \alpha + 9(\cos \alpha + \sqrt{3})^2]}}$; $\cos GPO = \frac{3(\cos \alpha + \sqrt{3})}{\sqrt{[\sin^2 \alpha + 9(\cos \alpha + \sqrt{3})^2]}}$

In the triangle GPO , we have $OG : OP = \sin GPO : \sin PGO$; $\therefore \sin PGO = \frac{\sqrt{3} \sin \alpha}{\sqrt{[\sin^2 \alpha + 9(\cos \alpha + \sqrt{3})^2]}}$; $\cos PGO = \frac{\sqrt{[9(\cos \alpha + \sqrt{3})^2 - 2\sin^2 \alpha]}}{\sqrt{[\sin^2 \alpha + 9(\cos \alpha + \sqrt{3})^2]}}$

But $\sin \varphi = \sin (GPO + PGO)$, therefore

$$\sin \varphi = \frac{\sin \alpha \sqrt{[9(\cos \alpha + \sqrt{3})^2 - 2\sin^2 \alpha]} + 3\sqrt{3} \sin \alpha (\cos \alpha + \sqrt{3})}{\sin^2 \alpha + 9(\cos \alpha + \sqrt{3})^2}$$

With regard to the trisection of an arc this formula gives a correct result only for an arc of 180° . For other arcs, I find, if $\alpha = 45^\circ$, $\varphi = 15^\circ 6'$; if $\alpha = 30^\circ$, $\varphi = 10^\circ 2'$, &c.

305. "Find the equation of the curve, rectangular co-ordinates (x, y) in which the length $s = \frac{ny}{x} = \int_0^x dx \sqrt{[1 + (\frac{dy}{dx})^2]}$, n being a constant."

SOLUTION BY W. E. HEAL, WHEELING, INDIANA.

Let the indefinite integral

$$\int dx \left[1 + \left(\frac{dy}{dx} \right)^2 \right]^{\frac{1}{2}} = \varphi(x), \text{ then } \int_0^x dx \left[1 + \left(\frac{dy}{dx} \right)^2 \right]^{\frac{1}{2}} = \varphi(x) - \varphi(0) = \frac{ny}{x}$$

Differentiating the above eq'n, (1), remembering that $\varphi(0)$ is constant,

$$dx \sqrt{\left[1 + \left(\frac{dy}{dx} \right)^2 \right]} = \frac{n(xdy - ydx)}{x^2}. \quad (2)$$

Put $dy \div dx = p$, and (2) becomes $x^2 \sqrt{(1+p^2)} = np x - ny$. (3)

Differentiating (3) we have

$$\frac{xpdp}{\sqrt{(1+p^2)}} + 2dx\sqrt{(1+p^2)} = ndp. \quad (4)$$

Divide by $2(1+p^2)^{\frac{1}{2}}$; $dx(1+p^2)^{\frac{1}{2}} + xpd p \div (1+p^2)^{\frac{1}{2}} = ndp \div 2(1+p^2)^{\frac{1}{2}}$. (5)

Integrating (5) and determining the constant $C=0$, we have

$$x(1+p^2)^{\frac{1}{2}} = \frac{n}{2} \int \frac{dp}{(1+p^2)^{\frac{1}{2}}}, \quad (6). \text{ Let } p = \frac{2^{\frac{1}{2}}v(1-\frac{1}{2}v^2)^{\frac{1}{2}}}{1-v^2}, \text{ then (6) becomes}$$

$$\begin{aligned} \frac{x}{(1-v^2)^{\frac{1}{2}}} &= \frac{n}{2^{\frac{1}{2}}} \int \frac{dv}{\sqrt{(1-v^2)^3(1-\frac{1}{2}v^2)}} = 2^{\frac{1}{2}}n \left[\frac{v(1-\frac{1}{2}v^2)^{\frac{1}{2}}}{(1-v^2)^{\frac{1}{2}}} \right. \\ &+ \frac{1}{2} \int \frac{dv}{(1-v^2)^{\frac{1}{2}}(1-\frac{1}{2}v^2)^{\frac{1}{2}}} - \left. \int \frac{(1-\frac{1}{2}v^2)^{\frac{1}{2}}dv}{(1-v^2)^{\frac{1}{2}}} \right] = 2^{\frac{1}{2}}n \left[\frac{v(1-\frac{1}{2}v^2)^{\frac{1}{2}}}{(1-v^2)^{\frac{1}{2}}} + \frac{1}{2}F(\frac{1}{2}2^{\frac{1}{2}},v) \right. \\ &- \left. E(\frac{1}{2}2^{\frac{1}{2}},v) \right] \quad (7), \text{ where } F \text{ and } E \text{ are elliptic funct's of the 1st and 2nd ord's.} \end{aligned}$$

But $v = \frac{[(1+p^2)^{\frac{1}{2}}-1]^{\frac{1}{2}}}{(1+p^2)^{\frac{1}{2}}}$, and from (3) $p = \frac{n^2y + x[n^2(x^2 + y^2) - x^4]}{x(n^2 - x^2)}$.

Substituting these values in (7) we have the equation of the curve.

[The solution of 306, by Prof. Johnson, of 307, by Prof. Seitz, and of 308, by Mr. Adcock, will be published in No. 4.]

PROBLEMS.

309. *By Prof. Beman.*—In a given circle, find the vertices of the inscribed square, pentagon, octagon, and decagon by using the dividers alone.

310. *By Prof. Edmunds.*—Required the locus of vertices of a right angled spherical triangle whose legs pass through two fixed points given on the surface of the sphere.

311. *By Prof. Casey.*—A uniform circular plate is placed with its centre upon a prop, to find at what points on its circumference three given weights p, q, r must be attached that it may remain at rest in a horizontal position.

312. *By Prof. Scheffer.*—To find the area of the loop of the curve $y^3 + x^2y - axy + bx^2 = 0$.

EDITORIAL NOTE.—It will be seen that this No. of the ANALYST contains 40 pages—8 pages more than we are usually able to present to our readers in one Number. As we have recently been favored with a greater number of acceptable papers for publication than we are able to publish immediately in our usual issue of 32 pages, we are pleased to state that Lieut. Ingalls has not only contributed to this No. a very elegant discussion of the Curve of Pursuit, but has generously permitted us to add 8 pages to the Number, at his expense.

SPECIAL NOTICE.—As several of our contributors have requested us to furnish them extra sheets containing articles contributed by them for publication, we have so adjusted our contract with our printer and binder that we can furnish copies of one sheet (8 pages), in lots of 50, or multiples of 50, copies, folded, stitched and covered with light cover paper, at the rate of \$3.00 per hundred copies. But as all the expenses incurred, in getting out each number, are paid by us before the number is mailed, and as we will realize no profit by supplying parts of Nos., at the above rate, no extra copies will be printed unless the order for them is accompanied by the money.

PUBLICATIONS RECEIVED.

- Coast Pilot of Alaska: Appendix 1, Meteorology.* By WM. H. DALL, acting assist. U. S. C. Survey, assisted by MARCUS BAKER, of the U. S. Coast Survey. 1879.
This is a magnificent quarto volume of 375 pp., finely illustrated with maps and engravings, and giving an exhaustive account of what is known of the Meteorology of Alaska.
- Ciel et Terre:* a popular review of Astronomy and Meteorology, published on the 1st and 15th of each month, at Brussels, Belgium, in the French language. 8vo. 24pp. 10fr.
- Three Approximate Solutions of Kepler's Problem.* By H. A. HOWE, A. M., assistant at the Cincinnati Observatory.
- The Observatory* (for April, 1880). A monthly review of Astronomy. W. H. M. CHRISTIE, Editor. London, England. 8vo. 32pp. Price, \$3.50 per annum.
- The Mathematical Visitor*, No. 4. ARTEMAS MARTIN, M. A., Editor. Erie, Pa.

ERRATA.

On page 65, line 3, from bottom, for "AB", read, AOB.

" " 67, " 9, for "this last", read, the last.

" " " " 11, for "multiplied", read, interpreted.

" " 68, " 4, for "must", read, might.

" " 71, " 12, for "also i^2 ", read, also = i^2 .

" " 85, " 5, after "then", insert, by $b_1, b_2, \dots b_m; l_1, l_2, \dots l_m$.

" " 88, for " $\frac{1}{2}\pi - \phi$ ", in eq. (10), read, $\frac{1}{2}(\pi - \phi)$.

[Besides the above corrections, Prof. Wood sent us a Note, correcting a remark in Sect. 9, p. 36, which came too late to be inserted with his Art., but will appear in No. 4.]

THE ANALYST.

VOL. VII.

JULY, 1880.

No. 4.

ON SOME PROPERTIES OF POLYNOMIALS.

BY E. L. DE FOREST.

(Continued from page 82.)

THE following property of polynomials of one variable deserves somewhat fuller treatment than we have yet given it. Suppose that the unlimited series

$$\dots u_{-2}, u_{-1}, u_0, u_1, u_2, \dots$$

is of algebraic form and of the r order, its terms being equidistant values of the function

$$u = A_0 + A_1x + A_2x^2 + \dots + A_rx^r, \quad (52)$$

and therefore ordinates to the curve of which this is the equation, the constant interval between them being Δx . Supposing these terms to be adjusted by an unsymmetrical formula,

$$u'_0 = l_0u_0 + l_1u_1 + l_2u_2 + \dots + l_mu_m \\ + l_{-1}u_{-1} + l_{-2}u_{-2} + \dots + l_{-m}u_{-m}, \quad (53)$$

let us inquire what conditions must be satisfied by the coefficients l , in order that the adjusted term u'_0 may be an ordinate to the curve (52), though not necessarily the same as u_0 . Let $\mu\Delta x$ be the interval, if any, between u'_0 and u_0 , μ being a number, integral or fractional. Taking the position of u'_0 as origin of coordinates, μ is essentially + or - according as u_0 lies to the right or left of u'_0 . Hence:

$$\begin{aligned} u_0 &= A_0 + A_1\mu\Delta x + A_2\mu^2(\Delta x)^2 + \dots + A_r\mu^r(\Delta x)^r, \\ u_1 &= A_0 + A_1(\mu+1)\Delta x + A_2(\mu+1)^2(\Delta x)^2 + \dots + A_r(\mu+1)^r(\Delta x)^r, \\ u_{-1} &= A_0 + A_1(\mu-1)\Delta x + A_2(\mu-1)^2(\Delta x)^2 + \dots + A_r(\mu-1)^r(\Delta x)^r, \\ &\vdots \\ u_m &= A_0 + A_1(\mu+m)\Delta x + A_2(\mu+m)^2(\Delta x)^2 + \dots + A_r(\mu+m)^r(\Delta x)^r, \\ u_{-m} &= A_0 + A_1(\mu-m)\Delta x + A_2(\mu-m)^2(\Delta x)^2 + \dots + A_r(\mu-m)^r(\Delta x)^r. \end{aligned}$$

Substitute these values in (53), and write for brevity

$$\left. \begin{aligned} b_0 &= l_0 + l_1 + l_{-1} + l_2 + l_{-2} + \dots + l_m + l_{-m}, \\ b_1 &= \mu l_0 + (\mu+1) l_1 + (\mu-1) l_{-1} + \dots + (\mu+m) l_m + (\mu-m) l_{-m}, \\ b_2 &= \mu^2 l_0 + (\mu+1)^2 l_1 + (\mu-1)^2 l_{-1} + \dots + (\mu+m)^2 l_m + (\mu-m)^2 l_{-m}, \\ b_r &= \mu^r l_0 + (\mu+1)^r l_1 + (\mu-1)^r l_{-1} + \dots + (\mu+m)^r l_m + (\mu-m)^r l_{-m}, \end{aligned} \right\} \quad (54)$$

and it will be found that (53) becomes

$$u'_0 = A_0 b_0 + A_1 b_1 \Delta x + A_2 b_2 (\Delta x)^2 + \dots + A_r b_r (\Delta x)^r. \quad (55)$$

This adjusted value u'_0 is the same value which u in (52) has at the origin, namely A_0 . But A_0 , A_1 , A_2 , &c., in (52), may be any arbitrary numbers, so that to make the second member of (55) always equal to A_0 , we must have

$$b_0 = 1, \quad b_1 = 0, \quad b_2 = 0, \dots b_r = 0. \quad (56)$$

These are the conditions which the coefficients l must fulfil. The first one requires that the sum of all the l 's shall be unity. The second one determines μ , giving

$$-\mu = l_1 - l_{-1} + 2(l_2 - l_{-2}) + \dots + m(l_m - l_{-m}), \quad (57)$$

so that $-\mu \Delta x$ is the lever arm of the system of parallel forces l , about the place of l_0 . But since it is also the distance from l_0 or u_0 to the place of u'_0 , we see that the adjusted term u'_0 is located at the centre of parallel forces of the coefficients l . In the third condition, $b_2 (\Delta x)^2$ being the sum of the products formed by multiplying each l into the square of its distance from the centre of forces, $b_2 = 0$ requires that the radius of gyration of the coefficients about their centre of forces, shall be equal to zero. We use the term radius of gyration, as before, in an extended sense, for the l may be either positive or negative, and indeed some of them must be negative to make the radius zero. In general, $b_n = 0$ reduces to zero the sum of the products formed by multiplying each l into the n th power of its abscissa reckoned from the centre of forces as an origin. Now if (53) is so constructed as to satisfy these conditions, any term in the adjusted series is separated by the constant interval $\mu \Delta x$ from the place of that term of the given series which had the coefficient l_0 when the adjusted term in question was computed, wherefore the adjusted terms are equidistant, and being ordinates to the curve (52), they form a series of the r order. Let this series be re-adjusted by a second formula like that in (4), and let its coefficients L satisfy conditions similar to the foregoing, so that it gives another adjusted series of the r order, whose terms are equidistant, the adjusted term u''_0 being coincident with the centre of forces, and separated from the place of u'_0 by an interval $\mu' \Delta x$ which is constant, being the distance which separates L_0 from the centre of forces of this system. But as we saw in connection with (4),

the two adjustments are equivalent to a single adjustment made by a resultant formula, whose coefficients are those of the powers of z in the product of the two polynomials (5), these two having the same coefficients l and L which the component adjustment formulas (4) had. Denoting the coefficients in the resultant formula by λ , we see that since this formula gives an adjustment of the original series like that which (53) gives, only having the interval $(\mu + \mu')dx$ in the place of μdx , it must satisfy similar conditions (56), so that the sum of all the λ 's is unity, and the sum of the products of each λ into the n th power of its abscissa reckoned from the centre of forces in this system, is = zero, n being any integer from 1 to r . The l and L are numbers which may have any values, subject only to the conditions named. Any equation $b_n = 0$, or, as (54) gives it,

$$\mu^n l_0 + (\mu+1)^n l_1 + (\mu-1)^n l_{-1} + \dots + (\mu+m)^n l_m + (\mu-m)^n l_{-m} = 0,$$

will still hold good when each l in it is multiplied by a constant number. Any polynomial is equal to the product of a constant number into a polynomial the sum of whose coefficients is unity. Hence, neglecting the condition $b_0 = 1$, we have proved that if the remaining conditions are satisfied by the coefficients in each of two polynomials, such as (17), they will also be satisfied by the coefficients in their product. Consequently, if any number of polynomials, each satisfying these conditions, are multiplied together, their final product will satisfy the same conditions. If the conditions are satisfied by any one polynomial, they will also be satisfied by any power of that polynomial. This is the case already illustrated (ANALYST, Jan. 1880, p. 4), where the polynomial used as an example was

$$\frac{1}{81}(-4 + 30z + 60z^2 - 5z^3).$$

The divisor 81 may as well be omitted, dispensing with the condition $b_0 = 1$. As a further illustration, let us take the two polynomials

$$-4 + 30z + 60z^2 - 5z^3, \quad 1 - 4z + 6z^2 + 2z^3 + z^4,$$

the centre of forces in the first one being $\frac{2}{3}$ of the way from the 30 to the 60, while in the second one it coincides with the 2. In the first, we have both for $n = 2$ and $n = 3$,

$$b_n = -4(-\frac{5}{3})^n + 30(-\frac{2}{3})^n + 60(\frac{1}{3})^n - 5(\frac{4}{3})^n = 0;$$

and likewise in the second,

$$b_n = 1(-3)^n - 4(-2)^n + 6(-1)^n + 2(0)^n + 1(1)^n = 0.$$

According to our theory, the same two conditions, $b_2 = 0$ and $b_3 = 0$, ought to be satisfied by the product of these polynomials, which is

$$-4 + 46z - 84z^2 - 73z^3 + 436z^4 + 120z^5 + 50z^6 - 5z^7.$$

Its centre of forces is $\frac{2}{3}$ of the way from the 436 to the 120, and we find actually, both for $n = 2$ and $n = 3$,

$$b_n = -4(-\frac{1}{3})^n + 46(-\frac{1}{3})^n - 84(-\frac{2}{3})^n - 73(-\frac{5}{3})^n + 436(-\frac{2}{3})^n + 120(\frac{1}{3})^n + 50(\frac{4}{3})^n - 5(\frac{7}{3})^n = 0.$$

Polynomials of two variables possess properties analogous to the foregoing. Suppose that an unlimited double series, such as (6), is of algebraic form, its terms being values of the function

$$u = \frac{A_{0,0} x^0 y^0 | A_{r,0} x^r y^0}{A_{0,0} x^0 y^0 | A_{r,0} x^r y^0}.$$

For greater convenience of notation, we will express such a function by means of only two terms diagonally opposite, thus,

$$u = A_{0,0} x^0 y^0 || A_{r,0} x^r y^0. \quad (58)$$

The terms of the double series are ordinates to the surface of which (58) is the equation, the constant intervals between them being Δx and Δy . If these terms are to be adjusted by an unsymmetrical formula like the first one in (11), which we write more compactly thus,

$$u'_{0,0} = l_{-m,-n} u_{-m,-n} || l_{m,n} u_{m,n}, \quad (59)$$

we will inquire what conditions must be satisfied by the coefficients l , in order that the adjusted term $u'_{0,0}$ shall be also an ordinate to the surface, and be separated from $u_{0,0}$ by the constant horizontal and vertical intervals $\mu \Delta x$ and $\nu \Delta y$. Taking the position of $u'_{0,0}$ as the origin, the coordinates of the term $u_{m,n}$ will be $x = (\mu + m) \Delta x$, $y = (\nu + n) \Delta y$, so that (58) gives

$$u_{m,n} = A_{0,0} (\mu + m)^0 (\nu + n)^0 (\Delta x)^0 (\Delta y)^0 || A_{r,0} (\mu + m)^r (\nu + n)^r (\Delta x)^r (\Delta y)^r, \quad (60)$$

which is a general expression for any term in the given series. For example, $u_{2,-1}$ is expressed by putting $m=2$ and $n=-1$. If (60) is multiplied throughout by $l_{m,n}$, it becomes a general expression for any product, $l_{m,n} u_{m,n}$, in the adjustment formula (59). Supposing these expressions to be substituted for all the products in (59), it is evident that any const. independent of m and n , such as $A_{r,0} (\Delta x)^r (\Delta y)^r$ for example, will enter (59) with a coefficient which is the sum of all the values of

$$(\mu + m)^r (\nu + n)^r l_{m,n}$$

which can be formed by giving to m any integral value from $-m$ up to m , and to n any value from $-n$ to n . Denoting this sum by $b_{r,0}$, we have,

$$b_{r,0} = (\mu - m)^r (\nu - n)^r l_{-m,-n} || (\mu + m)^r (\nu + n)^r l_{m,n}, \quad (61)$$

and the adjusted term will stand

$$u'_{0,0} = A_{0,0} b_{0,0} (\Delta x)^0 (\Delta y)^0 || A_{r,0} b_{r,0} (\Delta x)^r (\Delta y)^r. \quad (62)$$

This is the same value which u in (58) has at the origin, namely $A_{0,0}$, and since the A 's in (58) are supposed to be arbitrary numbers given in advance, the second member of (62) will not in general be equal to $A_{0,0}$ unless we have

$$b_{0,0} = 1, \quad b_{r,0} = 0. \quad (63)$$

This last is to be understood as equating to zero all the values of $b_{r,0}$ in

(61) which can be formed by assigning to r and s all the pairs of values which the exponents of x and y have in (58), excepting only the pair $r=0$, $s=0$. The condition $b_{0,0}=1$ requires that the sum of all the l 's shall be unity. The two next conditions, $b_{1,0}=0$ and $b_{0,1}=0$, determine the relations of the l to μ and ν ;

$$-\mu = -ml_{-m,-n} || ml_{m,n}, \quad -\nu = -nl_{-m,-n} || nl_{m,n}, \quad (64)$$

so that $-\mu \Delta x$ is the lever arm of the system of parallel forces l about a vertical axis passing through the position of $l_{0,0}$, and $-\nu \Delta y$ is the lever arm about a horizontal axis through the same point. But these are also the horizontal and vertical distances from $l_{0,0}$ or $u_{0,0}$ to $u'_{0,0}$, whence it appears that the adjusted term $u'_{0,0}$ is located at the centre of parallel forces of the coefficients l . In general, any one condition $b_{r,s}=0$ reduces to zero the sum of the products formed by multiplying each l into the r power of its abscissa and the s power of its ordinate, reckoned from the centre of forces as an origin. By reasoning quite similar to that already followed in the case of a single series, and which need not be repeated here in detail, we get the following results.

Suppose that (59) satisfies the conditions (63), and that the adjusted terms, forming a series of the same order as the given series, are readjusted by a second formula like that in (11), with coefficients L satisfying conditions similar to those which the l satisfied, μ' and ν' taking the places of μ and ν , so that we have a second adjusted series of the same order. The adjusted term $u''_{0,0}$ will be located at the centre of forces for the L , and separated from $u'_{0,0}$ by horizontal and vertical intervals $\mu' \Delta x$ and $\nu' \Delta y$. The two adjustments are equivalent to a single one made by a resultant formula, whose coefficients λ are those of the powers of x and y in the product of the two polynomials (12), these two having the same coefficients l and L which the component formulas had. The λ will satisfy the same conditions (63) as the l and L did, and the adjusted term $u''_{0,0}$ will coincide with the centre of forces for λ , and will be separated from $u_{0,0}$ by the intervals $(\mu + \mu') \times \Delta x$ and $(\nu + \nu') \Delta y$. All of the conditions (63) except $b_{0,0}=1$ will hold good when the coefficients, as for instance the l 's in (61) are all multiplied by a constant number. Neglecting then the $b_{0,0}=1$, if the remaining conditions are satisfied by the coefficients in each of the two polynomials in x and y , they will also be satisfied by the coefficients in their product. If therefore several polynomials, all satisfying those conditions, are multiplied together, their final product will satisfy the same conditions.

If the conditions are satisfied by one polynomial, they will be satisfied by any power of it. We will illustrate these properties by a single example.

Suppose that the equation of the surface is

$$u = \left\{ \begin{array}{l} +A_{0,2}y^2 \\ +A_{0,1}y + A_{1,1}xy + A_{2,1}x^2y \\ A_{0,0} + A_{1,0}x + A_{2,0}x^2 \end{array} \right\}, \quad (65)$$

its form being such that a transfer of the origin to any point in the plane of XY will not introduce any new term into the equation. The conditions to be satisfied by the l , omitting $b_{0,0} = 1$, are therefore

$b_{1,0} = 0, b_{0,1} = 0, b_{1,1} = 0, b_{2,0} = 0, b_{0,2} = 0, b_{2,1} = 0,$ (66)
the sub-indices of b being the same as the exponents of x and y . Two polynomials, whose coefficients are given in (67), have been constructed so that the coefficients of each will satisfy all these conditions. The coordinates

0	0	-2
3	6	3
3	6	-3

-6	12	18	6
-12	33	-18	57
18	-55	60	-33

(67)

of their centres of forces, reckoned from the lower left hand coefficient as an origin, are made $x' = \frac{1}{2}\Delta x$ and $y' = \frac{1}{2}\Delta y$ in the first polynomial, and $x' = \frac{5}{2}\Delta x$ and $y' = \frac{3}{2}\Delta y$ in the second. In the first one, with the centre of forces as an origin, $b_{m,n}$ cleared of fractions is expressed as in (68), and it

$$2^{m+n}b_{m,n} = \left\{ \begin{array}{l} 0(-1)^m(3)^n + 0(1)^m(3)^n - 2(3)^m(3)^n \\ + 3(-1)^m(1)^n + 6(1)^m(1)^n + 3(3)^m(1)^n \\ + 3(-1)^m(-1)^n + 6(1)^m(-1)^n - 3(3)^m(-1)^n \end{array} \right\} \quad (68)$$

will be found to reduce to zero when m and n receive any one of those pairs of values which form the sub indices of b in (66). The coefficients of the second polynomial have a like property, their centre of forces being taken as the origin. Now multiply the two polynomials together, and the coefficients of the product stand as in (69). As already proved, the coordinates or lever arms of the centre of forces in the product will be the sum

0	0	12	-24	-36	-12
-18	0	132	96	126	-96
-54	27	216	362	150	219
18	-30	84	60	378	-270
54	-57	-204	426	-378	99

(69)

of those in the factors, namely $x' = (\frac{1}{2} + \frac{3}{2})\Delta x = 3\Delta x$, $y' = (\frac{1}{2} + \frac{3}{2})\Delta y = 2\Delta y$, so that this centre coincides with the 362. Taking it as the origin, we express $b_{m,n}$ as in (70), a part being omitted to save space. It will be found

$$b_{m,n} = \left\{ \begin{array}{l} 0(-3)^m 2^n + 0(-2)^m 2^n + 12(-1)^m 2^n - \&c. \\ -18(-3)^m 1^n + 0(-2)^m 1^n + 132(-1)^m 1^n + \&c. \\ -54(-3)^m 0^n + 27(-2)^m 0^n + 216(-1)^m 0^n + \&c. \\ + \dots \dots \dots \end{array} \right\} \quad (70)$$

that here for the product, as before for the two factors, $b_{m,n}$ becomes zero when m and n have any one of those pairs of values which are the sub-indices of b in (66). Wherever 0^0 occurs in (70) it has the value unity. That this must be so, is evident because the condition $b_{0,0} = 1$ signifies that the sum of all the coefficients is unity, and if this condition is satisfied by each of two factors, it will be by their product. When one coefficient in this product happens, as in the case of (69), to coincide with the centre of forces, it will stand multiplied by $0^m 0^n$ in (70), and other coefficients in the same row or column will be multiplied by 0^n or 0^m . When m and n are zero, the coefficients in question must still be counted in forming the sum $b_{0,0} = 1$, and this cannot be done unless 0^0 is here equal to unity.

Let us now take an algebraic function of three variables, x, y and z , whose terms are supposed to be arranged in the form of a rectangular block, as in the demonstration of (51). We place the origin at one corner occupied by an initial term $A_{0,0,0}$, while the opposite or farthest corner is occupied by the general term $A_{r,s,t} x^r y^s z^t$. For convenience of notation, we will express such a function by

$$u = A_{0,0,0} x^0 y^0 z^0 || A_{r,s,t} x^r y^s z^t. \quad (71)$$

When the variables are each supposed to take any equidistant values, increasing by $\Delta x, \Delta y$ and Δz , the corresponding values of u form what may be called a triple series, any term of which may be taken as an initial term $u_{0,0,0}$, the position of the other terms being denoted by sub-indices reckoned from this in the three coordinate directions. Let any block of terms be adjusted by the unsymmetrical formula,

$$u'_{0,0,0} = l_{-m,-n,-p} u_{-m,-n,-p} || l_{m,n,p} u_{m,n,p} \quad (72)$$

and we inquire what conditions must be satisfied by the coefficients l , in order that the adjusted term $u'_{0,0,0}$ may be always a value of the function (71), but separated in position from $u_{0,0,0}$ by the constant intervals $\mu \Delta x, \nu \Delta y$, and $\pi \Delta z$. Taking the origin at the adjusted term, the coordinates of $u_{m,n,p}$ will be $x = (\mu + m)\Delta x$, $y = (\nu + n)\Delta y$, $z = (\pi + p)\Delta z$, and (71) gives

$$u_{m,n,p} = A_{0,0,0} (\mu + m)^0 (\nu + n)^0 (\pi + p)^0 (\Delta x)^0 (\Delta y)^0 (\Delta z)^0 || A_{r,s,t} (\mu + m)^r (\nu + n)^s (\pi + p)^t (\Delta x)^r (\Delta y)^s (\Delta z)^t, \quad (73)$$

a general expression for any term in the triple series. If it is multiplied throughout by $l_{m,n,p}$, it becomes an expression for any product $l_{m,n,p} u_{m,n,p}$ in the adjustment formula (72). Substituting such expressions for all the products in that formula, we see that any such quantity as

$$A_{r,s,t} (\Delta x)^r (\Delta y)^s (\Delta z)^t$$

will enter (72) with a coefficient which is the sum of all the values of

$$(\mu+m)^r (\nu+n)^s (\pi+p)^t l_{m,n,p}$$

which can be formed by giving to m any integral value from $-m$ to m , and to n any such value from $-n$ to n , and to p any such value from $-p$ to p . Denoting this sum by $b_{r,s,t}$, we have

$$b_{r,s,t} = (\mu-m)^r (\nu-n)^s (\pi-p)^t l_{-m,-n,-p} || (\mu+m)^r (\nu+n)^s (\pi+p)^t l_{m,n,p} \quad (74)$$

and the adjusted term is expressed thus ;

$$u'_{0,0,0} = A_{0,0,0} b_{0,0,0} (\Delta x)^0 (\Delta y)^0 (\Delta z)^0 || A_{r,s,t} b_{r,s,t} (\Delta x)^r (\Delta y)^s (\Delta z)^t. \quad (75)$$

This is the same value which u in (71) has at the origin, namely $A_{0,0,0}$ and the second member of (75) will not in general $= A_{0,0,0}$ unless we have

$$b_{0,0,0} = 1, \quad b_{r,s,t} = 0, \quad (76)$$

which is to be understood as equating to zero all the values of $b_{r,s,t}$ in (74) which can be formed by giving to r, s and t all the sets of values which the exponents of x, y and z have in the several terms of (71), excepting only the set $r=0, s=0, t=0$. The condition $b_{0,0,0} = 1$ makes the sum of all the l 's equal to unity. The three next conditions,

$$b_{1,0,0} = 0, \quad b_{0,1,0} = 0, \quad b_{0,0,1} = 0,$$

determine the relations of the l to μ, ν and π , giving

$$\left. \begin{aligned} -\mu &= -ml_{-m,-n,-p} || ml_{m,n,p} \\ -\nu &= -nl_{-m,-n,-p} || nl_{m,n,p} \\ -\pi &= -pl_{-m,-n,-p} || pl_{m,n,p} \end{aligned} \right\} \quad (77)$$

Thus $-\mu \Delta x, -\nu \Delta y$ and $-\pi \Delta z$ are the lever arms of the coefficients l with respect to three planes passing through the position of $l_{0,0,0}$ or $u_{0,0,0}$ and parallel to the coordinate planes. But since these lever arms are also the coordinate distances from $u_{0,0,0}$ to $u'_{0,0,0}$, it appears that the adjusted term is located at the centre of forces of the coefficients l . Any single condition $b_{r,s,t} = 0$ reduces to zero the sum of the products formed by multiplying each l into the r, s and t powers of its coordinates in the x, y and z directions respectively, reckoned from the centre of forces as an origin.

If the l in (72) satisfy the conditions (76), the adjusted triple series may be readjusted by another formula with coefficients L which satisfy the same conditions, only having μ', ν' and π' in the places of μ, ν and π . These two adjustments are equivalent to a single adjustment by a resultant

formula whose coefficients λ must satisfy the same conditions, μ, ν and π being here replaced by $(\mu + \mu'), (\nu + \nu')$ and $(\pi + \pi')$. By reasoning precisely analogous to that which we have already followed in the cases of single and double series, in connection with (4) and (11), it will appear that the resultant of two triple adjustment formulas such as (72), with coefficients l and L , is a formula in which any term $u_{r,s,t}$ of the given series enters with a coefficient λ which is the sum of all the products that can be formed by multiplying the L into the l so that in each product the sum of the first sub-indices shall be r , the sum of the second, shall be s , and the sum of the third, shall be t .

For example, denoting the two component formulas as in (72), by

$$u'_{0,0,0} = l_{-m,-n,-p} u_{-m,-n,-p} || l_{m,n,p} u_{m,n,p},$$

$$u''_{0,0,0} = L_{-h,-i,-j} u'_{-h,-i,-j} || L_{h,i,j} u'_{h,i,j},$$

it is evident that if a given term $u_{r,s,t}$ enters $u'_{0,0,0}$ through $u'_{0,0,0}$, it enters with the coefficient $L_{0,0,0} l_{r,s,t}$. If it also enters through

$$u'_{1,0,0}, \quad u'_{0,1,0}, \quad u'_{0,0,1}, \quad u'_{1,1,0}, \quad \&c.,$$

it has the respective coefficients

$$L_{1,0,0} l_{(r-1),s,t}, \quad L_{0,1,0} l_{r,(s-1),t}, \quad L_{0,0,1} l_{r,s,(t-1)}, \quad L_{1,1,0} l_{(r-1),(s-1),t}$$

and so on, each product Ll being such that the sums of the two first, the two second, and the two third sub-indices, are r, s and t respectively. The total resultant coefficient λ is the sum of all the products of the L into the l which can be formed in this way. The several values of λ will thus be identical with the coefficients in the product of the two polynomials

$$l_{-m,-n,-p} x^{-m} y^{-n} z^{-p} || l_{m,n,p} x^m y^n z^p,$$

$$L_{-h,-i,-j} x^{-h} y^{-i} z^{-j} || L_{h,i,j} x^h y^i z^j,$$

or what is the same thing, the coefficients in the product of

$$l_{-m,-n,-p} x^0 y^0 z^0 || l_{m,n,p} x^{2m} y^{2n} z^{2p},$$

$$L_{-h,-i,-j} x^0 y^0 z^0 || L_{h,i,j} x^{2h} y^{2i} z^{2j},$$

these two factors having coefficients which are the same and in the same order as the l and L of the two adjustment formulas.

All the conditions (76) except $b_{0,0,0} = 1$ will hold good when the l 's or L 's are multiplied by a constant number. Neglecting then the $b_{0,0,0} = 1$, if the remaining conditions are satisfied by the coefficients in each of two polynomials in x, y and z , they will also be satisfied by the coefficients in their product. If therefore a polynomial satisfying such conditions is raised to any power, or if several such polynomials are multiplied together, the final power or product will satisfy the same conditions. As regards the number of conditions (76) that may be applied, answering to the number of terms in the function (71), we remark that some of the highest terms and conditions may be absent, provided that those which remain form a complete

series as far as they go, so that if the origin is shifted by substituting $x+a$, $y+b$ and $z+c$ for x , y and z in (71), this will not introduce any terms of different degree from those already there. A similar remark applies in the cases of single and double series.

As an illustration of the foregoing, suppose that the function (71) has the form

$$u = A + Bx + Cy + Dz + Exz + Fy^2, \quad (78)$$

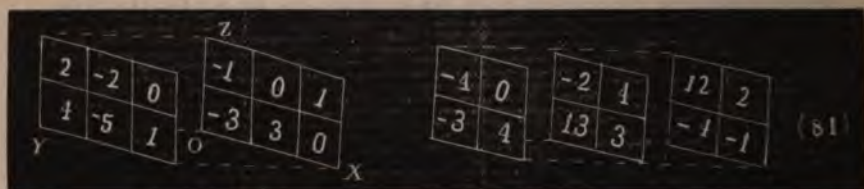
so that the conditions to be satisfied by the coefficients l , omitting $b_{0,0,0} = 1$, are

$$b_{1,0,0} = 0, b_{0,1,0} = 0, b_{0,0,1} = 0, b_{1,0,1} = 0, b_{0,2,0} = 0. \quad (79)$$

The two polynomials

$$\left. \begin{aligned} & -3 + 3x + 4y - z - 5xy + 2yz - 2xyz + x^2y + x^2z, \\ & -4 - x + 13y + 12z + 3xy + 2xz - 2yz + 4xyz - 3y^2 + 4xy^2 - 4y^2z, \end{aligned} \right\} \quad (80)$$

have been constructed so that their coefficients satisfy these conditions. Arranged in block form, they appear as in the diagram (81). The lines OX , OY , OZ give merely the directions, not the locations of the axes.



Starting from the first coefficient as an origin, that is, from the -3 in the one case and the -4 in the other, the centres of forces are placed in both cases at the point

$$x' = \frac{1}{2}Ax, \quad y' = \frac{1}{2}Ay, \quad z' = \frac{1}{2}Az,$$

being at the middle of the block of eight coefficients nearest the origin. Making this centre of forces an origin, we have for any value of b in the first polynomial for instance, the expression (82), which reduces to zero for

$$2^{m+n+p} b_{m,n,p} = \left\{ \begin{aligned} & 2(-1)^m(1)^n(1)^p - 2(1)^m(1)^n(1)^p + 0 \\ & + 4(-1)^m(1)^n(-1)^p - 5(1)^m(1)^n(-1)^p + 1(3)^m(1)^n(-1)^p \\ & - 1(-1)^m(-1)^n(1)^p + 0 + 1(3)^m(-1)^n(1)^p \\ & - 3(-1)^m(-1)^n(-1)^p + 3(1)^m(-1)^n(-1)^p + 0 \end{aligned} \right\} \quad (82)$$

any set of values of m , n , p which form the sub-indices of b in (79). The second polynomial has a similar property. Now when the two polynomials are multiplied together, the coefficients in their product stand as in the diagram (83). As previously shown, the lever arms or coordinates x' , y' , z' of the centre of forces in the product are the sums of

(83)

-8	8	0	0
-22	34	-12	0
-12	31	-23	4

0	12	-12	0
33	-10	-31	8
61	-74	10	3

26	-24	-6	4
33	-67	29	5
-55	46	10	-1

those in the two factors, so that the centre of forces here coincides with the —67. Estimating $b_{m,n,p}$ from this centre as an origin, we shall find that it reduces to zero for the same sets of values of m, n, p as it did in the cases of the two factors.

SOLUTION OF A SPECIAL CASE OF PROBLEM 288, WHERE THE CIRCLES ARE TANGENT AT THE SAME POINT OF AB.

BY PROF. W. W. BEMAN, ANN ARBOR, MICHIGAN.

First part. Geometrical Demonstration.—The triangle CMN being isosceles, $\angle CMN = \angle CNM$.

$\therefore \angle NMD = \angle MND' = \angle NSD'$; $\therefore$ the triangles

QMD and QSD are similar,

and $MD : SD :: r : R ::$

$QD : QD'$. By division,

$(R-r) : R :: DD' : QD'$,

or $(R-r) : R :: (R+r) : QD'$;

$$\therefore QD' = \frac{R(R+r)}{R-r},$$

i. e. QD' is constant, and M -

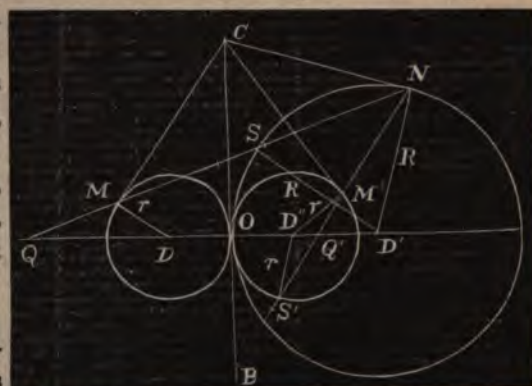
N in all positions will pass

through Q . Similarly it may be shown that NM' in all positions will pass

through the point Q' .

$$QO = \frac{2Rr}{R-r}, \quad Q'O = \frac{2Rr}{R+r}.$$

Analytical Demonstration.—Let O be the origin, QD' and BC the axes, (x_1, y_1) coordinates of M or M' , (x_2, y_2) coordinates of N , $(0, a)$ coordinates of the variable point C . Then we easily obtain



$$y_1^2 = 2rx_1 - x_1^2 \quad (1), \quad y_2^2 = 2Rx_2 - x_2^2; \quad (2)$$

$$yy_1 = r(x+x_1) - xx_1 \quad (3), \quad yy_2 = R(x+x_2) - xx_2; \quad (4)$$

$$y - y_1 = \frac{y_1 - y_2}{x_1 - x_2} (x - x_1). \quad (5)$$

Since $C(0, a)$ is a point in both (3) and (4), we have

$$ay_1 = rx_1 \quad (6), \quad ay_2 = Rx_2. \quad (7)$$

Combining (6) with (1) and (7) with (2), we find

$$x_1 = \frac{2a^2r}{a^2 + r^2}, \quad y_1 = \frac{2ar^2}{a^2 + r^2}, \quad x_2 = \frac{2a^2R}{a^2 + R^2}, \quad y_2 = \frac{2aR^2}{a^2 + R^2}.$$

Substituting these values in (5), and reducing,

$$a^4y - a^2[(r+R)x - 2rR] + a^2r(r-R)y - ar^2[(r+R)x - 2rR] - r^3Ry = 0. \quad (8)$$

This locus evidently passes through the point $[2rR \div (r+R), 0]$ for all values of a ; $\therefore$ this point is the envelope required. r is *positive* when the circles are tangent internally, *negative* when they are tangent externally.

Second part. To find the locus of the middle point of MN .—Letting ρ_1 and ω be the polar coordinates of any point in the circle D' , the pole being at Q' , we have

$$\rho_1 = -r \left\{ \frac{R-r}{R+r} \cos \omega \pm \frac{\sqrt{[(R+r)^2 - (R-r)^2 \sin^2 \omega]}}{R+r} \right\} \quad (1)$$

Again, letting ρ_2 and ω be the polar coordinates of any point in the circle D' , the pole being at Q' , we have

$$\rho_2 = R \left\{ \frac{R-r}{R+r} \cos \omega \pm \frac{\sqrt{[(R+r)^2 - (R-r)^2 \sin^2 \omega]}}{R+r} \right\}. \quad (2)$$

Now, if ρ and ω be the coordinates of any point in the locus desired, we easily get from (1) and (2),

$$\rho = \frac{1}{2} \left\{ \frac{(R-r)^2}{R+r} \cos \omega \pm \sqrt{[(R+r)^2 - (R-r)^2 \sin^2 \omega]} \right\}. \quad (3)$$

one form of the equation.

If the rectangular equation with origin at O be required, we make

$$\rho = \frac{\sqrt{(R+r)^2 y^2 + [(R+r)x - 2Rr]^2}}{R+r}, \quad \sin \omega = \frac{(R+r)y}{\sqrt{(R+r)^2 y^2 + [(R+r)x - 2Rr]^2}}$$

$$\cos \omega = \frac{(R+r)x - 2Rr}{\sqrt{(R+r)^2 y^2 + [(R+r)x - 2Rr]^2}}, \quad \text{and get, after some labor,}$$

$$(R+r)^2(y^2 + x^2)^2 - (R+r)(R^2 + 6Rr + r^2)x(x^2 + y^2) + (R+r)^2Rry^2 + 4Rr \times (R^2 + 3Rr + r^2)x^2 - 4R^2r^2(R+r)x = 0, \quad r \text{ being regarded as } \textit{positive} \text{ when the circles are tangent internally, and } \textit{negative} \text{ when tangent externally. We may write it } (R \pm r)^2(y^2 + x^2)^2 - (R \pm r)(R^2 \pm 6Rr + r^2)x(x^2 + y^2) \pm (R \pm r)^2Rry^2 \pm 4Rr(R^2 \pm 3Rr + r^2)x^2 - 4R^2r^2(R \pm r)x = 0, \text{ the upper sign being taken when the circles are tang't internally, and the lower when tang't ext'y.}$$

SOLUTION OF A PROBLEM IN CURVES OF PURSUIT.

BY LIEUT. JAS. M. INGALLS, FT. TRUMBULL, NEW LONDON, CONN.

Problem. A boat starts from a given point in the bank of a straight canal of uniform current, and is propelled continually toward a movable object, A , on the opposite bank. To find the path of the boat.

It will be seen that this is an extension of the problem discussed in the May No. of the ANALYST, page 89 et seq. It is here proposed to give a few of the results of an investigation of the above problem leaving their demonstration to be supplied by those of your readers who are interested in the subject.

Let P be the starting point of the boat; and A , that of the moveable object, A .

Put PO , the width of the canal = b ; $PA = a$; and angle $PAO = \beta$. All these are supposed to be given and are reg'ed as constants.



Let P' be the position of the boat at any instant, and A' the corresponding position of the object, A . Put $P'A' = z$; and angle $P'A'P'' = \varphi$. Take O as the origin of rectangular coordinates, and make $OD = x$, and $P'D = y$.

Designate the velocity of the boat relative to the water by u ; the velocity of the current by v ; and that of the object, A , by w . Consider motion positive in the direction AP'' . Also put $[(w-v) \div u] = m$; and $v \div u = n$. Then it may be shown that $x^m \tan \frac{1}{2}\varphi$ is constant, and therefore equal to $b^m \tan \frac{1}{2}\beta$. Make this equal to c^m , where c is the value of z when it is perpendicular to AP'' .

From the above we find the equation of the curve when $m < 1$, to be,

$$y = \frac{(1-n)x^{1+m}}{2c^m(1+m)} - \frac{(1+n)c^m x^{1-m}}{2(1-m)} + \frac{a[m+n-(1+mn)\cos\beta]}{1-m^2}. \quad (1)$$

When $m > 1$ we must take $m-1$ instead of $1-m$. When $m=1$, we have,

$$y = \frac{c(1+n)}{2} \log \frac{b}{x} - \frac{1-n}{4c} (b^2 - x^2); \quad (2)$$

and

$$c = b \tan \frac{1}{2}\beta = a(1 - \cos \beta).$$

The above formulas are true for all finite values of u , v and w , except when $v = w$ (and $\therefore m = 0$). In this case the path of the boat is a

straight line whose equation is, supposing $u > v$,

$$y = \frac{b-x}{\sin \beta} (n - \cos \beta).$$

When $v = 0$ (and $\therefore n = 0$), equations (1) and (2) represent the ordinary curve of pursuit.

If we make $m = \frac{1}{2}$ and $n = 1$ ($\therefore u = v = \frac{2}{3}w$), the path becomes a parabola whose focal distance is equal to $c = b \tan \frac{1}{2}\beta$.

As a final example, suppose $w = 0$; therefore $m = -n$. That is, the boat is directed toward a *fixed object*. Equation (1) reduces to

$$y = \frac{c^m x^{1-m}}{2} - \frac{x^{1+m}}{2c^m} - a \cos \beta;$$

and the expression for c becomes

$$c^m = b^m \cot \frac{1}{2}\beta.$$

If in the last equation we make $m = 1$, we shall have

$$y = \frac{a(1 - \cos \beta)}{2} - \frac{x^2}{2c};$$

the equation of a parabola whose axis is AP'' , and whose focal dist. is $\frac{1}{2}c$.

If $t =$ the time from the starting point, we shall have,

$$ut = \frac{a(1-m \cos \beta)}{1-m^2} - \frac{c^m x^{1-m}}{2(1-m)} - \frac{x^{1+m}}{2c^m(1+m)}.$$

This equation is true for all finite values of m , positive or negative. When $x = 0$, and $m < 1$, we have the whole time of crossing

$$T = \frac{a(1-m \cos \beta)}{u(1-m^2)}.$$

If $\cos \beta = m$, we have $T = a+u$. That is, if the boat, at starting, be directed toward an object at A , so that the angle $PAO = \cos^{-1}m$; then, as the object, A , moves from A to P'' , the boat will describe the curve $PP'P''$ in the same time that would be required to move from P to A along the line PA with the same velocity, if there were no current.

MR. GLAISHER'S ENUMERATION OF PRIMES.

COMMUNICATED BY PROF. JOHNSON.

IN the Report for 1879, of the committee on mathematical Tables to the British Association, Mr. J. W. L. Glaisher extends his enumeration of primes over the fourth million, by means of Mr. James Glaisher's factor table for this million, which has just been printed. The results of the enumeration of primes are given in the same form as those referred to at p. 7, Vol.

ANALYST; viz., the number of "centuries" which contain no primes, 1 prime, 2 primes, &c., is given for each group of 10,000 in the million, and the total number of primes in each group. These totals for the first million, and for the 7th, 8th and 9th millions were quoted at pages 7 and 8, and for the fourth million they are as follows:—

1st group	6676	+41
2nd "	6717	—26
3rd "	6691	—52
4th "	6639	—28
5th "	6611	—36
6th "	6575	+96
7th "	6671	—81
8th "	6590	+36
9th "	6624	—89
10th "	6535	

Total in 4th million 66,329.

In a paper published in the Proceedings of the Cambridge Philosophical Society, Mr. Glaisher has discussed his results in connection with Legendre's formula for the number of primes inferior to x , viz.,

$$\frac{x}{\log x - A},$$

in which Legendre's value of A was 1.08366, which it is supposed was originally determined chiefly from the value $x = 10,000$.

In order that the formula should agree with the logarithmic integral

$$\int_0^x \frac{du}{\log u},$$

Glaisher has shown that we should have, as a first approximation,

$$A = 1 + (1 \div \log x),$$

agreeing with what was shown by Tchebycheff, that when x is infinite we must have $A = 1$] accordingly he has compared the actual number of primes counted with the value of

$$\frac{x}{\log x - 1 - (1 \div \log x)^2} \quad (1)$$

as well as with Legendre's formula and $x \div (\log x - 1.08365)$ (2), and with $\log x - 1$ (3). The comparison is made for values of x at intervals of 1,000 and the final results are that, for $x = 4,000,000$, while Legendre's formula (2) gives 177.3 in excess, (3) gives 1,491.7 in defect and (1) gives 1.7 in defect. It may be remarked that the omitted terms in the value of A are all positive so that the second approximation to A which is

$$A = 1 + \frac{1}{\log x} + \frac{3}{(\log x)^2},$$

would give a result still nearer the actual number of primes counted.

QUATERNIONS.

BY PROF. DE VOLSON WOOD, HOBOKEN, NEW JERSEY.

(Continued from page 71.)

OBLIQUE VECTORS.

26. IN section 15 of the preceding article, it was stated that, comparing vector β with vector α consists of the compound operation of determining the angle through which α must be rotated in order to bring it into a position *parallel* to β and of comparing the *length* of β with that of α .

Parallelism implies a zero angle between the *directions* considered. Thus, if β is to be compared with $-\alpha$, then $-\alpha$ must be revolved to coincide in *direction* with $+\beta$. In section 19 a notation was adopted to represent the rotation—left handed being considered positive. The ratio of their lengths is simply a number—positive and abstract—and is expressed in the well known way; hence if a and b represent the lengths respectively of α and β , t the angle between the vectors (the right angle being unity), and i a positive unit vector perpendicular to the plane of $\alpha\beta$, we may write

$$\frac{\beta}{a} = \frac{T\beta \cdot U\beta}{Ta \cdot Ua} = \frac{T\beta}{Ta} \cdot \frac{U\beta}{Ua} = T \frac{\beta}{a} \cdot U \frac{\beta}{a} = \frac{b}{a} i. \quad (14)$$

For all values of α and β , we have

$$U \frac{\beta}{a} = i. \quad (15)$$

If $a = b$ equation (14) reduces to (15); hence for *unit* vectors U may be omitted without ambiguity, in which case we may write

$$\frac{\beta}{a} = i^t; * \quad (16)$$

hence $\beta \div \alpha$, as unit vectors, represents a versor. (See sec. 20.)

If the operation be observed on the side of the negative axis, $-i$, the rotation of α into β will be right-handed, or negative, and will be represented by $-t$; hence we have, as unit vectors

$$\frac{\beta}{a} = (-i)^{-t}; \quad (17)$$

which compared with (16) gives

$$i^t = (-i)^{-t}; \quad (18)$$

that is, the *value* of a versor remains the same when the signs of its base and exponent are both changed from $+$ to $-$, or the reverse.



*Hamilton's Lectures, p. 127.

It is well to observe that in the expression $(-i)^t$, the negative sign applies to the vector—or axis—only; but in the expression $-i^t$ the entire versor is negative and is the same as $-(i^t)$. The MINUS in the last expression has the effect of reversing* the *direction* of the result produced by i^t .

The angles between the vectors will be limited to those between zero and plus or minus two right angles, unless otherwise expressly stated. The angle between α and $-\beta$ (see the figure) is the same as between $-\alpha$ and β , and is $2 - t$ (being the supplement of the angle t between vectors $+\alpha$ and $+\beta$), and is positive in reference to the negative axis $-i$; hence we have

$$\frac{-\beta}{\alpha} = \frac{\beta}{-\alpha} = -\frac{\beta}{\alpha} = -\left(\frac{\beta}{\alpha}\right) = (-i)^{2-t} = i^{t-2}. \quad (19)$$

From (16) and (19) we find

$$i^t = -(-i)^{2-t}. \quad (20)$$

The second member of (20) [which is equivalent to $-(-i^2 \cdot -i^{-t})$] shows that the vector operated upon (as α) may be rotated positively about the negative axis $-i$ through two quadrants, followed by a negative rotation of t about the same axis, thus making it coincide with the vector $(-\beta)$ which was divided, after which the result is reversed, thus bringing it into the position of $+\beta$. The left member rotates α into β positively in one operation.

The value $\beta \div \alpha$ may now be written in various forms. Thus, as unit vectors, we have from the preceding equations,

$$\frac{\beta}{\alpha} = \beta \div \alpha = i^t = (-i)^{-t} = \frac{1}{(-i)^t} = \frac{1}{i^{-t}} = -(-i)^{2-t} = -\frac{1}{i^{2-t}}, \text{ etc.} \quad (21)$$

If vector β be operated upon, turning it to a parallelism with α , the rotation will be positive about $-i$ (the relative positions of the lines being as before), and we have [see (16), (18), and (20)]

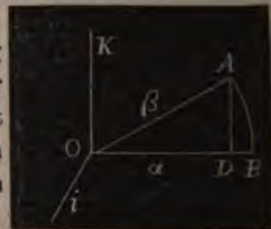
$$\frac{\alpha}{\beta} = (-i)^t = i^{-t} = \frac{1}{i^t} = -i^{2-t}, \text{ etc.;} \quad (22)$$

from which we have

$$-i^{-t} = -\frac{1}{i^t} = i^{2-t}. \dagger \quad (23)$$

27. We now proceed to decompose the versor.

Let α and β be two coinitial unit vectors making an angle θ between them. Take i , a unit vector perpendicular to the plane of $\alpha\beta$, positive in front of the plane, and k perpendicular to α and i . From the extremity of β drop the perpendicular AD upon α ; then as lines we have



*It is an INVERSOR, or its effect is an *in*-version. Lectures, p. 54. †Ibid. pp. 331, 536.

$$\begin{aligned} OD &= \cos \theta, & DA &= \sin \theta, \\ \text{and as vectors (§ 6), } DA &\text{ being parallel to } Ok, \\ OD &= \cos \theta.a, & DA &= \sin \theta.k. \end{aligned}$$

According to equation (1) we have

$$\beta = OD + DA = \cos \theta.a + \sin \theta.k. \quad (24)$$

Dividing a into both members of (24), observing that $k \div a$ is a quadrantal versor and positive about i [Eq. (3)], we have [see also Eq. (21)]

$$\begin{aligned} \frac{\beta}{a} &= \cos \theta + \sin \theta \frac{k}{a} \\ &= \cos \theta + i \sin \theta = i^t, \end{aligned} \quad (25)$$

where $t = \theta \div \frac{1}{2}\pi$. In a similar manner we may find directly from the figure, or by taking the reciprocal of both members of (25), observing that $i^2 = -1$,

$$\frac{a}{\beta} = \cos \theta - i \sin \theta = (-i)^t, \quad (26)$$

the last member of which is from eq. (22). The term $i \sin \theta$ will be negative if i be positive and θ negative, in which case the third member will be i^{-t} , as already shown in equation (22).

Multiplying a into both members of (24) gives

$$\begin{aligned} a\beta &= a^2 \cos \theta + ak \sin \theta \\ &= -\cos \theta + i \sin \theta = i^{2-t}, \end{aligned} \quad (27)$$

in which the exponent is $2 - t$ because $\cos \theta$ being negative and $\sin \theta$ positive, the angle of the product is the supplement of θ ; that is, the supplement of the angle between the vectors.

Multiplying both members of (24) into a gives

$$\begin{aligned} \beta a &= a^2 \cos \theta + ka \sin \theta \\ &= -\cos \theta - i \sin \theta = (-i)^{2-t}. \end{aligned} \quad (28)$$

The second members of the last four equations are versors. In division, the versor operating on the divisor line is conceived to turn it, positively, about the axis of the versor ($+i$, or $-i$) through an angle equal to that of the versor (θ) to coincide in direction with the dividend line. In multipl'n the versor operating upon the multiplier line [as a , eq. (27)] is conceived to turn it in a positive direction about the axis of the versor ($+i$) through an angle equal to the *supplement* of the angle of the versor [$\pi - (\pi - \theta) = \theta$], making it coincide in direction with the multiplicand line (as β).*

The preceding equations show that the versor may be decomposed into two parts, one of which is a number equal to $\pm \cos \theta$; the other, a multiple of a unit vector and equal $\sin \theta.i$, the unit vector being the axis of the quo-

*Lectures, p. 85.

tient, or of the product, of two unit vectors. Hence if q_1 represent the quotient (or product) of two unit vectors in space, w_1 the term in the result containing the *number* only, and ρ_1 the term containing the *vector*, we may write

$$q_1 = w_1 + \rho_1. \quad (29)$$

28. As unit vectors we may write

$$\frac{\beta}{a} = \beta a^{-1} \text{ (but not } \frac{\beta}{a} = a^{-1}\beta). \quad (30)$$

Observing that $a^{-1} = -a$ [eq. (9)], the product of β into a^{-1} is the same as β into $-a$, the rotation being positive, and the angle of the *product* the *supplement* of the angle between β and $-a$, that is t , the right angle being unity; and the *division* of a into β implies a positive rotation through the same angle t ; hence the two *results* are equal. But the product $a^{-1}\beta$ *reverses* the direction of the rotation, and hence cannot equal the former result. Generally, we have



$$U\frac{\beta}{a} = U\beta a^{-1} = -U\beta a^*.$$

Numerous other results present themselves, but we pass them for the present and proceed to the establishment of the general equations and the completion of the nomenclature.

29. If a and β be not unit vectors, let their lengths be a and b times that of the unit, then $Ta = a$, $T\beta = b$, and we have

$$\left. \begin{aligned} \frac{\beta}{a} &= \frac{b}{a}(\cos \theta + i \sin \theta) = \frac{b}{a}i^t \\ \frac{a}{\beta} &= \frac{a}{b}(\cos \theta - i \sin \theta) = \frac{a}{b}(-i)^t = \frac{a}{b}i^{-t} \\ a\beta &= ab(-\cos \theta + i \sin \theta) = ab.i^{2-t} \\ \beta a &= ab(-\cos \theta - i \sin \theta) = ab(-i)^{2-t}. \end{aligned} \right\} \quad (A)$$

These are general equations for the division, or for the multiplication, as the case may be, of any two directed lines in space; and such quotient, or product constitutes a quaternion. By a discussion of equations (A) all the practical equations of this science may be deduced, including the results of the preceding article and those thus far given in the present one.

Thus if $\theta = 90^\circ$, all the results for the multiplication, and for the division, of rectangular vectors are quickly found. The 3d and 4th give $a\beta = -\beta a$.

*Lectures, p. 411.

If $\theta = 0^\circ$ the 3rd and 4th will give $a\beta = +\beta a$. If $a = b$, the 1st and 4th of (A) will differ only by having contrary signs, hence involving the third member in the expression, we have

$$\frac{\beta}{a} = -\beta a = i^4 = -(-i)^{2-4}.$$

The angle between β and $-a$ is the supplement of the angle between a and β , the axis of rotation being positive, or $+i$; therefore for unit vectors we have

$$\beta(-a) = \cos \theta + i \sin \theta,$$

which will be the same as the 1st of (A) when $a = b$; hence we will have

$$\frac{\beta}{a} = \beta(-a) = \beta a^{-1},$$

as shown in eq. (30).

The parenthetical part of the second members of equations (A) is the versor, and we may write

$$U \frac{\beta}{a} = \cos \theta + i \sin \theta = i^4, \quad (31)$$

$$U a \beta = -\cos \theta + i \sin \theta = i^{2-4}; \quad (32)$$

and the 1st and 3rd of (A) give

$$U \frac{\beta}{a} = -U \beta a, \text{ as before shown.} \quad (33)$$

For a representative expression, we choose one, the elements of which are naturally positive. For this reason the 1st of (A) may be considered as TYPICAL — since a and b are essentially positive, the axis of rotation (i), positive, the direction of rotation, positive, and when θ is acute, $\cos \theta$ and $\sin \theta$ will be positive. If q represent the left member, we have as a TYPICAL FORM

$$\begin{aligned} Uq &= \cos \theta + i \sin \theta \\ &= \cos \theta + \sqrt{-1} \sin \theta, \end{aligned} \quad (34)$$

the last value of which is independent of any particular axis, and is suggestive of De Moivre's formula.

30. The second members of equations (A) are composed of two parts, one of which is a *number* equal to a multiple of $\cos \theta$, the other is a multiple of a unit vector, as $ab \sin \theta \cdot i$; hence if q represent the first member, w the *numerical term* and ρ the entire vector in the second term of the second member, we have the typical form

$$q = w + \rho,^* \quad (35)$$

which reduces to (29) when a and β are each unit vectors.

31. A SCALAR is a number, either positive or negative—it represents the *reals* of algebra. This name is applied to such quantities because they

*Lectures, p. 116.

may be laid off on a scale from $-\infty$ to $+\infty$.* It is indicated by the letter S . The numerical term in equations (A) is called the scalar part† of the quotient, or product (as the case may be) of two directed lines. Hence we have

$$S \cdot \frac{\beta}{a} = \frac{b}{a} \cos \theta, \quad S \cdot \frac{a}{\beta} = \frac{a}{b} \cos \theta, \quad (36)$$

$$S \cdot \beta a = -ab \cos \theta, \quad S \cdot a\beta = -ab \cos \theta. \quad (37)$$

The last is read "The scalar of the product of a into β equals minus the continued product of tensor a , tensor β , and the cosine of the angle between a and β "—and similarly for the others. The period after S indicates that the scalar of all that follows it in that term is to be taken. It may be omitted where no ambiguity will result, hence the preceding expressions may be written $S\frac{\beta}{a}$, $Sa\beta$, etc.

From (37) we have

$$Sa\beta = S\beta a. \quad (38)$$

From 3rd and 4th of (A), we have

$$a\beta + \beta a = 2Sa\beta. \quad (39)$$

From (31), (32), (33), and (A), we have

$$SU\frac{\beta}{a} = \cos \theta \dagger = SU\frac{a}{\beta} = -SU\beta a, ** \quad (40)$$

$$SUa\beta = -\cos \theta = SU\beta a. \dagger\dagger \quad (41)$$

The scalar of a number is the same number, hence

$$SSq = S^2q = Sq = \pm \cos \theta,$$

or briefly

$$S^2 = S.\dagger\dagger$$

And, generally,

$$Sq = Tq.SUq. \quad (42)$$

32. The second term of the second member of equations (A) is called the VECTOR PART†† of the quaternion and is indicated by the letter V . Thus

$$V\frac{\beta}{a} = \frac{b}{a} \sin \theta.i, \quad V\frac{a}{\beta} = -\frac{a}{b} \sin \theta.i; \quad (43)$$

$$Va\beta = ab \sin \theta.i, \quad V\beta a = -ab \sin \theta.i; \quad (44)$$

$$\therefore Va\beta = -V\beta a. \quad (45)$$

From 3rd and 4th of (A) we have

$$a\beta - \beta a = 2Va\beta. \quad (46)$$

From (32) we have

$$VUa\beta = \sin \theta.i. \quad (47)$$

Observing that i , unaffected by an exponent, is a quadrantal versor (§20),

*Lectures, p. 58. †Ibid., p. 392. †Ibid., p. 397. **Ibid., p. 412. †† & ††Ibid., p. 393.

the versor of $\sin \theta \cdot i$ is i ; hence from (47) we have

$$UVU_{a\beta} = i. \quad (48)$$

Similarly from (44) we have

$$UV_{a\beta} = i = -UV_{\beta a}, \quad (49)$$

which value is the same as (48). In (44), $ab \sin \theta$ is the tensor of i (or of $-i$), hence

$$TV_{a\beta} = TV_{\beta a} = ab \sin \theta. * \quad (50)$$

Similarly (32) gives

$$TVU_{a\beta} = \sin \theta. \dagger \quad (51)$$

Equation (50) shows that $TV_{a\beta}$ is the area of a parallelogram constructed on Ta and $T\beta$ as adjacent sides having an angle θ between them.

From (43) we have $V_{\frac{\beta}{a}} \cdot V_{\frac{\alpha}{\beta}} = \sin^2 \theta$. Numerous other relations may be readily deduced from the preceding equations.

33. According to the preceding notation, equations (A) may be written in the following form:

$$\text{Equations (A) [see (14)] give } q = Tq \cdot Uq; \quad (52)$$

$$\text{" (36), (37), (43), (44), } = Sq + Uq; \quad (53)$$

$$\text{" (36), (37), (40), (43), (44) } = Tq \cdot SUq + Tq \cdot \sin \theta \cdot i;$$

$$\begin{aligned} \text{factoring} &= Tq(SUq + \sin \theta \cdot i), \\ \text{substituting (47)} &= Tq(SUq + VUq), \end{aligned} \quad (54)$$

$$\text{or substituting (51)} = Tq(SUq + TVUq \cdot i),$$

$$\text{in which substitute (49)} = Tq(SUq + TVUq \cdot UVq),$$

$$\text{or subst. (48) instead of (49)} = Tq(SUq + TVUq \cdot UVUq).$$

NOTE.—For § 9, page 36, substitute the following :

9. A TENSOR is a numerical factor by which a unit vector is multiplied to produce a given vector. Literally the term implies a stretching, but its meaning is here enlarged so as to include the shortening of a line. It is the *ratio* of the *length* of a vector to that of a unit parallel vector, and hence is always positive, or more strictly speaking, *signless*, and may be an entire, fractional, or incommensurable number. The tensor of a vector is sometimes indicated by the letter T , as $T\gamma$. Thus in the equations $\gamma = b\gamma_1$, $\delta = -c\delta_1$, $\eta = 3\eta_1$, where γ_1 , δ_1 , η_1 , are unit-vectors, we have $T\gamma = b$, $T\delta = T(-\delta) = c$, $T\eta = 3$. Generally, we have

$$\gamma = T\gamma(\gamma_1).$$

[For want of sorts, γ , δ , and η are used in the foregoing Note, respectively instead of *Alpha*, *Beta* and *Gamma*, used by Prof. Wood.]

NOTE 2.—For the following corrections I am indebted to suggestions by Alex. S. Christie, communicated to me by the editor.

*Lectures, p. 418. †Ibid., p. 397.

At page 67, after the last line, add, When the direction of rotation of the fraction and its reciprocal are in opposite senses, the reciprocal of the axis is the axis of the reciprocal, as may be shown by equation (9).

At page 69, lines 7 and 8, expunge the sentence "As a versor it implies that β has been turned from some arbitrary direction into the given one."; for it conflicts with the definition given by Hamilton, which is "*U β is the versor of a right quotient.*" (Elements, p. 136.)

At page 70, line 3, for "has a fixed position," read, is confined to one plane.

Also, at page 68, line 3, for " $i+ - i$ " read, $-i \div i$; and at page 71, line 16, for " $-kij = -ijk = -jki$ " read, $-kij = -ikj = -jik$.

[Prof. Wood proposes to discuss some examples illustrating the foregoing principles, which will appear in future Nos., as space will permit.]

A FOURTH PROPORTIONAL.

BY JESSE APPLGATE, YONCALLA, OREGON.

If a straight line, ABC , be considered as composed of two parts, a and b , or the line a be laid upon the line b , the lines a and b , in neither case, form an angle at B , for the lines a and b are *extended* in a line with each other, and their relation to each other at their junction B , would be expressed as an angle of 180° , or a right line. In the second case, as the line a coincides with the line b , the angle at B may be represented by 0° .

Whatever the lengths of the lines a and b may be, a *proportion* exists of which a and b are the means, and their sum $a + b$, one of the extremes. That is,

$$a + b : b = a : x,$$

whence

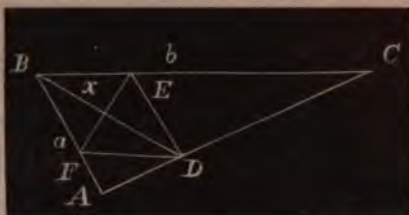
$$x = \frac{ab}{a + b},$$

x being the 4th proportional.

In the above cases, where a and b form a straight line, the 4th proportional depends on the *lengths* of the lines only, it will therefore also exist whatever *angle* these lines make with each other, from 180° to 0° .

To ascertain the utility, if there be any, in the 4th proportional; let the point B be the apex, and carry the line a around until it coincides with the line b , stopping at any point, say at 90° , or before or after passing 90° , to inquire what value the 4th proportional may have in solving problems in Geometry and Trigonometry.

Draw BD bisecting the angle B ; also draw DE and DF , respectively parallel to AB and BC . Then will $BEDF$ be an equilateral quadrilateral of which BD and EF are diagonals.



Let x represent the side of this quadrilateral. Then by similar trian.

$$(b-x) : x = b : a; \therefore ab - ax = bx, \text{ or } ax + bx = ab;$$

$$\therefore x = \frac{ab}{a+b},$$

which is the equation of the 4th proport'l existing between the lines a and b .

As the sides of a quadrilateral equilateral can be ascertained from the sides of the included angle, the diagonals of which are the measures of the angle, not by the degrees of an imaginary arc but by actual latitude and departure, two sides of a triangle with any data from which one of these diagonals may be determined, the unknown side and all other things about the triangle may be determined. That is, it solves all the cases of plane Trigonometry by measurement only, without the use of a table.

This method suggests a cheap and simple instrument for the field that would do all the work of the transit more simply and with equal accuracy. And for the observatory, where the quadrangle might be of any desir'd size, the latitude and departure to the $\frac{1}{106}$ part of every second in the semicircle might be read from the diagonals without the necessity of a single calcul'n.

SOLUTION OF PROB'S IN NO. 2; CONTINUED FROM P. 103.

306. "If N is an odd number and m the number of odd numbers less than N and prime to it (including 1), then N is a divisor of one of the two numbers $2^m \pm 1$."

SOLUTION BY PROF. WM. WOOLSEY JOHNSON.

The number of odd numbers less than and prime to an odd number is one half of the whole number of numbers less than and prime to the given number; for, corresponding to every odd number r less than and prime to N , there is an even number $N-r$, also less than and prime to N . Hence $\phi(N)$ denoting this whole number, the theorem is that when N is odd either

$$2^{\frac{1}{2}\phi(N)} - 1 \quad \text{or} \quad 2^{\frac{1}{2}\phi(N)} + 1$$

is divisible by N . But the theorem may be generalized thus, If a is prime to N , one of the quantities

$$a^{\frac{1}{2}\phi(N)} - 1 \quad \text{or} \quad a^{\frac{1}{2}\phi(N)} + 1$$

is divisible by N [except when $N = 2$, the only case in which $\phi(N)$ is an odd number].

By Euler's extension of Fermat's Theorem (see Serret's Course d'Algebre Supérieure, 3rd Ed., Vol. 2, p. 36)

$$a^{\phi(N)} - 1 \text{ is divisible by } N, \quad (1)$$

and if $N = a^\alpha \beta^\beta \gamma^\gamma \dots$, $a, \beta, \gamma \dots$ being the *different* prime factors of N , we have

$$\phi(N) = a^{\alpha-1} \beta^{\beta-1} \gamma^{\gamma-1} \dots (\alpha-1)(\beta-1)(\gamma-1),$$

whence if $N = N_1 N_2$, where N_1 and N_2 are prime to one another, we have

$$\phi(N) = \phi(N_1) \cdot \phi(N_2). \quad (2)$$

Now unless N is a power of a prime or double the power of a prime, it will be possible to take N_1 and N_2 each greater than 2, and therefore $\phi(N_1)$ and $\phi(N_2)$ will both be even numbers. It follows that $\frac{1}{2}\phi(N)$ is a common multiple of $\phi(N_1)$ and $\phi(N_2)$ and therefore $a^{\frac{1}{2}\phi(N)} - 1$ is divisible by $a^{\phi(N_1)} - 1$ and $a^{\phi(N_2)} - 1$; hence by (1) it is divisible by N_1 and by N_2 , and since N_1 and N_2 are prime to one another, it is divisible by $N_1 N_2$, that is by N .

The cases excluded above are those in which N is of the form a^r , $2a^r$ or 2^r , a denoting an odd prime, and $r > 1$.

$$\text{We have} \quad a^{\phi(N)} - 1 = [a^{\frac{1}{2}\phi(N)} - 1][a^{\frac{1}{2}\phi(N)} + 1]. \quad (3)$$

Now if $N = a^r$, since by (2) the first member of (3) is divisible by a^r , one of the factors in the second member is divisible by the prime a , and since these factors differ by 2 units, the other is *not* so divisible; therefore the former factor is divisible by a^r that is by N .

Again if $N = 2a^r$ the same proof shows that one of the factors is divisible by a^r , but since in this case the first member of (3) is divisible by $2a^r$, one of the factors is divisible by 2, and therefore since they differ by 2 *both* factors are so divisible, and hence one of them is divisible by $2a^r$ i. e. by N .

Again if $N = 4$, both the factors in the second member of (3) are proved as above to be even and since they differ by 2 one of them is divisible by 4.

Finally if $N = 2^r$ where $r > 2$, a , which is by hypothesis prime to N , is odd, and we may write $a = 2^k \pm 1$, where k is an integer. Squaring successively we have

$$a^2 = 2^{2k_1} + 1,$$

$$a^{2^2} = 2^{4k_2} + 1,$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$a^{2^{r-2}} = 2^{rk_{r-2}} + 1.$$

Now in this case $\phi(N) = 2^{r-1}$ and $\frac{1}{2}\phi(N) = 2^{r-2}$, hence the final equation expresses that

$$a^{\frac{1}{2}\phi(N)} - 1 = 2^{rk_{r-2}} = Nk_{r-2}.$$

307. "Three points are taken at random in the surface of the northern hemisphere; find the chance that the small circle through the points lies wholly north of the equator, supposing the earth a perfect sphere."

SOLUTION BY PROF. E. B. SEITZ, KIRKSVILLE, MO.

Let $NEGF$ be the equator, A, B, C , three random points in the surface of the northern hemisphere, M the center of the small circle through A, B, C , O the center of the earth, and OP a radius such that AB is parallel to the plane MOP . From P as a pole, describe $EKFL$, the circumference of a great circle.



Now instead of confining the three points to the surface of the hemisphere whose great circle is $NEGF$, we will confine them to the surface of the hemisphere whose great circle is $EKFL$, and consider the number of ways they can be taken, so that the circle through them shall lie wholly in the hemisphere. This will evidently give the same result as that required in the question.

Let $OA = r$, $\angle AOM = \theta$, $\angle BAM = \varphi$, $\angle CAM = \psi$, $\angle MOP = \mu$, and the angle the plane MOP makes with a fixed plane through $OP = \omega$. Then an element of surface at A is $2\pi r^2 \sin \theta d\theta$, at B it is $4r^2 \sin \theta \cos \varphi d\varphi d\mu$, and at C it is $4r^2 \sin \theta \sin \mu \cos \psi \sin(\varphi + \psi) d\psi d\omega$. The limits of θ are 0 and $\frac{1}{2}\pi$; of φ , $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$; of ψ , $-\varphi$ and $\frac{1}{2}\pi$, and doubled; of μ , 0 and $\frac{1}{2}\pi - \theta$; of ω , 0 and 2π . Hence, since the whole number of ways the three p'ts can be taken in the surface of a hemisphere is $(2\pi r^2)^3$, the required chance is,

$$\begin{aligned} & \frac{2}{(2\pi r^2)^3} \int_0^{\frac{1}{2}\pi} \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \int_{-\varphi}^{\frac{1}{2}\pi} \int_0^{\frac{1}{2}\pi - \theta} \int_0^{2\pi} 32\pi r^6 \sin^3 \theta d\theta \cos \varphi d\varphi \cos \psi \sin(\varphi + \psi) d\psi \\ & \quad \times \sin \mu d\mu d\omega \\ &= \frac{16}{\pi} \int_0^{\frac{1}{2}\pi} \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \int_{-\varphi}^{\frac{1}{2}\pi} \int_0^{\frac{1}{2}\pi - \theta} \sin^3 \theta d\theta \cos \varphi d\varphi \cos \psi \sin(\varphi + \psi) d\psi \sin \mu d\mu \\ &= \frac{16}{\pi} \int_0^{\frac{1}{2}\pi} \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \int_{-\varphi}^{\frac{1}{2}\pi} (1 - \sin \theta) \sin^3 \theta d\theta \cos \varphi \cos \psi \sin(\varphi + \psi) d\psi \\ &= \frac{8}{\pi} \int_0^{\frac{1}{2}\pi} \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} [(\frac{1}{2}\pi + \varphi) \sin \varphi + \cos \varphi] (1 - \sin \theta) \sin^3 \theta d\theta \cos \varphi d\varphi \\ &= 6 \int_0^{\frac{1}{2}\pi} (1 - \sin \theta) \sin^3 \theta d\theta = 4 - \frac{2}{3}\pi. \end{aligned}$$

[Mr. Adcock obtains, for the required probability, $2\pi(\frac{2}{3} - \frac{2}{16}\pi)$, the numerical value of which does not differ much from that obtained by Prof. S. The method he employs, however, is similar to that of his published solution of 286, which is incorrect, as we have before stated, because the units in the numerator and denominator are dissimilar, and therefore not necessa-

rily equally probable. For instance, a finite number of units in the numerator contains, as individual units, many *concentric* circles; whereas it may easily be shown that the chance that there will be *two* concentric circles in a finite number of units of the denominator, is infinitely small.

Mr. Adcock requests us to make the following corr. in his solution of 286:

"For number of positions use the number of combinations instead of the number of permutations. For instance, for $\pi^2 m^2 r^2$ read, $\frac{1}{2} \pi^2 m^2 r^2$."—Ed.]

308. "Evaluate

$$\int_0^{\frac{\pi}{2}} dx \sin [x(a-x)] \text{ and } \int_0^{\frac{\pi}{2}} dx \cos [x(a-x)]$$

by a rapidly converging series."

SOLUTION BY R. J. ADCOCK, ROSEVILLE, ILL.

Since $\sin (ax-x^2) = ax-x^2 - \frac{(ax-x^2)^3}{1.2.3} + \frac{(ax-x^2)^5}{1.2.3.4.5} - \&c.$, and by p'ts,

$$\int (a-x)^n x^n dx = -\frac{(a-x)^{n+1} x^n}{n+1} - \frac{n(a-x)^{n+2} x^{n-1}}{(n+1)(n+2)} - \frac{n(n-1)(a-x)^{n+3} x^{n-2}}{(n+1)(n+2)(n+3)} - \&c. + C.$$

$\therefore \int_0^{\frac{\pi}{2}} (a-x)x dx = -\frac{1}{2}(a-x)^2 x - \frac{1}{6}(a-x)^3 + C = \frac{1}{12}a^2$, and

$\int_0^{\frac{\pi}{2}} (a-x)^2 x^2 dx = \frac{a^7}{280}$, $\int_0^{\frac{\pi}{2}} (a-x)^4 x^4 dx = \frac{a^{11}}{5324}$; hence

$\int_0^{\frac{\pi}{2}} dx \sin [x(a-x)] = \frac{a}{12} - \frac{a^7}{1680} + \frac{a^{11}}{106480} - \&c.$,

which converges rapidly for values of a not greater than one.

SOLUTIONS OF PROBLEMS IN NUMBER THREE.

Solutions of problems in No. 3 have been received as follows:

From Geo. H. Bangs, 311; Alex. S. Christie, 310; Prof. W. P. Casey, 309, 310, 311; Geo. Eastwood, 310; Prof. E. J. Edmunds, 309, 310; H. Heaton, 310, 311, 312; Lieut. Jas. M. Ingalls, 312; Prof. J. H. Kershner, 309, 310, 311, 312 [Prof. Kershner should have been credited in No. 3 for a solution of 304, which was omitted by an oversight]; W. L. Marcy, 311; Prof. D. J. Mc Adam, 311; P. Richardson, 311; Prof. J. Scheffer, 310, 311, 312; J. L. Schock, 312; Prof. De Volson Wood, 311.

309. "In a given circle, find the vertices of the inscribed square, pentagon, octagon, and decagon by using the dividers alone."

passing through p ; x, y , the coordinates of q, x_1, y_1 , of r , and R the radius of the circle. Then by moments we have

$$pR = qx + rx_1,$$

$$qy = ry_1;$$

and from the circle

$$R^2 = x^2 + y^2 = x_1^2 + y_1^2.$$

Solving we find

$$x = \frac{1}{2}R \frac{r^2 + p^2 - q^2}{pr}, \quad y = \pm R \sqrt{\left(1 - \frac{(r^2 + p^2 - q^2)}{4p^2r^2}\right)};$$

in which if y be real, the solution will give two points, but if imaginary the problem will be physically impossible. Similarly values may be found for x_1 and y_1 .

312. "To find the area of the loop of the curve $y^3 + x^2y - axy + bx^2 = 0$."

SOLUTION BY J. L. SCHOCK, CADET MID'N, U. S. N. A.

The equation $y^3 + x^2y - axy + bx^2 = 0$ (1), gives

$$y = \frac{axy - bx^2}{x^2 + y^2}, \text{ or at the limit } y = \frac{axy - bx^2}{x^2 + y^2} \Big|_{y=0} = -b.$$

Hence $y = -b$ is an asymptote to the curve. $x(ay - bx) = 0$ are the eq^s of the tangents at the origin since these terms contain all the lower powers.

The curve then evidently is:—

$$A = \frac{1}{2} \int r^2 d\theta.$$

The limits of integration for polar coordinates evidently are

$$\tan^{-1} \frac{b}{a} \text{ and } \frac{\pi}{2},$$

as shown in the curve. Transforming (1) to polar coordinates,

$$r = \frac{\cos \theta}{\sin \theta} \left[a \sin \theta - b \cos \theta \right]. \quad (2)$$

$$\therefore \text{Area of loop} = \frac{1}{2} \int_{\tan^{-1} \frac{b}{a}}^{\frac{\pi}{2}} \frac{b \cos^2 \theta}{a \sin^2 \theta} \left[a \sin \theta - b \cos \theta \right]^2 d\theta. \quad (3)$$

The integral of (3) is

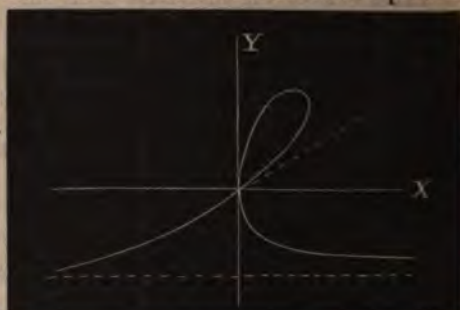
$$\frac{1}{4}(a^2 - 3b^2)\theta + \frac{1}{4}(a^2 - b^2) \sin \theta \cos \theta - ab(\log \sin \theta + \frac{1}{2} \cos^2 \theta) - \frac{1}{2} b^2 \cot \theta.$$

Substituting the limits we have

$$\frac{1}{4} \left[(a^2 - 3b^2) \left(\frac{\pi}{2} - \tan^{-1} \frac{b}{a} \right) + ab \left(2 \log \frac{b^2}{a^2 + b^2} + 3 \right) \right], \text{ or}$$

$$\frac{1}{4} \left[(a^2 - 3b^2) \left(\cot^{-1} \frac{b}{a} \right) + ab \left(2 \log \frac{b^2}{a^2 + b^2} + 3 \right) \right],$$

the required area.



QUERY BY PROF. A. HALL.—(1). If v be the potential function we have the equation given by Laplace

$$\frac{d^2 v}{dx^2} + \frac{d^2 v}{dy^2} + \frac{d^2 v}{dz^2} = 0,$$

which holds for a point outside the attracting body. For a point inside the body we have the equation given by Poisson,

$$\frac{d^2 v}{dx^2} + \frac{d^2 v}{dy^2} + \frac{d^2 v}{dz^2} = -4\pi\rho.$$

What is the value of the right hand side of this equation for a point on the surface of the attracting body? Moigno says, "in this case the expression has in reality eight distinct values." Statics, p. 460.

(2) Given a hemispherical dome turning about a pin at the top, and having a slit extending from the horizon to the zenith. Can a telescope be placed inside this dome in such a position that every point of the heavens can be seen through the telescope?

QUERY BY T. W. WRIGHT, U. S. LAKE SURV. DETROIT, MICH.—Can a satisfactory proof be given of the statement that "the weight of an observation varies as the square of the probability of its accuracy"? (Monthly Notices R. A. S., Vol. XIII.)

QUERY BY PROF. WM. WOOLSEY JOHNSON, ANNAPOLIS, MD.—"It is stated in Brande and Cox' Dictionary of Sc. and Lit., with respect to Fermat's 'lost theorem', viz.: that $x^n + y^n = z^n$ is insoluble in integers except for $n=2$, that no complete demonstration has yet been given. Yet Barlow in his Theory of Numbers gives what professes to be a demonstration of the impossibility of the equivalent equation $x^n - y^n = z^n$, p. 164.

What is the fallacy in Barlow's demonstration and by whom was it first exposed?"

PROBLEMS.

3. By Prof. De Volson Wood.—A piston, weight w , is dropped into the end of a vertical cylinder filled with air, length l ; how far will the piston descend, assuming no friction nor escape of air, nor heat from the compressed air?

4. By Prof. L. G. Barbour, Richmond, Ky.—A helix is coiled about a right circular cylinder the radius of whose base is 1. The projection of the evolute of the helix on the plane of the base of the cylinder encloses double the area of that base. Required the angle made with the plane of the base by a tangent to the helix.

315. *By T. W. Wright, U. S. Lake Survey, Detroit, Mich.*—Two marksmen compete for a prize. Three shots are to be fired by each. The distance of each shot hole from the bull's eye is measured and he is the winner whose total distance is least. *A*'s three shots measured respectively 1, 2, 3 in. from the bull's eye. *B* then made three shots and the bull's eye fell off leaving no mark by which it could be replaced.

An examination showed that the coordinates of *B*'s shot holes, referred to an origin arbitrarily taken, were

(0, 0) inches, (1, 1) in. and (2, 3) inches.

Which of the marksmen was the probable winner?

316. *By W. E. Heal, Wheeling, Ind.*—Required the value of

$$\frac{\log x \sqrt{1 - a^2 x^2}}{\sqrt{1 - x^2}}$$

where $x = 1$, and $a = \log(1 - x)$.

317. *By William Hoover, Wapakoneta, Ohio.*—If the scale of relation of a recurring series be $a_n - 7a_{n-1} + 12a_{n-2} = 0$, and if $u_0 = 2$, $u_1 = 7$, find u_n , and the sum of $u_0 + u_1 + \dots + u_{n-1}$.

318. *By G. H. Harvill, Colfax, La.*—A man takes hold of the end of a cart tongue and travels off at right angles to the direction in which the tongue originally lay. Required the equation of the curve made by the middle of the axle.

319. *By Prof. Scheffer.*—Within a triangle to determine a point so that if the three perpendiculars are let fall from it upon the sides of the triangle, the latter will be divided into three equal parts.

PUBLICATIONS RECEIVED.

On the Ghosts in Rutherford's Diffraction Spectra: A Quincuncial Projection of the Sphere: Note on the Theory of the Economy of Research: By C. S. PEIRCE. 4to. [Published by the authority of the Superintendent of the U. S. Coast and Geodetic Survey.]

Observations of the Satellites of Mars. [March, 1880.] By PROFESSOR A. HALL. [Reprinted from the Monthly Notices, R. A. S., Vol. XL, No. 5.]

On the Observations of Double Stars. By ASAPH HALL, of Washington, D. C. [From the Proceedings of the A. A. A. S., Vol. XXVIII, August, 1879.]

ERRATA.

On page 38, line 6 from bottom, for "*ABEF*" read, *ABED*.

" " 111, line 2 from bottom, for $(\triangle z)_0$ read, $(\triangle z)^0$.

In the diag. "(83)" p. 115, the figure 9 in the right hand section and lower row is reversed.

THE ANALYST.

VOL. VII.

SEPTEMBER, 1880.

No. 5.

MECHANICS BY QUATERNIONS.

BY PROF. E. W. HYDE, UNIVERSITY OF CINCINNATI.

STATICS.

(1). OWING to the fact that certain of the quantities treated in Mechanics possess *direction* as well as *magnitude*, and are thus in their very nature *vector* quantities, it appears that the Quaternion methods should be peculiarly fitted for dealing with mechanical problems. Such is indeed the case, and it is proposed in these papers to give an elementary quaternion treatment of the subject.

(2). According to the notation used by Hamilton a vector is always represented by some small Greek letter, except in the case of a system of 3 mutually perpendicular unit vectors which he generally represents by the letters i, j, k . Now usage has assigned to certain letters certain meanings in mechanics, such as P, Q, R, F for forces, v for velocity etc., and it seems desirable to adhere to this usage: hence it has been deemed best in these papers to indicate the fact that a letter is used as a vector by placing a subscript prime before the letter, thus:— a, b, P . This notation has the further advantage that a will be the tensor of a, P of P etc., so that the use of the symbol T can be to some extent dispensed with. For convenience the primes will be omitted in the case of i, j, k used in the sense given above, and also from the letters $\epsilon_1, \epsilon_2, \epsilon_3$ etc., used as a system of *unit* vectors in any directions, and perhaps occasionally in other cases when there is no danger of misunderstanding arising from such omission.

(3) Proceeding in the usual, though not, perhaps, the most strictly logical order, we will first discuss Statics, which treats of the laws of equilibrium; and afterwards Dynamics, which treats of the laws of motion.

The fundamental quantities of Mechanics are Space (by which is meant the path of a moving particle), Time and mass (or quantity of matter); all the other quantities to which names are given are functions of these three.

Since the path of a moving particle possesses length, and direction either constant or varying from point to point, it is evidently a vector quantity. Time and mass have no direction, and are therefore scalar quantities.

Force is that which is postulated as the cause of any change, or tendency to change, in the rate of motion. It is evidently of the nature of a vector, since the change of motion, or tendency to change, must have some direction, as well as magnitude. It will be assumed as self evident that a force may be considered as applied at any point in its line of action; whence it follows that a force ρ is completely determined when we know the vector π of some point in its line of action.

A body is said to be in *equilibrium* when the external forces acting on it balance, or neutralize each other.

FORCES ACTING ON A PARTICLE.

(4). The parallelogram of forces follows at once from the law of addition and subtraction of vectors. Thus, by Fig. 1,

$$\rho_1 + \rho_2 = \rho_1 \quad \text{and}$$

$$\rho_1 - \rho_2 = \rho_2.$$

Similarly we have for any number of forces

$$\rho_1 + \rho_2 + \rho_3 + \&c. = \Sigma(P) = \rho. \quad (1)$$

This equation contains in itself the propositions concerning the polygon of forces whether in a plane or in space, since $\rho_1, \rho_2, \&c.$, can have any magnitudes and directions whatever. The condition that the particle shall be in equilibrium is then evidently

$$\rho = \Sigma(P) = 0. \quad (2)$$

(5) *Equilibrium of a particle constrained to remain on a smooth curve.*—In this case it is only necessary that the resultant should have no component along the tangent to the curve at the point where the particle is situated, i. e., the resultant must be perpendicular to this tangent.

Let $\rho = \varphi(t)$, in which φt is some vector function of the scalar variable t , be the equation of a curve; then $d\rho = \varphi'(t)dt$ is a vector along the tangent. Hence the required condition may be written

$$Sd\rho, \rho = Sd\rho, \Sigma(P) = 0, \quad (3)$$

or

$$S, \varphi'(t) \Sigma(P) = 0. \quad (4)$$

Suppose for instance the particle to be on a right line whose equation is

$$\rho = a + bt.$$



Then $\frac{d_{,}\rho}{dt} = ,\varphi'(t) = ,b$, and $S_{,}b\Sigma(,P) = 0$ is the condition of equilibrium.

As a further illustration of the use of equation (4) we will solve the following problem.

A particle is constrained to remain on the diagonal of a parallelopiped, and is acted on by three forces represented in magnitude and direction by the edges of the parallelopiped which meet at a corner not on the diagonal: determine the condition of equilibrium.

Let $,a$, $,b$ and $,c$ be the vector edges and at the same time the three forces. Then the equation of the diagonal will be,

$$,\rho = ,a + t(,b + ,c - ,a),$$

if we assume that it passes through the extremity of $,a$. Hence

$$,\varphi'(t) = ,b + ,c - ,a,$$

and by eq. (4)

$$S(,b + ,c - ,a)(,a + ,b + ,c) = 0;$$

$$\therefore -,a^2 + ,b^2 + ,c^2 + 2S_{,}b_{,}c = 0, \text{ or } a = \sqrt{-(,b + ,c)^2} = T(,b + ,c).$$

(6) *Particle constrained to remain on a smooth surface.*—In this case let $,\nu$ be a vector along the normal to the surface; then the condition of equilibrium is

$$V_{,}\nu\Sigma(,P) = V_{,}\nu R = 0. \quad (5)$$

The equation of the surface may be either a scalar or vector equation. If the former, then, as the equation will be linear in $d_{,}\rho$ after differentiation, it may be written

$$S\phi(, \rho)d_{,}\rho = 0, \quad (6)$$

in which $\phi(, \rho)$ is some vector function of $, \rho$. Now $\phi(, \rho)$ is evidently perpendicular to $d_{,}\rho$ and hence $V_{,}\nu\phi(, \rho) = 0$, or we may write $,\nu = \phi(, \rho)$.

As an example let the equation of a surface be

$$S^2_{,}a_{,}\rho + S_{,}a_{,}b_{,}\rho - S_{,}a_{,}\rho S_{,}b_{,}\rho = C;$$

differentiating

$$2S_{,}a_{,}\rho S_{,}ad_{,}\rho + S_{,}a_{,}bd_{,}\rho - S_{,}a_{,}\rho S_{,}bd_{,}\rho - S_{,}ad_{,}\rho S_{,}b_{,}\rho = 0,$$

or taking out $d_{,}\rho$,

$$S[2_{,}aS_{,}a_{,}\rho + V_{,}a_{,}b - ,aS_{,}b_{,}\rho - ,bS_{,}a_{,}\rho]d_{,}\rho = S\phi(, \rho)d_{,}\rho = S_{,}\nu d_{,}\rho = 0.$$

If the given equation be a vector equation it will be of the form

$$,\rho = ,\varphi(t, u). \quad (7)$$

In this case $D_{,}\rho$ and $D_{u,}\rho$ will be vectors along tangents to the surface at the extremity of $, \rho$, and hence we shall have

$$,\nu = VD_{,}\rho D_{u,}\rho \quad (8)$$

as the value of $,\nu$ to substitute in equation (5).

(7) We will now give several examples illustrative of the preceding formulæ.

Ex. 1. Let ABC be a triangle, and let A', B', C' be the middle points of the sides opposite to A, B and C : then if forces, represented in magnitude and direction by AA', BB', CC' , act on a particle it will be in equilib'm.

Let the vector $CB = a$ and the vector $CA = b$; then the three forces, $AA' = \frac{1}{2}a - b$, $BB' = \frac{1}{2}b - a$ and $CC' = \frac{1}{2}(a + b)$. The sum of these is zero, which proves the proposition.

Ex. 2. Let $\epsilon_1, \epsilon_2, \epsilon_3$ be any three unit vectors, and $P_1\epsilon_1, P_2\epsilon_2, P_3\epsilon_3$ any three forces acting along ϵ_1, ϵ_2 and ϵ_3 : then if they are in equilibrium we have

$$P_1\epsilon_1 + P_2\epsilon_2 + P_3\epsilon_3 = 0. \quad (9)$$

$$\text{Operate by } S.V\epsilon_1\epsilon_2, \therefore S\epsilon_1\epsilon_2\epsilon_3 = 0. \quad (10)$$

Hence the three forces must be coplanar. Next operate by $V\epsilon_1$, therefore $P_2V\epsilon_1\epsilon_2 + P_3V\epsilon_1\epsilon_3 = 0$, or $P_2V\epsilon_1\epsilon_2 = P_3V\epsilon_1\epsilon_3$. But since $\epsilon_1, \epsilon_2, \epsilon_3$ are coplanar, $UV\epsilon_1\epsilon_2 = UV\epsilon_1\epsilon_3$, hence $P_2TV\epsilon_1\epsilon_2 = P_3TV\epsilon_1\epsilon_3$; as a similar relation can be found between P_1 and P_2 , we have the equations

$$\frac{P_1}{TV\epsilon_2\epsilon_3} = \frac{P_2}{TV\epsilon_3\epsilon_1} = \frac{P_3}{TV\epsilon_1\epsilon_2} = \frac{P_1}{\sin \alpha} = \frac{P_2}{\sin \beta} = \frac{P_3}{\sin \gamma}, \quad (11)$$

if α, β, γ are the angles between ϵ_2 and ϵ_3, ϵ_3 and ϵ_1, ϵ_1 and ϵ_2 respectively.

Ex. 3. Let $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4$ be any four unit vectors, and $P_1\epsilon_1, P_2\epsilon_2, P_3\epsilon_3, P_4\epsilon_4$ any four forces along these vectors: then for equilibrium;

$$P_1\epsilon_1 + P_2\epsilon_2 + P_3\epsilon_3 + P_4\epsilon_4 = 0. \quad (12)$$

Operate by $S.V\epsilon_3\epsilon_4, \therefore P_1S\epsilon_3\epsilon_4\epsilon_1 + P_2S\epsilon_3\epsilon_4\epsilon_2 = 0$, or

$$\frac{P_1}{S\epsilon_3\epsilon_4\epsilon_1} = -\frac{P_2}{S\epsilon_3\epsilon_4\epsilon_2}.$$

Operating by $S.V\epsilon_4\epsilon_1$ and $S.V\epsilon_1\epsilon_2$, we obtain

$$\frac{P_1}{S\epsilon_2\epsilon_3\epsilon_4} = \frac{-P_2}{S\epsilon_3\epsilon_4\epsilon_1} = \frac{P_3}{S\epsilon_4\epsilon_1\epsilon_2} = \frac{-P_4}{S\epsilon_1\epsilon_2\epsilon_3}. \quad (13)$$

These scalars are respectively proportional to the volumes of the pyramids whose edges are the vectors involved. We have also $S\epsilon_1\epsilon_2\epsilon_3 \sin \alpha_4 \times \cos \beta_4$, if α_4 is the angle between ϵ_2 and ϵ_3 , and β_4 that between ϵ_1 and $V\epsilon_2\epsilon_3$, and so for the others; whence

$$\frac{P_1}{\sin \alpha_1 \cos \beta_1} = \frac{-P_2}{\sin \alpha_2 \cos \beta_2} = \frac{P_3}{\sin \alpha_3 \cos \beta_3} = \frac{-P_4}{\sin \alpha_4 \cos \beta_4}. \quad (14)$$

Equations (13) and (14) are analogous to equation (11).

Ex. 4. Through a point within a sphere three chords are drawn at right angles; forces act upon this point, outwards, represented in magnitude and direction by the portions of the chords between the point and spherical surface: find the resultant.

Let the equation of the sphere be $T\rho = a$, or $\rho^2 = -a^2$, let i, j, k be unit vectors parallel to the chords, and let δ be the vector from the center of the sphere to the point. The equations of the three chords will be

$$\rho = \delta + xi, \quad \rho = \delta + yj, \quad \rho = \delta + zk.$$

To find the vectors of the extremities of the chords, eliminate ρ between the equations of the chords and sphere. Hence

$$\delta^2 + 2xSi\delta - x^2 = -a^2, \therefore x = Si\delta \pm \sqrt{(S^2i\delta + \delta^2 + a^2)},$$

$$\therefore P_1 = i[Si\delta + \sqrt{(S^2i\delta + \delta^2 + a^2)}]$$

$$\text{and } P_2 = i[Si\delta - \sqrt{(S^2i\delta + \delta^2 + a^2)}].$$

By substituting j for i we obtain P_3 and P_4 , and by substituting k we have P_5 and P_6 . Therefore

$$\Sigma(P) = 2(iSi\delta + jSj\delta + kSk\delta) = -2\delta = R.$$

Ex. 5. Find a point on the surface

$$\frac{S^2i\rho}{a^3} + \frac{S^2j\rho}{b^3} + \frac{S^2k\rho}{c^3} + 1 = 0$$

where a particle attracted by a force to the origin will remain at rest.

By differentiation we find

$$\nu = \frac{iS^2i\rho}{a^3} + \frac{jS^2j\rho}{b^3} + \frac{kS^2k\rho}{c^3},$$

and by eq. (5) $V_\nu R = -nV_\nu\rho = 0$, since R acts along ρ towards the origin. This equation is equivalent to

$$\nu = \frac{iS^2i\rho}{a^3} + \frac{jS^2j\rho}{b^3} + \frac{kS^2k\rho}{c^3} = m_\rho.$$

Operating successively by $S.i$, $S.j$ and $S.k$, and also by $S.\rho$, we find

$$\frac{Si\rho}{a^3} = \frac{Sj\rho}{b^3} = \frac{Sk\rho}{c^3} = \frac{1}{\rho^2} = \frac{-1}{\sqrt{(a^6 + b^6 + c^6)}}.$$

(8). *Moment.*—When a force acts on a particle it always produces, or tends to produce, motion in its own direction. If it be opposed by an equal force in opposite direction, the effect is neutralized.

Suppose now two equal forces to act in opposite directions on different points of a rigid body, say at the ends of a uniform bar, as in the Fig., then as before each force will tend to move its point of application straight forward in its own direction, and hence the resulting effect must be a tendency to turn the bar about its middle point. If the bar is free to move, ac-



tual rotation will be produced. If it is fastened, motion will be prevented by a pair of equal forces similar to those shown but producing an equal tendency to turn in the opposite direction. Such a pair of forces as this is called a *couple*, and all rotations are produced by the action of couples.

The tendency toward rotation of a couple is evidently proportional to the forces acting, and to the perpendicular distance between them, and hence to the product of these quantities. Thus if ρ and ρ' are the forces of the couple and ρ and ρ' vectors from some fixed point to any point in the line of direction of ρ and ρ' ; then the effect of the couple is measured by

$$M = V(\rho - \rho'), P. \quad (15)$$

This quantity is called the *moment* of the couple. When the moment of a force about a point is spoken of, it is tacitly understood that two equal and opposite forces are inserted at the point each equal and parallel to the given force, one of which forms a couple with the given force, while the other tends to produce motion of translation at the point the same in amount that the given force would produce if acting there.

Since $M = V(\rho - \rho'), P = V\rho, P - V\rho', P$, it is evident that the moment of the couple is equal to the algebraic sum of the moments of the component forces about any point. It is also plain that the moment is the same about all points, since $\rho - \rho'$ is independent of the position of the fixed point O .

We see that the moment is a vector perpendicular to the plane of the couple, i. e., parallel to the axis of the couple, whose tensor is $PT(\rho - \rho') \times \sin \theta$, if θ is the angle between $\rho - \rho'$ and P . Thus it appears that any number of couples may be combined by adding the vectors which represent them. Hence

$$\Sigma(M) = \Sigma[V(\rho - \rho'), P] = G = \text{resultant couple:}$$

and the condition of equilibrium when a number of couples act upon a body is

$$\Sigma(M) = \Sigma[V(\rho - \rho'), P] = G = 0. \quad (16)$$

Let ρ be the vector of a particle on which any forces, P_1, P_2 , &c. act: then since $V\rho, R = V\rho, \Sigma(P) = \Sigma(V\rho, P)$, it follows that the moment of the resultant about any point is equal to the sum of the moments of the forces.

If the origin be at a point of R , the vector ρ will coincide in direction with R , so that $V\rho, R = 0$: hence the moment about any point of the resultant is zero.

(9). *Parallel Forces*.—Let ϵ be a unit vector in any direction; then a system of parallel forces, P_1, P_2 , etc., may be written $P_1\epsilon, P_2\epsilon$, etc.

For equilibrium we must have

$$\Sigma(P) = \Sigma(P\epsilon) = \epsilon\Sigma(P) = 0, \text{ i. e., } \Sigma(P) = 0; \quad (17)$$

and $\Sigma(V_{,i}\rho_i P) = V\Sigma(P_{,i}\rho)\epsilon = 0$, or since $\Sigma(P_{,i}\rho)$ is not parallel to ϵ ,
 $\Sigma(P_{,i}\rho) = 0$. (18)

It has been shown in the last article that the moment of the resultant of a system of forces acting at a point is equal to the sum of the moments of the forces. This is equally true of a system of parallel forces; for let some point of each force in the first system be fixed and then let the common point of the forces move off in some direction to infinity. The law holds up to the limit, and then the forces have become parallel. Hence we have, if $_{,i}\rho_0$ is a vector to some point of the resultant,

$$\begin{aligned} V_{,i}\rho_0 R &= V\Sigma(P_{,i}\rho)\epsilon = R V_{,i}\rho_0 \epsilon, \text{ or} \\ V[R_{,i}\rho_0 - \Sigma(P_{,i}\rho)]\epsilon &= 0. \end{aligned} \quad (19)$$

Now, as ϵ is not in general parallel to the vector in parentheses, we must have $R_{,i}\rho_0 = \Sigma(P_{,i}\rho)$ or $_{,i}\rho_0 = \Sigma(P_{,i}\rho) \div R$; (20)

thus we have determined the position of the resultant. As eq. (20) is independent of ϵ , it is plain that the value of $_{,i}\rho_0$ remains unchanged whatever direction the system of parallel forces may have, so long as $_{,i}P_1, _{,i}P_2$, etc., pass through the extremities of $_{,i}\rho_1, _{,i}\rho_2$, etc., the latter being supposed const.

Thus if w_1, w_2 , etc. be the weights of a system of heavy particles at the extremities of $_{,i}\rho_1, _{,i}\rho_2$ etc.; then the vector of the center of gravity of the system will be

$$_{,i}\rho_0 = \frac{\Sigma(w_{,i}\rho)}{\Sigma(w)}. \quad (21)$$

If in (20) $R = \Sigma(P) = 0$, then $_{,i}\rho_0 = \infty$, i. e., when the system reduces to a couple the resultant is a force $= 0$ applied at ∞ . If $\Sigma(P_{,i}\rho) = 0$, $_{,i}\rho_0 = 0$, and the origin is at a point of the resultant, which agrees with the last statement of Art. 8, since if $\Sigma(P_{,i}\rho) = 0$ the moment about the origin is nothing by eq. (18). In case of equilibrium $\Sigma(P_{,i}\rho) = R = 0$, and $_{,i}\rho_0$ becomes indeterminate.

(10). *Resultant of any system of forces acting on a rigid body.*—Let π be the vector of some fixed point, and, as before, $_{,i}\rho_1, _{,i}\rho_2$ etc., vectors from the origin to points upon the lines of action of $_{,i}P_1, _{,i}P_2$ etc. Suppose that at the extremity of π there act the forces $_{,i}P_1$ and $-_{,i}P_1, _{,i}P_2$ and $-_{,i}P_2$ etc.: these will balance each other, and therefore will not affect the system. Now $_{,i}P_1$ at $_{,i}\rho$ will form a couple with $-_{,i}P_1$ at π whose moment will be $_{,i}M = V(_{,i}\rho_1 - \pi)_{,i}P_1$, while there will remain the force $_{,i}P_1$ acting at π , and the same statement holds for each of the other forces. Hence the whole system of forces will be reduced to a single force $_{,i}R = \Sigma(_{,i}P)$ acting at π , and a couple $_{,i}G = \Sigma(_{,i}M) = \Sigma[V(_{,i}\rho - \pi)_{,i}P] = V\Sigma[(_{,i}\rho - \pi)_{,i}P]$
 $= V\Sigma(_{,i}\rho, P) - V\pi\Sigma(_{,i}P) = V\Sigma(_{,i}\rho, P) - V\pi_{,i}R.$

The quantity $V\Sigma(\rho, P)$ being independent of π will be the same for all values of π so long as the position of the origin is unchanged. Let us call it G_0 , so that $G = G_0 - V\pi, R$.

The general conditions of equilibrium then are

$$R = \Sigma(P) = 0, \quad (22)$$

$$G = \Sigma(M) = \Sigma[V(\rho - \pi), P] = G_0 - V\pi, R = 0. \quad (23)$$

But when $R = 0$, $G = G_0$, so that (23) becomes $G = G_0 = V\Sigma(\rho, P) = 0. \quad (23_a)$

(11). Before proceeding to develop certain conditions and relations from the preceding formulæ we will give a few examples illustrative of their use.

Ex. 1. Show that if a given system of forces be reduced in any manner to two forces which do not meet and are not parallel, the volume of the tetrahedron of which these two forces form opposite vector edges is constant.

Let the two forces be P and P' , and let ρ and ρ' be vectors to points in their respective lines of action. Then since we have a given system

$$P + P' = R = \text{a constant vector},$$

and $V\rho, P + V\rho', P' = G = \text{a constant vector}.$

Operate by S, P and S, P' , $\therefore S, P, \rho, P' = S, P, G$, and $S, P', \rho, P = S, P', G$. Adding these two equations we have

$$S, P, P'(\rho - \rho') = S, P + P', G = S, R, G = \text{constant},$$

which proves the proposition.

Ex. 2. Show that if forces act at the middle points of the sides of any plane polygon, the forces being perpendicular and proportional to the respective sides, and directed either all inward or all outward, they will be in equilibrium.

Let the sides of the polygon taken around in order be a_1, a_2 , etc., and let ϵ be a unit vector perpendicular to the plane of the polygon. Then the forces may be taken as $\epsilon, a_1, \epsilon, a_2$ etc. Now as

$$a_1 + a_2 + \dots + a_n = \Sigma(a) = 0,$$

it follows that

$$\Sigma(P) = \epsilon \Sigma(a) = 0.$$

Next taking moments about O we have $\pi = 0$ in eq. (23), and therefore

$$G = \Sigma(V\rho, P) = \frac{1}{2}V, a_1 \epsilon, a_1 + V, (a_1 + \frac{1}{2}a_2) \epsilon, a_2 + \dots \\ \dots + V, (a_1 + a_2 + a_3 + \dots + a_{n-1} + \frac{1}{2}a_n) \epsilon, a_n.$$

But $V, a_1 \epsilon, a_1 = -\epsilon, a_1^2$, etc., so that

$$\begin{aligned} -2G &= \epsilon, a_1^2 + \epsilon, a_2^2 + \dots + \epsilon, a_n^2 + 2\epsilon(S, a_1, a_2 + S, a_1, a_3 + \dots + S, a_1, a_n) \\ &\quad + 2\epsilon(S, a_2, a_3 + S, a_2, a_4 + \dots + S, a_2, a_n) + \dots + 2\epsilon S, a_{n-1}, a_n \\ &= \epsilon[\Sigma(a)]^2 = 0. \end{aligned}$$

A GENERAL SOLUTION OF PROBLEM 292.

BY CHAS. H. KUMMELL, U. S. LAKE SURVEY, DETROIT, MICH.

Two surveyors measure a plane field of n sides, one measuring the courses (1.2), (2.3), . . . (n-1.n), (n.1) with a chain, and the other measuring the bearings $z_{1.2}, z_{2.3}, \dots z_{n-1.n}, z_{n.1}$ with a theodolite. From former experience it is known that the first is liable to a probable error of c inches per chain and the other to w'' per angle. Required, normal equations for determining the most probable corrections to the measured quantities.

Denoting the angles by (1), (2), . . . (n-1), (n) and corrections to observed quantities by prefixing δ we have

$$z_{1.2} + \delta z_{1.2} = \pi + z_{n.1} + \delta z_{n.1} - (1) - \delta(1), (1_1)$$

$$z_{2.3} + \delta z_{2.3} = 2\pi + z_{n.1} + \delta z_{n.1} - (1) - \delta(1) \\ - (2) - \delta(2), (1_2)$$

.

$$z_{n-1.n} + \delta z_{n-1.n} = (n-1)\pi + z_{n.1} + \delta z_{n.1} - (1) - \\ \delta(1) - (2) - \delta(2) - \dots - (n-1) - \delta(n-1), (1_{n-1})$$

$$2\pi + z_{n.1} + \delta z_{n.1} = n\pi + z_{n.1} + \delta z_{n.1} - (1) - \delta(1) - (2) - \delta(2) - \dots - (n) - \delta(n), (1_n)$$

Placing in (1_n) $(n-2)\pi - (1) - (2) - \dots - (n) = E$, we have

$$\delta(1) + \delta(2) + \dots + \delta(n) - E = 0 \quad (I)$$

We have the sum of the Northings and Southings

$$[(1.2) + \delta(1.2)] \cos(z_{1.2} + \delta z_{1.2}) + [(2.3) + \delta(2.3)] \cos(z_{2.3} + \delta z_{2.3}) + \dots \\ + [(n.1) + \delta(n.1)] \cos(z_{n.1} + \delta z_{n.1}) = 0.$$

This condition is true whatever $z_{n.1}$ is. The same is the case with the condition of Eastings and Westings. There are therefore no conditions to determine its correction, that is we cannot orient the figure without another determination of absolute azimuth.

We have, regarding (1)

$$[(1.2) + \delta(1.2)] \cos[z_{1.2} + \delta z_{1.2} - (1)] + [(2.3) + \delta(2.3)] \cos[z_{2.3} + \delta z_{2.3} - \delta(1) - \delta(2)] + \dots \\ + [(n-1.n) + \delta(n-1.n)] \cos[z_{n-1.n} + \delta z_{n-1} - \delta(1) - \delta(2) - \dots - \delta(n-1)] + [(n.1) + \delta(n.1)] \cos(z_{n.1} + \delta z_{n.1}) = \cos z_{1.2} \delta(1.2) + \cos z_{2.3} \delta(2.3) \\ + \dots + \cos z_{n-1.n} \delta(n-1.n) + \cos z_{n.1} \delta(n.1) + W_{1.n} \sin 1'' \delta(1) + W_{2.n} \sin 1'' \delta(2) \\ + \dots + W_{n-1.n} \sin 1'' \delta(n-1) - W_{1.n+1} \sin 1'' \delta z_{n.1} + S_{1.n+1} = 0, \quad (II)$$

where $W_{r.n}$ denotes the projection of the diagonal (r.n) on the East and West axis and $S_{r.n}$ that on the North and South axis.



From the condition that the corrected eastings and westings must balance we have similarly

$$\sin z_{1.2} \delta(1.2) + \sin z_{2.3} \delta(2.3) + \dots + \sin z_{n-1.n} \delta(n-1.n) + \sin z_{n.1} \delta(n.1) - S_{1.n} \sin 1'' \delta(1) - S_{2.n} \sin 1'' \delta(2) - \dots - S_{n-1.n} \sin 1'' \delta(n-1) + S_{1.n+1} \sin 1'' \times \delta z_{n.1} + W_{1.n+1} = 0. \quad (\text{III})$$

Because the coefficients of $\delta z_{n.1}$, viz., $W_{1.n+1}$ and $S_{1.n+1}$ must be nearly = 0, these terms may be omitted, being small quantities of the second order.

If (1.2) is given in inches there are $\gamma \frac{1}{2} (1.2)$ or (putting $\gamma \frac{1}{2} = m$) $m(1.2)$ chains in it; it is therefore liable to a probable error = $c\sqrt{m(1.2)}$ and has the absolute weight = $792 + (1.2)c^2$. Similarly the other sides.

The weights of the angles (1), (2), ... (n) are, on the same scale = $1 + w$.

We have then the following minimum equation:

$$\frac{792}{c^2} \left\{ \frac{\delta(1.2)^2}{(1.2)} + \frac{\delta(2.3)^2}{(2.3)} + \dots + \frac{\delta(n.1)^2}{(n.1)} \right\} + \frac{1}{w^2} \left\{ \delta(1)^2 + \delta(2)^2 + \dots + \delta(n)^2 \right\} - 2[A^*(1) + A(2) + \dots + A(n) - E]C_1 - 2[\cos z_{1.2} A(1.2) + \cos z_{2.3} A(2.3) + \dots + \cos z_{n-1.n} A(n-1.n) + \cos z_{n.1} A(n.1) + W_{1.n} \sin 1'' A(1) + W_{2.n} \sin 1'' A(2) + \dots + W_{n-1.n} \sin 1'' A(n-1) + S_{1.n+1}]C_2 - 2[\sin z_{1.2} A(1.2) + \sin z_{2.3} A(2.3) + \dots + \sin z_{n-1.n} A(n-1.n) + \sin z_{n.1} A(n.1) - S_{1.n} \sin 1'' A(1) - S_{2.n} \sin 1'' A(2) - \dots - S_{n-1.n} \sin 1'' A(n-1) + W_{1.n+1}]C_3, \quad (2)$$

where C_1, C_2, C_3 are undetermined factors or correlates.

Taking partial differential coefficients and solving for the corrections we obtain:

$$A(1) = w^2 [C_1 + W_{1.n} \sin 1'' C_2 - S_{1.n} \sin 1'' C_3], \quad (3_1)$$

$$A(2) = w^2 [C_1 + W_{2.n} \sin 1'' C_2 - S_{2.n} \sin 1'' C_3], \quad (3_2)$$

$$\dots \dots \dots A(n-1) = w^2 [C_1 + W_{n-1.n} \sin 1'' C_2 - S_{n-1.n} \sin 1'' C_3], \quad (3_{n-1})$$

$$A(n) = w^2 [C_1], \quad (3_n)$$

$$A(1.2) = mc^2(1.2) [\cos z_{1.2} C_2 + \sin z_{1.2} C_3], \quad (4_1)$$

$$A(2.3) = mc^2(2.3) [\cos z_{2.3} C_2 + \sin z_{2.3} C_3], \quad (4_2)$$

$$\dots \dots \dots A(n-1.n) = mc^2(n-1.n) [\cos z_{n-1.n} C_2 + \sin z_{n-1.n} C_3], \quad (4_{n-1})$$

$$A(n.1) = mc^2(n.1) [\cos z_{n.1} C_2 + \sin z_{n.1} C_3], \quad (4_n)$$

Substituting these expressions into (I), (II) and (III) we obtain the normal equations:

$$0 = -E + nw^2 C_1 + w^2 \sin 1'' [W_{r.n}] C_2 - w^2 \sin 1'' [S_{r.n}] C_3$$

$$0 = S_{1.n+1} + w^2 \sin 1'' [W_{r.n}] C_1 + \{ w^2 \sin^2 1'' [W_{r.n}^2] + mc^2 [(r.r+1) \cos^2 z_{r.r+1}] \} C_2 - \{ w^2 \sin 1'' [W_{r.n} S_{r.n}] - mc^2 [(r.r+1) \sin z_{r.r+1} \cos z_{r.r+1}] \} C_3,$$

*Used to represent δ .

$$0 = W_{1,r+1} - w^2 \sin 1'' [S_{r,n}] C_1 - \{ w^2 \sin^2 1'' [W_{r,n} S_{r,n}] - mc^2 [(r,r+1) \times \sin z_{r,r+1} \cos z_{r,r+1}] \} C_2 + \{ w^2 \sin 1'' [S_{r,n}] + mc^2 [(r,r+1) \sin^2 z_{r,r+1}] \} C_3. \quad (5)$$

Having solved these equations, formulæ (3) and (4) will give the required corrections to the observed quantities. The weighted sum of squares of these corrections may then be computed, either directly or indirectly by the formula

$$\frac{1}{w^2} [d(r)^2] + \frac{792}{\phi^2} \left[\frac{d(r,r+1)^2}{(r,r+1)} \right] = EC_1 - S_{1,r+1} C_2 - W_{1,r+1} C_3. \quad (6)$$

The probable error of weight 1 is then

$$r = 0.6745 \sqrt{\frac{1}{3} (EC_1 - S_{1,r+1} C_2 - W_{1,r+1} C_3)}. \quad (7)$$

The published solution of this problem as given by Mr. Adcock at p. 60 is not correct for several reasons.

1. He adjusts the angles independent of their relation to the sides and then uses the resulting angle of this partial adjustment as absolutely correct in the adjustment for the sides.

2. He considers the sides all equally good although I only said that the measurement of the sides is liable to c inches per chain; therefore if for instance the side a contains a feet it is measured by $(a+66)$ chains, each of which is liable to c inches probable error. Then the sides a, b , etc., are liable to the probable errors

$$c \sqrt{\frac{a}{66}}, \quad c \sqrt{\frac{b}{66}}, \quad \text{etc., respectively.}$$

These are well recognized principles all over the world.

THE NINE-LINE CONIC.

BY MISS CHRISTINE LADD, BALTIMORE, MARYLAND.

THE properties of the nine-line conic may be obtained directly, or by reciprocation from those of the nine-point circle. I shall make use of the latter method. Some of these properties are well known; those which I suppose to have been unnoticed hitherto I shall indicate by *Italics*. Large letters will, as usual, represent points and small letters lines. As two large letters, BC , denote a line, so do two small letters, bc , denote a point.

$ABC \equiv$ triangle. $abc \equiv$ trilateral.

$O \equiv$ centre of circumscribed circle. $o \equiv$ directrix of inscribed conic with arbitrary focus, F .

$O_1, O_2, O_3 \equiv$ middle points of sides.

$o_1, o_2, o_3 \equiv$ third harmonics to pairs of sides; that is, o_1 is fourth harmonic to b, c , and $(bc)F$, &c. The points on them which subtend a right angle at the focus are on the straight line o .

$P_1, P_2, P_3 \equiv$ feet of perpendiculars from vertices on opposite sides.

$p_1, p_2, p_3 \equiv$ lines through each vertex and the point on the opposite side which subtends with it a right angle at the focus.

$P \equiv$ intersection of perpendiculars.

$p \equiv$ line through these three points.

$N \equiv$ centre of nine-pt. circle.

$n \equiv$ directrix of nine-line conic; that is, a conic may be drawn with focus F and tangent to o_1, o_2, o_3 ; to p_1, p_2, p_3 ; and to the third harmonics respectively to p, a ; p, b ; p, c .

$$ON = NP.$$

n passes through the intersection of o and p , and is their third harmonic.

$G \equiv$ centroid (centre of gravity), is on OP .

The lines o_1, o_2, o_3 intersect the sides a, b, c respectively in three points on one straight line, g , which passes through the intersection of o and p .

$I, E_1, E_2, E_3 \equiv$ centres of inscribed and exscribed circles.

The bisectors, external and internal, of the angles which the sides of the triliteral subtend at the focus cut those sides in six points on four right lines, i, e_1, e_2, e_3 . These lines are the directrices of the four conics with focus F which may be circumscribed about $a b c$. (Salmon, *Conic Sections*, p. 367.)

The nine-point circle is tangent to the in- and exscribed circles.

The nine-line conic is tangent to each of the four circumscribed conics.

IA bisects the angle A .

The focal chords of two points on a conic are equally inclined to the line joining to the focus the intersection of the directrix with the line through the two points.

The nine-point circles of the $\triangle s BIC, CIA, AIB$ pass through the points of contact of the circles I, N .

The nine-line conics of the trilaterals bic, cia, aib , are tangent to the one real common tangent to the conics whose directrices are i and n .

The quadrangles $IBC-E_1, E_1BCE_2$, &c., are inscriptible in circles whose centres are on the circumference of the circle O .

The conics with focus F inscribed in the quadrilaterals $ibce_1, e_1bce_2$, &c., have their directrices tangent to the conic inscribed in abc .

P is the radical centre of the circles on *BC*, *CA*, *AB* respectively as diameters.

p is the only real axis of similitude to the three conics with focus *F* tangent to *a* and *b*, *b* and *c*, *c* and *a* respectively, and having for their respective directrices the third harmonics to those pairs of lines.

The feet of perpendiculars from any point *M*, of a circumference on the sides of an inscribed triangle lie on the line of Simson. The line of Simson bisects *MP*.

Lines drawn from the vertices of a triangle to the points on a tangent, *m*, to the inscribed conic which subtend with them a right angle at the focus meet in a point, which we shall call the point of Simson. The point of Simson is on the third harmonic to *m* and *p*.

Lines from the vertices of a triangle to the middle points of corresponding sides of its orthocentric triangle meet in a point.

Through the vertices of the trilateral *p*₁*p*₂*p*₃, draw the third harmonics to its sides. They intersect the corresponding sides of the trilateral *a* *b* *c* in three points on one straight line.

The nine-point circle touches the twelve circles inscribed and exscribed to the triangles *APB*, *BPC*, *CPA*.

The nine line conic touches the twelve conics with focus *F* that may be circumscribed to the trilaterals *a* *p* *b*, *b* *p* *c*, *c* *p* *a*.

The circle *O* touches the sixteen circles inscribed and exscribed to the four triangles *AFB*, *AE₁B*, &c.

The inscribed conic touches the sixteen conics with focus *F* that may be circumscribed to the four trilaterals *a* *f* *b*, *a* *e*₁ *b*, &c.

Mr. Casey gives 1024 remarkable points on the nine-point circle, and M. Catalan has increased largely the number. Each point has, of course, a corresponding remarkable line tangent to the nine-line conic, but they are not so remarkable as to make it worth while to write down any more of them.

Miscellaneous Problems.—(1). Required to sum the series of squares of the Binomial coefficients, or to prove that

$$1 + n^2 + \left(\frac{n(n-1)}{1.2} \right) + \dots = \frac{2n(2n-1) \dots (2n-n+1)}{1.2.3 \dots n}.$$

(2). Required to transform the first of the following series into the second, where *v* may be assumed at pleasure.

$$S = ax + bx^2 + cx^3 + dx^4 + ex^5 + \dots$$

$$S = a \left(\frac{x}{1-vx} \right) - (av-b) \left(\frac{x}{1-vx} \right)^2 + (av^2-2bv+c) \left(\frac{x}{1-vx} \right)^3 - \dots$$

A QUASI PROOF OF THE ARITHMETICAL MEAN.

BY ORMOND STONE.

GIVEN any number of observations made under precisely similar circumstances, for the determination of a single unknown quantity, and let it be assumed that the most probable value which can be obtained from two such observations will be the arithmetical mean of the individual determinations; to prove that the most probable value which can be obtained from any number of such observations is also the arithmetical mean.

Suppose three values a , b , c given by observations made under similar circumstances. By hypothesis,

$$\frac{1}{2}(a + b), \frac{1}{2}(b + c), \frac{1}{2}(c + a)$$

are the most probable values which can be obtained from any two of the observations.

Each of these results may in turn be considered as a direct observation made under more favorable circumstances, from which we can again obtain the most probable value taking them two and two; viz.:

$$\frac{1}{3}(a + 2b + c), \frac{1}{3}(a + b + 2c), \frac{1}{3}(2a + b + c).$$

Combining these in a similar manner, we have

$$\frac{1}{3}(2a + 3b + 3c), \frac{1}{3}(3a + 2b + 3c), \frac{1}{3}(3a + 3b + 2c);$$

or, in general, for any even number n of such combinations,

$$\frac{1}{2} \left[\frac{2^n - 1}{3} a + \left(\frac{2^n - 1}{3} + 1 \right) b + \frac{2^n - 1}{3} c \right] = \frac{1}{3} \left[\left(1 - \frac{1}{2^n} \right) a + \left(1 + \frac{1}{2^{n-1}} \right) b + \left(1 - \frac{1}{2^n} \right) c \right],$$

etc.;

and for any odd number,

$$\frac{1}{2^n} \left[\left(\frac{2^n + 1}{3} - 1 \right) a + \frac{2^n + 1}{3} b + \frac{2^n + 1}{3} c \right] = \frac{1}{3} \left[\left(1 - \frac{1}{2^{n-1}} \right) a + \left(1 + \frac{1}{2^n} \right) b + \left(1 + \frac{1}{2^n} \right) c \right],$$

etc.

If n be infinite the corresponding values will evidently be the best that can be obtained upon the hypothesis which forms the basis of this investigation. For $n = \infty$, however, the three values are each equal to

$$\frac{1}{3}(a + b + c).$$

In a similar manner it can be shown that if the arithmetical mean is the most probable value for any number of observations p , the same will be true for the number $p+1$. Hence, the proposition is true for any number whatever.

CORRESPONDENCE.—EDITOR ANALYST: Permit me to say thro' your columns that I do not wish to be considered responsible for the corrections appended by Prof Wood, p. 128, to his article on quaternions in the last ANALYST. I have heretofore sent, either to Prof. W. directly, or to the Editor, criticisms on what I deemed errors in these papers on quaternions, viz., on the following:

- p. 36, §9.
- p. 67, the last two lines.
- p. 68, the transformation $kk = \frac{j}{i} \cdot \frac{-i}{j} = \frac{-i}{i}$.
- p. 69, the sentence beginning on line 7.
- p. 70, "The eq. $k^2 = -1$, when k has a fixed position, is the equation of a circumference".
- p. 71, line 16. And I would now respectfully call Prof. Wood's attention to the following errors, &c., in the last article:
 - p. 123, two sentences beginning line 11 from bottom.
 - p. 128, "when the direction of rotation of the fraction and its reciprocal are in opposite senses".
 - p. 128, line 6, the reason assigned for the correction.

ALEX. S. CHRISTIE.

U. S. Coast & Geodetic Survey Office.

[As the editor does not profess to be *read* in the science of Quaternions, and as he supposes many of his readers are in the same condition, he has hitherto admitted to the pages of the ANALYST only papers, on that science, of an elementary character and by responsible authors. Mr. Christie's criticisms, above referred to, were therefore, with his permission, sent to Prof. Wood with the hope that any material defect in the papers of Prof. Wood might be corrected without further discussion.

As Mr. Christie's criticisms were not returned to us by Prof. Wood, we are not able to insert them here that our readers might judge whether they have been properly responded to by Prof. Wood, and as the subject is one upon which clear and accurate knowledge is important, we hereby tender to Mr. Christie space in our next issue for an exposition of the passages above cited.—Ed.]

INTRODUCTION TO DIFFERENTIATION.

BY JOHN MACNIE, A. M., EASTON, CONN.

In the identity

$$\frac{r^n - 1}{r - 1} = r^{n-1} + r^{n-2} + \dots + r + 1, \quad (1)$$

since r may have any value, let $r = \frac{a^{\frac{1}{m}}}{b^{\frac{1}{m}}}$; then by substituting this value for

r in (1), multiplying both members by $b^{\frac{n-1}{m}}$, and simplifying, we obtain:

$$\frac{a^{\frac{n}{m}} - b^{\frac{n}{m}}}{\frac{a^{\frac{1}{m}}}{b^{\frac{1}{m}}} - 1} = a^{\frac{n-1}{m}} + a^{\frac{n-2}{m}} b^{\frac{1}{m}} + \dots + a^{\frac{1}{m}} b^{\frac{n-2}{m}} + b^{\frac{n-1}{m}}. \quad (2)$$

A result from which, since m may have any value (± 1 included), can be deduced all the theorems regarding the divisibility of $x^n \pm y^n$.

Dividing both members of (2) by the factor that renders $a^{\frac{1}{m}} - b^{\frac{1}{m}}$ rational, we obtain

$$\frac{a^{\frac{n}{m}} - b^{\frac{n}{m}}}{a - b} = \frac{a^{\frac{n-1}{m}} + a^{\frac{n-2}{m}} b^{\frac{1}{m}} + \dots + a^{\frac{1}{m}} b^{\frac{n-2}{m}} + b^{\frac{n-1}{m}}}{a^{\frac{n-1}{m}} + a^{\frac{n-2}{m}} b^{\frac{1}{m}} + \dots + a^{\frac{1}{m}} b^{\frac{n-2}{m}} + b^{\frac{n-1}{m}}}, \quad (3)$$

since, by (2), $a - b = (a^{\frac{1}{m}} - b^{\frac{1}{m}})(a^{\frac{n-1}{m}} + \dots + b^{\frac{n-1}{m}})$.

In (3), if we suppose $b = a$, then, since in the numerator of the second member we shall have n terms each $= a^{\frac{n-1}{m}}$, and, in the denominator, n terms each $= a^{\frac{n-1}{m}}$ we obtain

$$\left[\frac{a^{\frac{n}{m}} - b^{\frac{n}{m}}}{a - b} \right]_{a=b} = 0 = \frac{na^{\frac{n-1}{m}}}{ma^{\frac{n-1}{m}}} = \frac{n}{m} a^{\frac{n-1}{m}}, \quad (4)$$

the first member assuming the indeterminate form from the presence, in the numerator and denominator, of the factor $a^{\frac{1}{m}} - b^{\frac{1}{m}}$ which becomes zero by the supposition. Hence, since m may have any value, the formula

$$\left[\frac{x^n - y^n}{x - y} \right]_{x=y} = nx^{n-1} \quad (5)$$

holds true for every value of n .

Let $u = x^n$, and $u' = (x+h)^n$, then the ratio of the increase of x^n to that of x is expressed, for all values of h , by

$$\frac{u' - u}{h} = \frac{(x+h)^n - x^n}{(x+h) - x}. \quad (6)$$

If now we suppose $x = x+h$, i. e., $h = 0$, then by (5), we obtain

$$\frac{0}{0} = nx^{n-1}, \quad (7)$$

the first member assuming the indeterminate form, as in (4), from the presence of a factor that has become zero in both its terms. Supposing this common factor to be removed by division, and denoting the result by

$$\frac{dx^n}{dx}, \text{ we have the result } \frac{dx^n}{dx} = nx^{n-1}$$

(where n may have any value), which may be thus defined: *If in the function x^n , x be supposed to increase continuously, then, when x reaches any assigned value x' the rate at which x^n is increasing, as compared with that of x , is nx'^{n-1} , that is, if we denote an infinitesimal change in x by dx , the corresponding change in x^n is $nx'^{n-1}dx$. This last expression is called &c.*

The difficulty which here besets us arises from our inability to conceive an absolute infinitesimal in time or space, and is really identical with that we meet in questions regarding motions. Thus, for example, the velocity attained by a body falling under certain conditions, is expressed by $v = gt$. Now before the end of the t^{th} unit of time $v < gt$, and after the end of the t^{th} unit of time $v > gt$, nor are we able to conceive any portion of time, however minute, during which the body moves with the velocity gt . Yet, as the velocity increases continuously from less than to greater than gt , we are forced to the conclusion that for some infinitesimal portion of time $v = gt$. By performing the division indicated in the second member of (6) we obtain:

$$\frac{u' - u}{h} = (x+h)^{n-1} + (x+h)^{n-2}x + \dots + (x+h)x^{n-2} + x^{n-1}; \quad (8)$$

from which we see that $[(u - u') \div h] > nx^{n-1}$, if h , however conceivably small, be positive, and $[(u - u') \div h] < nx^{n-1}$, if h be negative. Hence, if we conceive x^n to vary continuously we are forced to conclude that, when x reaches any assigned value x' , then, corresponding to any infinitesimal change in x' there is a simultaneous change in x^n that is nx'^{n-1} times as great, which is expressed by the formula

$$\frac{dx^n}{dx} = nx^{n-1}.$$

ON THE PHENOMENA OF THE TIDES.

BY PROF. ELIAS SCHNEIDER, MILTON, PA.

IN Vol. VI, No. 6 of the ANALYST, a problem, No. 281, was proposed for solution. The solution was commenced in No. 2 and completed in No. 3, Vol. 7. The problem was given with the expectation that its solution would throw additional light on a subject, in the treatment of which our books do not, in my opinion, include all the facts relating to it. It is now proposed to show how the solution gives this additional light.

According to the result obtained in the solution, the ball would pass a point 180° from the initial point, nearly 3 hours before the point A will pass the same point in space. If the gravitating power of the sun has such an effect upon a smooth solid body resting on an equally smooth surface of the earth, it must also have a similar effect upon a fluid, resting on the earth's surface. Suppose, then, that instead of a ball situated as stated in the problem, the earth be surrounded by a belt of water, about 2000 m's wide and 5 miles deep. Now suppose the earth to rotate on an axis once in 24 hours, the water and earth being subject to no influence but that of their mutual attraction. Then every particle of this water will be relatively at rest with respect to the earth.

But now suppose the sun to come in to position and to act upon the water at A in the direction of the diameter AE . When, from the rotation of the earth, the diameter AE ceases to point towards the sun the particles of water at A will cease to be in equilibrium, and will move over the earth's surface toward the attracting body; and, according to the solution referred to, the water at A would acquire such an increased velocity as to pass a point 180° from the initial point, nearly 3 hours before the point A will pass the same point in space. Therefore, on account of increased centrifugal force, there will be a tendency in the water to move along a line tangent to the earth's surface at D , causing the water to swell out there. In other words, there will be a tide wave at D . This takes place according to the statement and solution of the problem, under the supposition that the earth has no other motion than rotation upon an axis. It follows therefore, that, if the earth had no other motion than axial rotation, the tide wave would be produced by the same cause that is supposed to move the ball, and that there would be no necessity for seeking the cause of the phenomena of the solar tide in the power of the sun to lift the water above its gen'l level,



and that the occurrence of this tide two or three hours after meridian wo'd not be due to the resistance made by the inertia of the water.

But the earth, in addition to an axial rotation, has also an orbital motion around the sun. Would this motion, if recognized in the prob. 281, change materially the motion of the ball? * * * *

[We proceed to answer Prof. Schneider's question as here proposed; but desire, first, to say that we dissent entirely to his conclusions deduced above.

In entering this protest, it is sufficient to say that the water in the supposed belt is *not* free to move forward as the ball is, because a limited portion of the waters of the sea cannot advance horizontally without first displacing an equal quantity of water, and the entire "belt" cannot move around the earth because the attraction upon the water on *both sides* of the diameter AD is in the same direction. Moreover, *analysis*, which is our only *safe* guide, leads to entirely different results in the two cases.

In reply to the above inquiry, we answer, it would. If we assume in prob. 281 that the earth has also an orbital motion around the sun, the ball, when at the initial point A , will also have an orbital motion, the centrifugal force from which will not only equal the centripetal force F , but will slightly exceed it for all points on the surface more remote from the sun than E (the center of the earth), and therefore the tendency to motion in the ball, while it remains within the first quadrant from the point A , in space, will be backward, or *from* the sun; for the only force that tends to roll the ball is the *difference* between its centripetal and centrifugal forces, which, in this case may be regarded as *negative*, the positive direction being toward the sun.

When, from the motion of the earth on its axis, the ball shall be transferred to a point in space which is nearer the sun than the point E , the difference between the centripetal and centrifugal forces will become *positive* and the ball will tend to move forward or toward the sun.

It follows therefore that matter, which is subject to the attraction of a remote body, and which is free to move horizontally over the earth's surface, will tend to accumulate both *under* and *opposite* the attracting body. This conclusion is in harmony with the theory of the tides, as it should be; for though the waters are *not* free to move horizontally, their equilibrium is continually disturbed by the attracting body, the distortion tending to a like accumulation both under and opposite the attracting body.

Because, in the solution of prob. 281, the earth is supposed to be stationary in its orbit, the centrifugal force developed by the rolling of the ball, being at right angles with its motion, need not be considered; and because the *difference* of the attracting force of the sun, for different positions of the ball, is small in comparison with the attracting force at any particular point,

the force which tends to roll the ball is regarded as constant, which renders the solution comparatively simple.

But in discussing the equilibrium of the sea, subject to a like attraction by the sun while the earth is moving in its orbit, the disturbing force is obviously the *difference* between the centripetal and centrifugal forces at any particular point, and hence is a *variable* force; therefore an equation involving the theory of the tides is much more complicated than that representing the solution of problem 281.

On revising his numerical calculation for the determination of t in the solution of 281, Mr. Kummell finds

$$t_* = 9^h 05^m 42^s.5.$$

Equation (9), p. 88, gives, by expanding in a series and integrating,

$$t_* = \frac{1.188}{2} 12^h \left[1 + \frac{1}{4}e^2 + \frac{9}{64}e^4 + \frac{25}{256}e^6 + \frac{1225}{16382}e^8 + \dots \right].$$

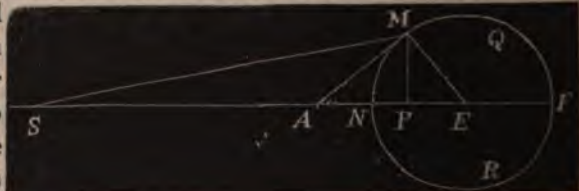
Summing twelve terms of this series we get

$t_* = 9^h 05^m 27^s$; being a little less than the answer obtained from eq. (10) in consequence of the slow convergence of this series.—Ed]

NOTE.—“Proof” of the preceding Art. and comments was sent to Prof. Schneider with the hope that the suppression of the latter part of his paper would be approved by him. As he insists, however, upon the printing of the entire paper and claims that we have omitted “the strongest part”; in order that no injustice be done to the argument of Prof. Schneider, we here insert, without comment, that part of Prof. Schneider’s paper which was omitted above.

* * * One of the computers says that, “the only *effect* of considering the progressive motion of the earth will be to shorten the time (say 2 minutes) of the $\frac{1}{2}$ day required in the problem.” It follows then, that the orbital motion of the earth does not effect materially the result of the solution, and therefore not the application.

The books tell us that “if the waters every where yielded immediately to the attractive force, it would always be high-water when the attractive force is on the meridian. But, owing to the inertia of the water, some time is necessary for so slight a force to set it in motion.” This declaration seems to me contrary to actual facts. I will refer to an illustration in one of our books. “Suppose E to represent the centre of the earth, the circle FQN its



circumf., M a particle of water on the earth's surface, and S the sun. The entire earth being rigid, each part of it will move under the influence of the sun's attraction as if the whole were concentrated at its centre. But the attraction of the sun upon the particle M , being different from its mean attraction on the earth, will tend to make it move differently from the earth. The force which causes this difference of motion, as already explained, will be represented by the line MA . It is true that this same disturbing force is acting on that portion of the solid earth at M as well as upon the water. But the earth cannot yield on account of its rigidity; the water therefore tends to flow along the earth's surface from M towards N . There is therefore a residual force tending to make the water higher at N than at M . Now, is it not evident that the inertia of the water has already been overcome, before it reaches the point N ? Its increased motion has commenced even before it reaches the point M , and this velocity is continually becoming more rapid until it has reached its maximum at N , and by virtue of its inertia it does not *resist* motion but *continues* moving in a line tang't to the earth at N , and thus causes the tide wave 2 or 3 hours after merid'n.

I have made reference to the solar tide only. The lunar tide is produced in the same way, the attracting power being the moon.

PROOF OF A PROPOSITION IN SOLID GEOMETRY.

BY PROF. E. W. HYDE, UNIVERSITY OF CINCINNATI.

It is probable that the proposition given below has been previously stated. I do not remember, however, to have seen it, and it may be of interest to some of the readers of the ANALYST.

If two different surfaces of the same order are tangent to each other at every point in which they are pierced by a given right line, then the surfaces intersect each other in two *plane curves*, real or imaginary.

Let $V = 0$ and $W = 0$ be two planes passing through the given line, and let $U_1 = 0$, $U_2 = 0$, $U_3 = 0$ etc., be any surfaces whatever. Also let $\varphi(U_1, U_2$ etc.) be any rational integral function of U_1, U_2 etc. Then the equations

$$\varphi(U_1, U_2 \text{ etc.}) + \lambda_1 V^2 + \mu_1 W^2 = 0, \quad (1)$$

$$\varphi(U_1, U_2 \text{ etc.}) + \lambda_2 V^2 + \mu_2 W^2 = 0, \quad (2)$$

represent two different surfaces each of which is tangent to $\varphi = 0$ at every point where it is pierced by the intersection of V and W ; for (1) and (2) are each satisfied by such values of x, y, z as make φ, V and W simultaneously equal to zero; also, V and W being squared, the given line is the

intersection of two pairs of coincident planes, i. e., it is a quadruple line, and the points where it pierces φ are quadruple points common to the surfaces (1) and (2). If we subtract (2) from (1) the resulting equation will be that of some surface passing through the intersection of (1) and (2). Thus we have

$$\begin{aligned} &(\lambda_1 - \lambda_2)V^2 + (\mu_1 - \mu_2)W^2 = 0, \\ \text{or} \quad &V\sqrt{(\lambda_1 - \lambda_2)} = \pm W\sqrt{(\mu_2 - \mu_1)}. \end{aligned} \quad (3)$$

Equation (3) represents two real or imaginary planes, according to the signs and relative values of λ_1 , λ_2 and μ_1 , μ_2 . But these planes contain the intersection of (1) and (2), which consists therefore of two plane curves, real or imaginary. Q. E. D.

The particular case of this proposition when $\varphi(U_1, U_2 \text{ etc.}) = U_1 U_2$, and U_1 and U_2 are of the 1st degree, or, in general, when φ is of the 2nd degree, proves directly the property at which I arrived indirectly in an article published in the ANALYST for July, 1876; viz., Two quadrics will intersect each other in plane curves when they have common tangent planes at the extremities of a common chord.

We may also write the reciprocal proposition to this as follows:

If two surfaces of the same class are tangent to each other at the points of contact of all planes that can be drawn tangent to them through a given right line, then they will have two common circumscribing cones; i. e., every common tangent plane of the two surfaces will pass through one or the other of two fixed points on the given line.

SOLUTION OF A PROBLEM.

BY GEORGE EASTWOOD, SAXONVILLE, MASS.

Problem.—Now that there are three tickets before the people, from which to elect our next President, let it be supposed that the votes will be divided between the candidates, on the assumption that the voters will attribute to the three tickets every degree of merit comprised within the limits 0 and 100. Supposing the voters to find in each ticket some—*more* or *less*—of every degree of merit contained in the other two, it is required to assign the mean value of *all* the the merits of the three tickets.

Solution.—That the question may have the utmost generality, let us suppose the number of tickets to be n , and the limits of their merits to be between 0 and a . Suppose $\beta_1, \beta_2, \dots, \beta_r$ represent the respective merits of the n tickets, β_1 being the merit attributed to the ticket which the voter

deems the most worthy, β_2 the next &c., β_r being the merit ascribed to the r th ticket in point of merit.

First, assume that β_1 shall have any positive value comprised within the limits 0 and a , and that the mean value of all such values of β_1 is required. Let v = the number of values taken and $\Sigma\beta_1$ = their sum; then the req'd mean of all the values is

$$M = \Sigma\beta_1 \div v.$$

If we suppose the values of β_1 to increase by equal portions δ , and that they become 0, δ , 2δ a , then

$$v = \Sigma\delta \div \delta, \text{ and } M = \delta\Sigma\beta_1 \div \Sigma\delta.$$

Now because β_1 is supposed to receive all values between 0 and a , the portion δ may be taken indefinitely small, say $= d\beta_1$, and the sums Σ may be replaced by integrals.

$$\therefore M = \int_0^a \beta_1 d\beta_1 \div \int_0^a d\beta_1 = \frac{\frac{1}{2}a^2}{a} = \frac{1}{2}a.$$

Hence the mean value of all these values is equal to the mean of the two extremes.

Similarly, suppose β_1, β_2 to be the merits of two tickets according to the opinion of any independent voter, β_1 being the highest estimate he places on the most worthy ticket; so that β_1 is greater than β_2 . In this case β_2 may have any values between 0 and β_1 , and β_1 any value between $\beta_2 = 0$ and $\beta_2 = \beta_1$. The sum of all the values of β_1 contained between $\beta_2 = 0$ and $\beta_2 = \beta_1$ is equal to

$$\beta_1 \int_0^{\beta_1} d\beta_2 \div d\beta_2,$$

and the sum of all the values of β_1 is equal to

$$\int_0^a (\beta_1 d\beta_1 \int_0^{\beta_1} d\beta_2) \div d\beta_1 d\beta_2.$$

The number of these values is

$$\int_0^a d\beta_1 \int_0^{\beta_1} d\beta_2 \div d\beta_1 d\beta_2,$$

therefore the mean value of β_1 is

$$\int_0^a \int_0^{\beta_1} \beta_1 d\beta_1 d\beta_2 \div \int_0^a \int_0^{\beta_1} d\beta_1 d\beta_2.$$

Thirdly, if there be n tickets and β_r be the merit attributed to the r th ticket, the mean value of β_1 will be

$$\begin{aligned} & \int_0^a \int_0^{\beta_1} \int_0^{\beta_2} \dots \int_0^{\beta_{n-1}} \beta_1 d\beta_1 d\beta_2 \dots d\beta_n \div \int_0^a \int_0^{\beta_1} \int_0^{\beta_2} \dots \int_0^{\beta_{n-1}} d\beta_1 d\beta_2 \dots d\beta_n \\ &= \frac{1}{2} \cdot \frac{1}{3} \dots \frac{1}{n-r} \cdot \frac{1}{1-r+2} \cdot \frac{1}{1-r+3} \dots \frac{1}{n+1} a^{n+1} + \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{n} a^n = \frac{n-r+1}{n+1} a. \end{aligned}$$

If n be an odd No. $\beta_{\frac{n}{2}+1}$ will be the middle term and its mean value will be $\frac{1}{2}a$. If n be an even No. $\beta_{\frac{n}{2}}$ and $\beta_{\frac{n}{2}+1}$ are middle terms; their mean values are $\frac{1}{2}a + \frac{\frac{1}{2}a}{n+1}$ and $\frac{1}{2}a - \frac{\frac{1}{2}a}{n+1}$, and the mean of these values is $\frac{1}{2}a$. The sum of n values is $\frac{1}{2}an$, and hence the mean of all the values is $\frac{1}{2}a$.

Suppose now we take the values of the three tickets, as estimated per question by the independent voters, and that these values are as the Nos. 1, 2, 3. Then, substituting n by 3, a by 100, and r severally by 1, 2, 3, we shall have for the mean value of the most worthy ticket, $\frac{1}{3} \times 100 = 75$,
 for the mean value of the next best, $\frac{1}{3} \times 100 = 50$,
 for the mean value of the third " , $\frac{1}{3} \times 100 = 25$;
 while the *mean* of all the values will be $\frac{1}{3} \times \frac{1}{3} \times 100 = 50$.

SOLUTION OF A PROBLEM.

BY WILLIAM HOOVER, SUPERINTENDENT OF SCHOOLS, WAPAKONETA, O.

Problem.—A triangle is formed by joining the centers of the escribed circles of a triangle; the same is done for the new triangle, and so on *ad infinitum*. Show that the final triangle is equilateral.

Solution.—If A be one of the angles of the given triangle, and A_1, A_2, A_3 , &c., the successive corresponding angles, we have

$$A_1 = (\frac{1}{2})(\pi - A), A_2 = (\frac{1}{2})^2(\pi + A), A_3 = (\frac{1}{2})^3(3\pi - A), A_4 = (\frac{1}{2})^4(5\pi + A).$$

The fraction for the n th angle is $(\frac{1}{2})^n$, the coefficient of π is the general term of the recurring series 1, 1, 3, 5, &c., $\frac{1}{3}[2^n + (-1)^{n-1}]$, and of A , it is $(-1)^n$. Hence

$$A_n = \frac{1}{2^n} \left[\frac{1}{3}\pi[2^n + (-1)^{n-1}] + (-1)^n A \right].$$

When n is infinite $A_n = \frac{1}{3}\pi$, and the triangle is equilateral.

SOLUTION OF PROB. 282 WITH CONDITION (3) CHANGED.

BY CHAS. H. KUMMELL, U. S. LAKE SURVEY.

The solution of this problem (see pp. 25, 26) is in accordance with the given conditions. However it cannot be admitted, as assumed in condition (3), that it is equally probable that only one loom should be out of order as

that any other number should be out of order. I propose to solve the problem by assuming a looms, each netting m dollars a day when in order, x weavers, each getting n dollars a day, whether working or not, and p the probability of a single loom being out of order. We have then to take the general mean of the following cases:

No. of looms in order.	Profits.	Probability.	
a	mx	$(1-p)^a$	$= P_a$
$a-1$	mx	$\frac{a}{1}(1-p)^{a-1} p$	$= P_{a-1}$
$a-2$	mx	$\frac{a}{1} \cdot \frac{a-1}{2} (1-p)^{a-2} p^2$	$= P_{a-2}$
.	.	.	.
x	mx	$\frac{a}{1} \cdot \frac{a-1}{2} \dots \frac{x+1}{a-x} (1-p)^x p^{a-x}$	$= P_x$
$x-1$	$mx-(m+n)$	$\frac{a}{1} \cdot \frac{a-1}{2} \dots \frac{x}{a-x+1} (1-p)^{x-1} p^{a-x+1}$	$= P_{x-1}$
$x-2$	$mx-2(m+n)$	$\frac{a}{1} \cdot \frac{a-1}{2} \dots \frac{x-1}{a-x+2} (1-p)^{x-2} p^{a-x+2}$	$= P_{x-2}$
.	.	.	.
1	$mx-(x-1)(m+n)$	$\frac{a}{1} (1-p) p^{a-1}$	$= P_1$
0	$mx-x(m+n)$	p^a	$= P_0$

We have then the average profit

$$y_x = mx - (m+n) \sum_1^x [s P_{x-s}].$$

If this is a maximum then

$y_x - y_{x-1} = \Delta y_x = m - (m+n) \sum_1^x [P_{x-s}]$ is small and positive,
& $y_{x+1} - y_x = \Delta y_{x+1} = m - (m+n) \sum_1^{x+1} [P_{x+1-s}]$ is small and negative,

Placing $a = 60$, $m = 3$, $n = 1$ and $p = 0.1$ we find $\Delta y_{56} = + 0.021$ and $\Delta y_{57} = - 0.113$. Therefore $x = 56$ weavers.

ANSWERS TO QUERIES IN NUMBER FOUR.

"QUERY BY PROF. A. HALL.—(1). If v be the potential function we have the equation given by Laplace

$$\frac{d^2 v}{dx^2} + \frac{d^2 v}{dy^2} + \frac{d^2 v}{dz^2} = 0,$$

which holds for a point outside the attracting body. For a point inside this body we have the equation given by Poisson,

$$\frac{d^2 v}{dx^2} + \frac{d^2 v}{dy^2} + \frac{d^2 v}{dz^2} = -4\pi\rho.$$

What is the value of the right hand side of this equation for a point on the surface of the attracting body? Moigno says, "in this case the expression will have in reality eight distinct values." Statics, p. 460."

ANSWER BY ORMOND STONE, A. M., DIRECTOR OF CINCINNATI OBSERVATORY.

In order that the expression may have eight values at the surface of the attracting body it is necessary to assume that the differential coefficients

$$\frac{d^2 v}{dx^2}, \quad \frac{d^2 v}{dy^2}, \quad \frac{d^2 v}{dz^2}$$

are independent of one another, an hypothesis which is inadmissible, since each is a function of the same variable

$$r = \sqrt{(\xi-x)^2 + (\eta-y)^2 + (\zeta-z)^2}.$$

The expression has in general only two values, $-4\pi\rho$ and 0, according as the displacement of the point attracted is such as to cause it to enter the attracting mass or not. For if a line be drawn through the given point, then for two points in this line, one within, the other outside the mass, there will be two potentials v_1 and v_2 , each of which will have a value whose limit will be v . The simplest illustration is the case of the sphere, in which we have for a point at the surface $r = R$, and

$$v_1 = 2\pi\rho(R^2 - \frac{1}{3}r^2) = v_2 = \frac{4}{3} \frac{\pi\rho R^3}{r} = v = \frac{4}{3}\pi\rho R^2.$$

(2) "Given a hemispherical dome turning about a pin at the top, and having a slit extending from the horizon to the zenith. Can a telescope be placed inside this dome in such a position that every point of the heavens can be seen through the telescope?"

ANSWER BY PROF. W. P. CASEY, SAN FRANCISCO, CAL.

A telescope may be placed inside the dome as required in various ways; thus, let ABC be a section of the dome, having BC the slit and Bz the axis on which the dome revolves and which supports it, and let this axis be encircled by a cylinder attached to the dome and also supported by the wheels o, x , having a small platform y , upon which a telescope may be placed which will revolve with the dome as required. Or a floor may be attached to the dome and be supported upon wheels.



QUERY BY T. W. WRIGHT, U. S. LAKE SURV., DETROIT, MICH.—“Can a satisfactory proof be given of the statement that “the weight of an observation varies as the square of the probability of its accuracy”? (Monthly Notices R. A. S., Vol. XIII.)”

ANSWER BY CHAS. H. KUMMELL, U. S. LAKE SURVEY.

The proof, at the place cited rests essentially on the incorrect expression for the probability of an error x , viz.,

$$\varphi(x) = \frac{h}{\sqrt{\pi}} e^{-h^2 x^2},$$

which is so given in most treatises on the method of Least Squares (some modern authors excepted) since Gauss first gave it in “Theoria Motus” in this form. It is clear that such an expression is approximately proportional to h , the precision, and therefore to the square root of the weight. Hence the pretended proof. The correct expression for the probability of an error x (see for ex. ANALYST, Vol. III, p. 135 et seq.) is however:

$$\varphi(x) = \frac{h dx}{\sqrt{\pi}} e^{-h^2 x^2},$$

and since as I have proved l. c., the factor

$$\frac{h dx}{\sqrt{\pi}} = \frac{dx}{\varepsilon \sqrt{2\pi}} = \varphi(0) = \frac{1}{\sqrt{(2\pi)}}$$

is an absolutely constant infinitesimal, being the same for all classes of observations, the fallacy of this proof is at once apparent. The very elegant writer of the Monthly Notices, J. W. L. Glaisher, F. R. S., proposed this method of reweighting observations of the same quantity, of which, for a first approximation, the arithmetic mean has been taken. The propriety of such a process (first suggested by Prof. De Morgan), provided weights could be used which are derived from correct principles, I do not altogether question. At present I have several plans of reweighting, each of which has some reason in its favor. One consists in weighting the observations by their respective probabilities or what is the same, by the finite variable factor $e^{-\Delta^2 \div 2\varepsilon^2}$ ($d\Delta \div \varepsilon \sqrt{2\pi}$ being an absolutely constant infinitesimal, ANALYST, Vol. III, p. 135). These numbers, used as weights, are proportionate to the number of times each of the errors $\Delta_1, \Delta_2, \dots \Delta_m$ should have occurred if there had been an infinite number of observations of the same precision. If the observations have different weights $p_1, p_2, \dots p_m$ we compute the second weights $p'_1, p'_2, \dots p'_m$ by the same formula, ε denoting mean error of an observation of weight 1. This process is theoretically to be repeated until the weights do not change any more and the mean value reaches a certain limit.—[No answer to Prof. Johnson's Query.]

SOLUTIONS OF PROBLEMS IN NUMBER FOUR.

SOLUTIONS of problems in No. 4 have been received as follows:

From R. J. Adcock, 315; David Barrow, Jr., 319; Prof. W. P. Casey, 313, 319; George Eastwood, 313, 314, 316, 319; William Hoover, 317; G. H. Harvill, 318; Chas. H. Kummell, 315; Prof. E. B. Seitz, 317, 318; Prof. J. Scheffer, 317, 319.

313. "A piston, weight w , is dropped into the end of a vertical cylinder filled with air, length l ; how far will the piston descend, assuming no friction nor escape of air, nor heat from the compressed air?"

SOLUTION BY GEORGE EASTWOOD, SAXONVILLE, MASS.

Let r = radius of cylinder, l = its length, w = weight of the piston, ϵ = elasticity of *any* fluid, with which the cylinder may be filled, and filling a given length h . Then, for any variable length x , of the cylinder, the elasticity for the space $\pi r^2 x$ is $\epsilon \times (h+x)$; the moving force upon the piston is $w - \epsilon \times (h+x)$ and the accelerating force is

$$F = 1 - \frac{\epsilon}{w} \cdot \frac{h}{x}.$$

Now from the formula for accelerating forces we have

$$\int v dv = -\int F dx,$$

from which we deduce

$$v^2 = 2 \left[\frac{\epsilon h}{w} \cdot \log x - x \right] + C = \left[\frac{\epsilon h}{w} \cdot \log \frac{x}{h} + h - x \right].$$

Let s = space through which the piston descends to acquire the velocity v , and let $x = h - s$. Then

$$v^2 = 2 \left[\frac{\epsilon h}{w} \cdot \log \frac{h-s}{h} + s \right].$$

Suppose now the fluid to be atmosphere with an elastic pressure of 15lbs upon the square inch; we shall have in this case,

$$\epsilon = \pi r^2 \times 15 \text{ lbs.}$$

Adding to w the atmospheric pressure upon it, we get

$$v^2 = 2 \left[\frac{15\pi r^2 h}{15\pi r^2 + w} \cdot \log \frac{h-s}{h} + s \right],$$

from which to determine s . In this case we may write l for h ; and because when the accelerating force becomes zero the descent of the piston ceases,

$$F = 1 - \frac{15\pi r^2 l}{(15\pi r^2 + w)x} = 0. \quad \text{Hence } x = \frac{15\pi r^2 l}{15\pi r^2 + w}.$$

314. "A helix is coiled about a right cylinder the radius of whose base is 1. The projection of the evolute of the helix on the plane of the base of the cylinder encloses double the area of that base. Required the angle made with the plane of the base by the tangent to the helix."

SOLUTION BY EASTWOOD.

The evolute of the helix is the contour of the osculating surface of the helix. If the projection of the contour of this surface be required on the plane of xy , it will be found to be the involute of the projection of the helix on the base of the cylinder. (De Morgan's Calculus, p. 416.) For a general solution, let a = radius of the cylinder, h = step of the helix, r, θ , the coordinates of any point of the involute, and p the perpendicular on the tangent. We have for the polar equation

$$a\theta = \sqrt{r^2 - a^2} - a \cos^{-1}(a/r);$$

and for the area

$$\begin{aligned} A &= \frac{1}{2} \int r^2 d\theta = \frac{1}{2} \int \frac{pr dr}{(r^2 - p^2)^{3/2}} = \frac{1}{2a} \int (r^2 - a^2)^{1/2} r dr, \\ &= \frac{4}{3} \pi^3 a^3, \end{aligned}$$

where p is taken between the limits $p = 0$, and $p = 2\pi a$.

When $a = 1$, this result gives $\frac{4}{3}\pi^3$ for the area enclosed by the projection of the evolute of the helix on the base of the cylinder, which differs from that specified in the enunciation of the question.

The angle which the tangent to the helix makes with the plane of xy is found from the eq'n $\tan \phi = h/2\pi a$, whatever value may be given to a .

315. "Two marksmen compete for a prize. Three shots are to be fired by each. The distance of each shot hole from the bull's eye is measured and he is the winner whose total distance is least. A 's three shots measured respectively 1, 2, 3 in. from the bull's eye. B then made three shots and the bull's eye fell off leaving no mark by which it could be replaced.

An examination showed that the coordinates of B 's shot holes, referred to an origin arbitrarily taken, were (0, 0) in. (1, 1) in. and (2, 3) inches.

Which of the marksmen was the probable winner?"

SOLUTION BY R. J. ADCOCK, ROSEVILLE, ILL.

Let x and y be the coordinates of the bull's eye with reference to the arbitrary origin, and $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ those of A 's shots. Then its most probable position is that which makes

$$\begin{aligned} x^2 + y^2 + (x-1)^2 + (y-1)^2 + (x-2)^2 + (y-3)^2 + (x-x_1)^2 + (y-y_1)^2 \\ + (x-x_2)^2 + (y-y_2)^2 + (x-x_3)^2 + (y-y_3)^2 = \text{a min.}, \end{aligned}$$

that is, $x^2 + y^2 + (x-1)^2 + (y-1)^2 + (x-2)^2 + (y-3)^2 + 1 + 4 + 9 = a \text{ min.}$
 $\therefore x = 1, y = \frac{4}{3}$, and the sum of B 's distances is 3.96 in., and as the sum of A 's distances is 6, $\therefore B$ is the probable winner.

[Mr. Kummell's solution of this problem is too extended for our space in this No. but will appear in a future No.]

316. "Required the value of

$$\frac{\log x \sqrt{1-a^2x^2}}{\sqrt{1-x^2}}, \text{ where } x=1, \text{ and } a = \log(1-x)."$$

SOLUTION BY EASTWOOD.

Put $\sqrt{1-a^2x^2} = \sqrt{1+ax}\sqrt{1-ax} = u$, and $\sqrt{1-x^2} = v$. We shall find

$$\begin{aligned} \frac{du}{dx} = \frac{\sqrt{1-ax}}{2\sqrt{1+ax}} \log x \left[\log a - \frac{x}{1-x} \right] + \frac{\sqrt{1+ax}}{2\sqrt{1-ax}} \log x \left[\frac{x}{1-x} - \log a \right] \\ + \frac{\sqrt{1+ax}\sqrt{1-ax}}{x}, \end{aligned}$$

$$\frac{dv}{dx} = -\frac{x}{\sqrt{1-x^2}}.$$

When $x=1, u=0, v=0, du \div dx = 0$, since $\log x = 0, dv \div dx = \infty$.

Hence $\left[\frac{u}{v} \right] = \frac{0}{0}, \left[\frac{du}{dv} \right] = \frac{0}{\infty} = 0$, and so on for higher diff. coefficients.

317. "If the scale of relation of a recurring series be $a_n - 7a_{n-1} + 12a_{n-2} = 0$, and if $u_0 = 2, u_1 = 7$, find u_n , and the sum of $u_0 + u_1 + \dots + u_{n-1}$."

SOLUTION BY WILLIAM HOOVER, WAPAKONETA, O.

The given scale of relation is an equation in Finite Differences. Integrating and denoting by u_n the general term,

$$u_n = C_1(4)^n + C_2(3)^n. \quad (1)$$

When $n=0, u_n = u_0 = 2$, and when $n=1, u_n = u_1 = 7$, hence by substitution in (1) we have

$$C_1 + C_2 = 2, \quad (2)$$

$$4C_1 + 3C_2 = 7, \quad (3)$$

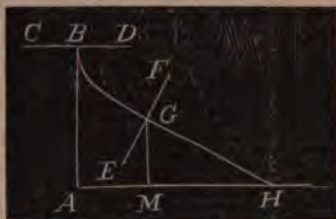
whence $C_1 = 1$, and $C_2 = 1$. $\therefore u_n = 4^n + 3^n$. We have $u_{n-1} = (4)^{n-1} + (3)^{n-1}$. $\therefore \Sigma u_{n-1} = \Sigma(4)^n + \Sigma(3)^n = \frac{1}{3}(4)^n + \frac{1}{2}(3)^n + C$.

When $n=1, u_0 = 2$; $\therefore C = -\frac{5}{6}$. $\therefore u_0 + u_1 + \dots + u_{n-1} = \frac{1}{3}(4)^n + \frac{1}{2}(3)^n - \frac{5}{6}$, the sum required.

318. "A man takes hold of the end of a cart tongue and travels off at right angles to the direction in which the tongue originally lay. Required the equation of the curve made by the middle of the axle."

SOLUTION BY PROF. E. B. SEITZ, KIRKSVILLE, MISSOURI.

Let ACD represent the original position of the cart, AB being the tongue, and CD the axle, and let HEF represent any other position of the cart. Draw GM perpendicular to AH . Let $HG = AB = a$, $AM = x$, $GM = y$. Then we have $MH = \sqrt{(a^2 - y^2)}$, and $GM = HM \tan GHM$, or $y = -\sqrt{(a^2 - y^2)} \times dy \div dx$, or $dx = -\sqrt{(a^2 - y^2)} dy \div y$. Integrating, and observing that when $y = a$, $x = 0$, we have



$$x = a \log \left(\frac{a - \sqrt{(a^2 - y^2)}}{y} \right) - \sqrt{(a^2 - y^2)}, \text{ the req'd eq'n.}$$

319. "Within a triangle to determine a point so that if the three perpendiculars are let fall from it upon the sides of the triangle, the latter will be divided into three equal parts."

SOLUTION BY PROF. CASEY.

Let ABC be the $\triangle$; take BC , BA as axes of x, y . Let O be the required point; draw On , Om , parallel to the axes, and let $Bn = x$ and $On = y$. The area $BsOr$ is given. $y \times \sin B = Or$ and $y \cos B = nr$; $\therefore x + y \cos B = Br$, $x \sin B = Os$ and $y + x \cos B = Bs$. Whence $(x + y \cos B) y \sin B + (y + x \cos B) \times x \sin B = 2^{\text{ce}} BsOr = \frac{2}{3} \triangle BAC$, or $x^2 + y^2 + 2xy \sec B = \frac{2}{3} \triangle \div \sin B \cos B =$ a given quantity, and $\therefore$ the locus of O is a conic, and for a like reason in regarding the area $ApOs$, the locus of O is also a conic, which determines the point O .



PROBLEMS.

320. By Octavian L. Mathiot, Baltimore, Md.—The transverse and conjugate axes of an ellipse being given, to find the diameter of the circular base, and the altitude, of the right cone, and where to pass a plane so as to produce the given ellipse.

321. *By Prof. Johnson.*—From any point B of the circle $x^2 + y^2 = a^2$ BR is drawn perpendicular to the straight line $x = b$. Find the locus of P the symmetrical point of R with respect to the tang't at B and its evolute.

322. *By request.*—The end of a prism is an isosceles triangle, altitude a and base b . The end of this prism is welded to the vertical face of a block of elastic material, the base of the triangular end of the prism being parallel with the horizontal face of the block and projecting a given height h above the block. A given force F_1 is applied at the projecting extremity the direction of which is normal to the vertical face of the block and which is in equilibrium with the forces F_2 and F_3 , of tension and pressure, above and below the neutral axis. Find the position of the neutral axis.

323. *By W. E. Heal, Marion, Indiana.*—Describe the spiral of Archimedes by continuous motion.

324. *By Prof. Kerahner.*—Find the envelope of the straight line

$$x \cos \varphi + y \sin \varphi = a (\cos n\varphi)^{\frac{1}{n}}.$$

PUBLICATIONS RECEIVED.

Principles of the Algebra of Logic, with Examples. By ALEXANDER MACFARLANE, M. A., D. Sc. (Edin.), F. R. S. E. 12mo. 155pp. Edinburgh: David Douglas. 1879.

We can, perhaps, best indicate the character and scope of this book, in the space at our command, by the following quotation from the preface: "The work, in its present state, forms an elementary treatise on the science of Formal Logic.

"I consider it proper to state that the theory of the operation of the mind in reasoning about Quality, which is advanced in this work, occurred to me five years ago; and that I have directed towards its development the whole of my subsequent study of the Mathematical, Physical, and Natural Sciences, which are embraced in the curriculum for the degree of Doctor of Science (Mathematics) at the University of Edinburgh."

Measurements of Gravity at Initial Stations in America and Europe. By C. S. PEIRCE. 4to. 145 pp. [Appendix No. 15, U. S. Coast Survey Report of 1876.]

On the Dynamics of a "Curved Ball." By ORMOND STONE, Cincinnati, O. [From the American Journal of Mathematics.]

On the Action of Jets of Water on Curved Vanes. By PROF. I. P. CHURCH, Cornell University. [From Van Nostrand's Engineering Magazine.]

The French Roof Truss. By PROF. P. H. PHILBRICK, Iowa State University. [From Van Nostrand's Engineering Magazine.]

ERRATA.

On page 125, line 1, for "If $a = b$ ", read If $a = b = 1$.

" " 133, line 6 from bot., for $0 = Sa\alpha S\beta\rho - Sa\beta$ read $Sa\alpha S\beta\rho + Sa\beta$.

THE ANALYST.

VOL. VII.

NOVEMBER, 1880.

No. 6..

ON A THEOREM IN PROBABILITY.

BY E. L. DE FOREST, WATERTOWN, CONN.

It should be noticed in connection with the first property discussed in my paper on polynomials (ANALYST, March, 1880), that component adjustment formulas, such as (4) and (11) for example, need not satisfy the condition mentioned at page 41, of having the sum of the coefficients l or L equal to unity, unless the formulas are to be used for adjustments or other special purposes. The l and the L may have any values whatever, and the application of the two components will still have the same effect as that of one resultant formula with coefficients which are those of the product of the two polynomials corresponding to the components. So too, if any number of components are successively used, and in any order, they are together equivalent to a single resultant whose coefficients are those of the continued product of the corresponding polynomials.

The whole property seems to be related to a theorem in probability which was first given in its most general form by Lagrange. (See the complete edition of his works, Paris, 1868, Vol. II, p. 198.) With some changes of notation and method, it may be stated thus. Suppose that observations of any class are subject to given errors, or deviations from the true value of the observed quantity, and that each of these errors may occur in a given number of ways or cases, and that all the ways are equally probable. The errors a, b, c , &c., may be essentially either positive or negative, and they can occur in α, β, γ , &c., ways respectively. Then in k observations, the number of ways or cases in which the algebraic sum of the errors can be v , is the coefficient of x^v in the expansion of the polynomial

$$(ax^a + \beta x^b + \gamma x^c + \&c.)^k, \quad (1)$$

The probability that the sum will be v is therefore the coefficient of x^v in the expansion of

$$\left(\frac{ax^a + \beta x^b + \gamma x^c + \&c.}{a + \beta + \gamma + \&c.} \right)^k, \quad (2)$$

which may be written

$$(px^a + qx^b + rx^c + \&c.)^k, \quad (3)$$

where $p, q, r, \&c.$, are the probabilities of the occurrence of the errors $a, b, c, \&c.$, respectively, in a single observation, and

$$p + q + r + \&c. = 1. \quad (4)$$

The same coefficient of x^v in the expansion of (3) is also the probability that the error of the mean of the k observations is $v \div k$.

The most probable total effect of the error a , in k observations, is kpa , that of b is kqb , that of c is krc , and so on, wherefore the most probable algebraic sum of the errors is

$$k(pa + qb + rc + \&c.), \quad (5)$$

and this is approximately the exponent of that power of x whose coefficient in the expansion of (3) is a maximum. The most probable error in the mean of the k observations is therefore

$$pa + qb + rc + \&c. \quad (6)$$

The above is Lagrange's theorem in substance. An abstract of it may be found in Todhunter's *History of the Theory of Probability*, p. 304. We now observe further, that the exponents $a, b, c, \&c.$, which represent the errors, can be regarded as whole numbers, for if they were fractions, they might be reduced to whole numbers by multiplying them all by a common multiple of their denominators, which amounts simply to measuring the errors on a reduced scale. And the exponents may all be made positive, for if the largest negative one is $-h$ for example, we can add h to each of them, and so convert (3) into the form

$$(p_0 + p_1 z + p_2 z^2 + \dots + p_m z^m)^k, \quad (7)$$

where z^0 and z^{+h} take the places of x^{-h} and x^v respectively. Each term in the first power of (3) is now represented by some term in the first power of (7). Those terms in the first power of (7) which do not represent any term in the first power of (3), will of course have their coefficients p equal to zero, so that

$$p_0 + p_1 + p_2 + \dots + p_m = 1. \quad (8)$$

The coefficients in the expansion of (3) to the k power are not altered by multiplying all the exponents of the first power of the polynomial by a constant number, or by adding a constant number to all of them, or by both these changes together. Hence it follows that the properties of the lever arm and the radius of gyration of the coefficients in powers and products of polynomials, demonstrated by me in the *ANALYST*, March and May, 1880,

pp. 44 and 74, will hold good alike whether the exponents are positive or negative, integral or fractional, provided that we locate the coefficients along the axis or lever at distances from a given point which are equal or proportional to the corresponding exponents. The lever arm of the coefficients in the product of two or more polynomials, about the place of exponent zero, is equal to the sum of the lever arms of all the factors, about their respective zero places. We represent by Δx that interval on the lever arm which corresponds to a difference of unity in the exponents of (7).

It may be well to prove the above directly for the two polynomials

$$\left. \begin{aligned} & \alpha_{-m} z^{-a_m} + \dots + \alpha_{-2} z^{-a_2} + \alpha_{-1} z^{-a_1} + \alpha_0 z^0 + \alpha_1 z^{b_1} + \alpha_2 z^{b_2} + \dots + \alpha_m z^{b_m}, \\ & \beta_{-n} z^{-c_n} + \dots + \beta_{-2} z^{-c_2} + \beta_{-1} z^{-c_1} + \beta_0 z^0 + \beta_1 z^{d_1} + \beta_2 z^{d_2} + \dots + \beta_n z^{d_n}, \end{aligned} \right\} \quad (9)$$

which have both positive and negative exponents. Let the sum of the coefficients α in the first polynomial be S_1 , and let their lever arm about the place of α_0 as a fulcrum be h_1 . Also let the sum of the coefficients β in the second polynomial be S_2 , and let their lever arm about the place of β_0 be h_2 . The statical moments of the two systems about these fulcrums are

$$\left. \begin{aligned} S_1 h_1 &= (-\alpha_{-m} a_m - \alpha_{-m+1} a_{m-1} - \dots - \alpha_{-1} a_1 + \alpha_1 b_1 + \alpha_2 b_2 + \dots + \alpha_m b_m) \Delta x, \\ S_2 h_2 &= (-\beta_{-n} c_n - \beta_{-n+1} c_{n-1} - \dots - \beta_{-1} c_1 + \beta_1 d_1 + \beta_2 d_2 + \dots + \beta_n d_n) \Delta x. \end{aligned} \right\} \quad (10)$$

Now when the first one of the polynomials (9) is multiplied by the second, the moments about the place of $\alpha_0 \beta_0$, of the coefficients in the parts of the product due to $\beta_0, \beta_1, \beta_2, \dots, \beta_n$, are

$$\beta_0 S_1 h_1, \beta_1 S_1 (h_1 + d_1 \Delta x), \beta_2 S_1 (h_1 + d_2 \Delta x), \dots, \beta_n S_1 (h_1 + d_n \Delta x),$$

and those for the parts due to $\beta_{-1}, \beta_{-2}, \dots, \beta_{-n}$ are

$$\beta_{-1} S_1 (h_1 - c_1 \Delta x), \beta_{-2} S_1 (h_1 - c_2 \Delta x), \dots, \beta_{-n} S_1 (h_1 - c_n \Delta x).$$

The sum of all the coefficients in the product is $S_1 S_2$, and denoting by H their lever arm about the place of $\alpha_0 \beta_0$, we have, by addition of the foregoing moments,

$$S_1 S_2 H = S_1 h_1 (\beta_{-n} + \beta_{-n+1} + \dots + \beta_{-1} + \beta_0 + \beta_1 + \beta_2 + \dots + \beta_n)$$

$$+ S_1 (-\beta_{-n} c_n - \beta_{-n+1} c_{n-1} - \dots - \beta_{-1} c_1 + \beta_1 d_1 + \beta_2 d_2 + \dots + \beta_n d_n) \Delta x,$$

which by the help of (10) is reduced to

$$\begin{aligned} S_1 S_2 H &= S_1 S_2 h_1 + S_1 S_2 h_2, \\ \therefore H &= h_1 + h_2, \end{aligned} \quad (11)$$

so that the lever arm in the product is equal to the sum of the lever arms in the two factors.

We will now make another statement and application of Lagrange's theorem. An urn contains a number of balls, some of which are marked 0, others 1, others 2, &c., in such proportion that the probabilities of drawing a single ball marked 0, 1, 2, ... m are $p_0, p_1, p_2, \dots, p_m$, respectively.

The occurrence of an error b , under the first statement, is here represented by the drawing of a ball marked $b+h$. If each ball is returned as soon as drawn, and they are all mixed so as to keep the chances the same, then after k drawings, the probability that the sum of the numbers drawn shall be a given number s , is the coefficient of z^s in the expansion of (7). Proof of this will be given farther on. The most probable shares which the numbers 0, 1, 2, &c., will respectively have in the formation of the sum, are

$$kp_0 \times 0, \quad kp_1 \times 1, \quad kp_2 \times 2, \quad \&c., \quad kp_m \times m,$$

so that the most probable sum will be

$$k(p_1 + 2p_2 + 3p_3 + \dots + mp_m). \quad (12)$$

This is the value, approximately, of the exponent of z in that term of the expansion whose coefficient is a maximum. It is in general approximate only, because it is not usually a whole number, as the exponents here are supposed to be. It will be seen that the portion of (12) within the parentheses is the lever arm of the coefficients p in the first power of (7), about the place of p_0 as a fulcrum, the constant interval Δx between the coefficients being unity. This answers to the property noticed by Lagrange in his first corollary. I have shown further, that the lever arm of the coefficients in the expansion to the k power will be k times what it is in the first power. Hence according to (12), the maximum coefficient in the expansion will be located approximately at the centre of forces, or centre of gravity, of the whole expanded series of coefficients, even when k is not a large number. I have before shown that this is true at the limit, when k becomes large or infinite, and proved that the series of coefficients then takes the form of the probability curve. (ANALYST, Sept. and Nov., 1879.) For a given value of k , the place or rank of the vertex is known and the curve can be constructed. Since k in my article last mentioned is in strictness infinite, we may write k instead of $k+1$. Also $b_2(dx)^2$ there represents the square of the radius of gyration r , of the coefficients of the first power of the polynomial, about their centre of forces. The equation of the curve then is

$$h^2 = \frac{1}{2kr^2}, \quad y = \frac{hdx}{\sqrt{\pi}} e^{-\frac{1}{2}x^2}, \quad (13)$$

where dx is what the interval Δx reduces to when the coefficients, regarded as ordinates, are brought together so as to be consecutive, and kr^2 is the square of the radius of gyration of the coefficients in the expanded polynomial, about their centre of forces. This radius is here seen to be identical with the quadratic mean error; as we have already noticed in the case of the expansion of the binomial $(p+q)^n$. (ANALYST, Jan., 1880, pp. 8 and 22.) The probable error, or probable deviation of the sum of the numbers drawn, from its most probable value, is

When the exponents k are tolerably large, there is no need to actually perform the multiplications. My theorems respecting the lever arm and the radius of gyration of the coefficients enable us easily to find the centre of forces for the final product, giving the most probable value of s , and to find the radius of gyration about this centre, determining h in the probability curve (13). The proposition that the most probable value is at the centre of forces approximately, is also in accordance with the known theorem in mechanics, that the radius of gyration (here the mean error) is a minimum when the axis of rotation is taken through the centre of gravity.

To illustrate the foregoing, suppose that two urns contain ten balls each. The first has four balls marked -1, five marked 0, and one marked 2; while the second has one marked -2, two marked -1, four marked 1, and three marked 2. Twelve drawings are made from the first and eight from the second. The two polynomials are

$$\left(\frac{4z^{-1} + 5 + z^2}{10}\right)^{12}, \quad \left(\frac{z^{-2} + 2z^{-1} + 4z + 3z^2}{10}\right)^8, \quad (17)$$

and the lever arms of the coefficients in their first powers, about the places of z^0 as fulcrum, are

$$\left. \begin{aligned} \frac{1}{10}[4(-1) + 5(0) + 1(2)]\Delta x &= -\frac{1}{5}\Delta x, \\ \frac{1}{10}[1(-2) + 2(-1) + 4(1) + 3(2)]\Delta x &= \frac{3}{5}\Delta x. \end{aligned} \right\} \quad (18)$$

Hence, in the product of the expansions of (17), the lever arm, about the place of z^0 as a fulcrum, is

$$12(-\frac{1}{5}\Delta x) + 8(\frac{3}{5}\Delta x) = \frac{1}{5}\Delta x. \quad (19)$$

The integer nearest to $\frac{1}{5}$ is 2, which is therefore the most probable algebraic sum of the numbers drawn. The squares of the radii of gyration of the coefficients in the first powers of the polynomials (17), about the places of the extremity of the lever arm, or the centre of parallel forces, are

$$\left. \begin{aligned} \frac{1}{10}[4(\frac{1}{5})^2 + 5(\frac{0}{5})^2 + 1(\frac{2}{5})^2](\Delta x)^2 &= \frac{1}{25}(\Delta x)^2, \\ \frac{1}{10}[1(\frac{2}{5})^2 + 2(\frac{1}{5})^2 + 4(\frac{0}{5})^2 + 3(\frac{2}{5})^2](\Delta x)^2 &= \frac{4}{25}(\Delta x)^2. \end{aligned} \right\} \quad (20)$$

Therefore in the product of the expansions of (17), the square of the radius of gyration of the coefficients, about their centre of forces, is

$$[12(\frac{1}{25}) + 8(\frac{4}{25})](\Delta x)^2 = 23.84(\Delta x)^2. \quad (21)$$

The probable error then is

$$.6745\sqrt{23.84} = 3.29,$$

so that it is about an even chance that the sum of the numbers drawn will fall within the limits $\frac{1}{5} \pm 3.29$.

The product of the expansions of (17) will approximate to the form of a probability curve. I have already shown that this is true for any high power of a single polynomial whose coefficients are all positive, so that the two expansions in (17) will each be of that form. Now in the curve (13) the parameter h is constant so long as kr^2 is constant, though k and r^2 should

vary inversely as each other; and many different polynomials may give the same value of r^2 . Hence a curve of given parameter may be regarded as approximately representing the form of the expansion, to some power, of an indefinite number of different polynomials. The two expansions in (17) may thus be taken as expansions of some one undetermined polynomial, to different powers. Their product will thus be an expansion of that polynomial, and consequently will take the form of the probability curve.

To construct its equation from (13), putting dx for Δx in the radius of gyration, we have

$$h^2 = \frac{1}{2 \times 23.84(dx)^2} \quad \therefore hdx = .14482,$$

and writing $i = x \div dx$, we get

$$\log y = \log(hdx) - \frac{1}{2} \log \pi - i^2(hdx)^2 \log e \quad \left. \vphantom{\log y} \right\} \quad (22)$$

$$= 2.91226 - .0091085 i^2.$$

The vertex of the curve, where $i = 0$, is at the centre of forces in the product of the expansions of (17), and the rank of that centre from z^0 , as given in (19), is $\frac{1}{2}$. Hence the place of the exponent 2 corresponds to $i = -\frac{2}{5}$, so that if we assign to i the values $-\frac{7}{5}, -\frac{2}{5}, \frac{3}{5}, \frac{8}{5}$, &c., the resulting values of y will be the approximate probabilities that the sum of the numbers drawn will be 1, 2, 3, 4, &c., respectively. The larger the exponents k are in such cases, the more accurate will the results be.

To show the general correctness of the above approximation, the computed values of y , for each value of i , are shown in the subjoined table. In the column headed C are also placed the values of the coefficients in the product of the expansions (17), as found by actual multiplication.

i	y	C	i	y	C	i	y	C	i	y	C
-15.4	.001		-7.4	.026	.027	0.6	.081	.083	8.6	.017	.018
-14.4	.001		-6.4	.035	.035	1.6	.077	.077	9.6	.012	.011
-13.4	.002		-5.4	.044	.045	2.6	.071	.072	10.6	.008	.006
-12.4	.003	.004	-4.4	.054	.054	3.6	.062	.061	11.6	.005	.005
-11.4	.005	.005	-3.4	.064	.065	4.6	.052	.052	12.6	.003	
-10.4	.008	.007	-2.4	.072	.074	5.6	.042	.042	13.6	.002	
- 9.4	.013	.013	-1.4	.078	.078	6.6	.033	.033	14.6	.001	
- 8.4	.019	.019	-0.4	.081	.083	7.6	.024	.023	15.6		

Only three places of decimals were used in doing this part of the work, so that the series C is not perfectly smooth and regular, but its general agreement with the series y is very close.

The probabilities that an event will happen or fail certain numbers of times in m trials, are represented, as is well known, by terms in the expansion of $(p+q)^m$. (ANALYST, May 1879, p. 65.) This is the simplest case under our present theorem, where there is but a single urn, containing balls marked either 0 or 1 for example, so that (16) reduces to

$$(p'_0 + p'_1 z)^{k'}. \quad (23)$$

The coefficient of z^s in the expansion is the probability that the event 1 will happen s times, and consequently that the event 0 will happen $k'-s$ times.

Reverting now to our first statement of Lagrange's theorem, as applied to the probable effect of a system of given errors $a, b, c, \&c.$, it appears that the most probable error in the mean result of k observations is the k th part of that total error, or algebraic sum of the errors, whose probability occupies the centre of forces in the series of coefficients in the expansion of the polynomial (3), when the coefficients in this expansion are regarded as masses acting about the point of no error as a fulcrum, with lever arms equal to the total errors whose probabilities are represented by the coefficients. The probable sum of the total deviations from the centre of forces or centre of gravity on one side, is equal to the probable sum of those on the other, so that the most probable mean result of the k observations is the arithmetical mean of all the mean results possible under the given system of errors, each possible result having a weight proportional to the prob'y of its occurrence.

If the algebraic sum of the given elementary errors, each weighted in proportion to the probability of its occurrence, is zero, the most probable mean result will be the true value of the observed quantity; and not otherwise. The most probable result will be greater or less than the true value according as the $+$ or $-$ elementary errors preponderate. When these errors are unknown, we have no reason to expect a preponderance on one side rather than on the other, and hence we conclude that as far as the true value of an observed quantity can be inferred from the results of numerous observations, the arithmetical mean of all the observed values is the most likely to be the true value. It is to be presumed that the observed deviations from the mean will be so distributed as to represent approximately the distribution of all the possible deviations.

As the number of observations is increased, the series of coeff's in the expansion of (3) approaches rapidly, as I have shown, to the form of the probability curve (13), and attains this form at the limit, when k becomes very large, or infinite. It appears to me that we have here the most general and complete proof of the validity of that exponential law of facility of error, or probability of deviation from the most probable value, whose geometrical expression is called the probability curve. The elementary errors are not restricted here by assuming that they follow an algebraical law of distribution, nor are positive and negative errors of equal amount required to be equally probable.

I will take a future occasion to show the application of similar principles in determining the law of errors in two dimensions, or errors in the position of points in a plane.

MECHANICS BY QUATERNIONS.

BY PROF. E. W. HYDE, UNIVERSITY OF CINCINNATI.

(12) RETURNING now to equations (22) and (23), let us first find the condition that the system of forces may have a single resultant. Evidently if G be perpendicular to R the forces composing G may be taken as parallel to R , and we shall then have a system of three parallel forces whose sum is not zero, which may therefore be combined into a single resultant.

Hence the required condition is

$$S, R, G = S, R, G_0 = 0. \quad (24)$$

This equation is satisfied by $R = 0$ and $G = 0$, as well as by R perpendicular to G . If the first is true there is no resultant, and the system is equivalent to a couple only.

If $G = G_0 - V, \pi, R = 0$, the locus of π will be the locus of all points at which there is a single resultant. G_0 being a constant vector this is the equation of a straight line as appears by solving the equation for π . Since the Scalar part of the quaternion π, R is indeterminate call it x , then

$$\begin{aligned} \pi, R &= x + V, \pi, R = x + G_0; \\ \pi &= x, R^{-1} + G_0, R^{-1}. \end{aligned} \quad (25)$$

The last term of this equation is a vector because

$$S, G_0, R^{-1} = S, R^{-1}, V, \pi, R = 0.$$

It is also evident that if G be any constant vector whatever the locus of π is a right line parallel to R , i. e., the moment is the same about all p'ts of any line parallel to R .

(13). We will next find the locus of π when $G = T, G$ is a constant quantity; i. e., $T, (G_0 - V, \pi, R) = G = \text{constant}$, or squaring and transposing, and remembering that $V^2, \pi, R = S^2, \pi, R - \pi^2, R^2$,

$$\pi^2, R^2 - S^2, \pi, R + 2S, G_0, \pi, R = G^2 - G_0^2. \quad (26)$$

This is easily seen to be the equation of a circular cylinder whose axis is parallel to R .

To find the position of the axis, let us change the origin by putting $\pi + \delta$ for π , with the condition $S, \pi, R = 0$; so that δ is the vector of the point where the axis pierces the plane $S, \pi, R = 0$. Substituting in (26) with this condition we have

$$(\pi + \delta)^2, R^2 + 2S, G_0, (\pi + \delta), R = G^2 - G_0^2.$$

If the new origin is to be at a point of the axis, the terms of the 1st degree in π must disappear; therefore

$$\begin{aligned} R^2, S, \pi, \delta + S, G_0, \pi, R &= 0 = S, (R^2, \delta - V, G_0, R), \pi; \\ \delta &= R^{-2}, V, G_0, R = V, G_0, R^{-1}. \end{aligned} \quad (27)$$

The equation of the axis of the cylinder is then

$$, \rho = V, G_0, R^{-1} + x, R. \quad (28)$$

Let us next find the value, or rather the locus, of $, \pi$ when $G = T, G$ is a minimum. The value of G increases with π to ∞ , so that there is no maximum. If G is a minimum we must have

$$S, G d, G = 0, \text{ i. e., } S, (G_0 - V, \pi, R) V, R d, \pi = 0.$$

Expanding and reducing we have

$$S(V, G_0, R - , \pi, R^2 + , RS, \pi, R) d, \pi = 0,$$

which can only be justified in general by equating the quantity in parentheses to zero. Therefore

$$\begin{aligned} , \pi, R^2 &= V, G_0, R + , RS, \pi, R \\ \text{or} \quad , \pi &= V, G_0, R^{-1} + R^{-1} S, \pi, R. \end{aligned} \quad (29)$$

This is the equation of a straight line, and is identical with equation (28) and also with eq. (25), which is the particular case when $S, G_0, R^{-1} = 0$.

The identity between (28) and (29) will be evident on replacing $, \rho$ in the former by $, \pi$, and then operating by S, R , when we shall find

$$x = , R^{-2} S, \pi, R.$$

It appears therefore that the axis of the cylinder $T, G = G = \text{const.}$ is the locus of $, \pi$ when G is a minimum; and that if the system of forces is capable of reduction to a single resultant, the line of action of this resultant is this same axis, i. e., in this case, the minimum value of G is zero.

This axis is called the *Central Axis* of the system. If we substitute the value of $, \pi$ from (29) in the value of $, G$ we have

$$\begin{aligned} , G &= , G_0 - V, \pi, R = , G_0 - V(V, G_0, R^{-1} + , R^{-1} S, \pi, R), R \\ &= , G_0 + V, R V, G_0, R^{-1} = , G_0 + , R^{-1} S, R, G_0 - , G_0 = , R^{-1} S, R, G_0, \end{aligned} \quad (30)$$

for the value of $, G$ when G is a minimum. Eq. (30) shows that in this case $, G$ is a vector parallel to $, R$, i. e., the plane of the resultant couple is perpendicular to $, R$. Let us see if there is any other position of $, R$ for which $, G$ is parallel to $, R$, i. e., $V, G, R = 0$. Operating on $, G$ by V, R , we have $V, R, G_0 - V, R V, \pi, R = 0$, $\therefore V, R, G_0 - , RS, \pi, R + , \pi, R^2 = 0$, or

$$, \pi = V, G_0, R^{-1} + , R^{-1} S, \pi, R,$$

which is (29) again, so that the central axis is the only line such that when the resultant acts along it the plane of the resultant couple is perp. to $, R$.

(14). Let us determine the conditions that must hold in order that the resolved parts along any straight line whatever of a system of forces acting on a rigid body may be in equilibrium.

Let ϵ be a unit vector in any direction, then $\epsilon^{-1} S, \epsilon, P$ is the resolved part of $, P$ along ϵ . In equation (22) and (23) put $\epsilon^{-1} S, \epsilon, P$ for $, P$; therefore

$$\Sigma[\epsilon^{-1} S, \epsilon, P] = \epsilon^{-1} S, \epsilon, \Sigma(P) = 0, \therefore S, \epsilon, \Sigma(P) = 0,$$

and as this must hold for all directions of ϵ

$$\Sigma(\rho P) = \rho R = 0. \quad (31)$$

Also $\Sigma[V, \rho \epsilon^{-1} S \epsilon, P] = 0 = V, \epsilon \Sigma[\rho S \epsilon, P]. \quad (32)$

If this is to be true for *all* directions of ϵ we must have

$$n \epsilon = \Sigma[\rho S \epsilon, P]. \quad (33)$$

Now the general value of ϵ in terms of any three non-coplanar vectors $\epsilon_1, \epsilon_2, \epsilon_3$ is

$$\epsilon = \frac{1}{S \epsilon_1 \epsilon_2 \epsilon_3} [\epsilon_1 S \epsilon_2 \epsilon_3 \epsilon + \epsilon_2 S \epsilon_3 \epsilon_1 \epsilon + \epsilon_3 S \epsilon_1 \epsilon_2 \epsilon],$$

and hence $\Sigma[\rho S \epsilon, P]$ must be capable of reduction to such a form if (33) is to be satisfied.

(15). If a rigid body be in equilibrium under the action of a system of forces and if equilibrium be maintained when the body has been turned through the angle θ about any axis (θ being not a multiple of two right angles, and the system of forces being constant in magnitude and direction), then the system will be in equilibrium whatever value θ may have.

We have the conditions

$$\Sigma(\rho P) = 0 \text{ and } \Sigma(V, \rho, P) = 0. \quad (34)$$

Let ϵ be a unit vector along the axis about which the body is to be revolved, then every ρ will be changed to $\epsilon^{\theta+\pi} \rho \epsilon^{-\theta+\pi}$. (See Tait's Quaternions, Art. 351.) Substituting in the second of (34) we have

$$\Sigma[V \epsilon^{\theta+\pi} \rho \epsilon^{-\theta+\pi}, P] = 0,$$

$$\text{i. e. } \Sigma[V(\cos \frac{1}{2}\theta + \epsilon \sin \frac{1}{2}\theta), \rho(\cos \frac{1}{2}\theta - \epsilon \sin \frac{1}{2}\theta), P] = 0.$$

Expanding, reducing and making use of (34) this becomes

$$2 \cos \frac{1}{2}\theta [\Sigma(\epsilon S, \rho, P) - \Sigma(\rho S \epsilon, P)] - V, \epsilon \Sigma(\rho S \epsilon, P - P S \epsilon, \rho) \sin \frac{1}{2}\theta = 0. \quad (35)$$

This being an equation between two vectors multiplied by scalar coefficients can only be satisfied by making both coefficients zero, or else both vectors zero. As the coefficients cannot be zero together the vectors must be;

$$\therefore \Sigma(\epsilon S, \rho, P) = \Sigma(\rho S \epsilon, P), \quad (36)$$

$$V \epsilon \Sigma(\rho S \epsilon, P) = V \epsilon \Sigma(P S \epsilon, \rho). \quad (37)$$

Eliminate $\Sigma(\rho S \epsilon, P)$ between (36) and (37),

$$V \epsilon \Sigma(\epsilon S, \rho, P) = V \epsilon \Sigma(P S \epsilon, \rho);$$

the first member is zero because $V \epsilon \epsilon = 0$, $\therefore$ the second member is also zero. We have supposed (35) to be true for *some* value of θ , hence the conditions (36) and (37) must be satisfied, and if they *are* satisfied (35) is true for *any* value of θ , Q. E. D.

(16). We will now show how the general equations of equilibrium of a rigid body may be deduced from those for forces acting on a particle.

Suppose a system of n rigidly connected particles. The action between any two particles to be taken along the straight line joining them, and to be mutual, i. e., the action of the first on the second equal and opposite to that of the second on the first. Let P_1 be the external force acting on the particle 1 and let Q_1, Q_2 &c., be the forces acting between it and other particles. Then there will be n equations

$$P_1 + \Sigma(Q') = 0, P_2 + \Sigma(Q'') = 0, \text{ etc.,}$$

where the primes indicate a different set of Q 's in each case. From the hypotheses above however it is clear that for every Q , as Q_i , we shall have also a force $-Q_i$; hence if we add up the equations above, the Q 's will all cancel out and leave us

$$\Sigma(P) = 0. \quad (38)$$

Let ρ_1, ρ_2 &c., be the vectors of the particles; then from the equations above we have

$$V_{\rho_1} P_1 + V_{\rho_1} \Sigma(Q') = 0, V_{\rho_2} P_2 + V_{\rho_2} \Sigma(Q'') = 0, \text{ etc.}$$

On adding these equations it appears from what has been already shown that we shall have a series of terms containing Q 's such as

$$V_{\rho_l} Q_{lm} - V_{\rho_m} Q_{lm} = V_{(\rho_l - \rho_m)} Q_{lm},$$

in which ρ_l and ρ_m are the vectors of the l th and m th particles and Q_{lm} is the force between them. But as, by hypothesis, Q_{lm} acts along $\rho_l - \rho_m$, it follows that every such vector is zero. Hence on summing we shall have

$$\Sigma(V_{\rho} P) = 0. \quad (39)$$

(17). Some additional illustrative examples will now be given.

Ex. 1. A cube is acted upon by 4 forces; one is in a diagonal, and the others in edges no two of which are in the same plane, and which do not meet the diagonal: find the condition that there may be a single resultant.

Let i, j, k be the unit vector edges of the cube, and let the four forces be $P_1 i, P_2 j, P_3 k$ and $P_4 U(i+j+k)$;

$$\therefore R = P_1 i + P_2 j + P_3 k + P_4 U(i+j+k).$$

Take moments about the origin so that $\pi = 0$, the origin being at a corner of the cube;

$$\begin{aligned} \therefore G &= V_j P_1 i + V_k P_2 j + V_i P_3 k \\ &= -i P_2 - j P_3 - k P_1. \end{aligned}$$

The condition is by eq. (24), $S, R, G = 0$; therefore

$$S[i P_2 + j P_3 + k P_1][i P_1 + j P_2 + k P_3 + P_4 U(i+j+k)] = 0.$$

But

$$U(i+j+k) = \frac{i+j+k}{T(i+j+k)} = \frac{1}{\sqrt{3}}(i+j+k);$$

$$\therefore S[i P_2 + j P_3 + k P_1][i P_1 + j P_2 + k P_3 + P_4 \sqrt{\frac{1}{3}}(i+j+k)] = 0$$

$$\therefore P_1P_2 + P_2P_4\sqrt{\frac{1}{3}} + P_2P_3 + P_3P_4\sqrt{\frac{1}{3}} + P_3P_1 + P_1P_4\sqrt{\frac{1}{3}} = 0;$$

$$\therefore P_4(P_1 + P_2 + P_3) + \sqrt{3}(P_1P_2 + P_2P_3 + P_3P_1) = 0.$$

Ex. 2. Six equal forces act along the edges of a cube which do not meet a given diagonal taken in order; find the resultant action.

Let P be the tensor of each force, i, j, k , unit vectors along the edges, the length of each edge being a

$$\therefore R = P(i+j+k-i-j-k) = 0,$$

and taking moments about a corner at the extremity of the above mentioned diagonal,

$$\begin{aligned} G &= G_0 = PaV(-ij + ji + ik - ki - jk + kj) = -2Pa(i+j+k) \\ &= -2\sqrt{3}.PaU(i+j+k). \end{aligned}$$

Ex. 3. Find the central axis of two forces P and Q .

Let the perpendicular distance between the forces be $2a$ and take the origin at its middle point, so that $S_aP = S_aQ = 0$, and

$$G = G_0 - V\pi R = Va(P-, Q) - V\pi(P+, Q).$$

Then, by eq (29), $\pi = -V.(P+, Q)^{-1}Va(P-, Q) + (P+, Q)^{-1}S\pi(P+, Q)$

$$\begin{aligned} &= \frac{-1}{(P+, Q)^2} \left[V.(P+, Q)Va(P-, Q) - (P+, Q)S\pi(P+, Q) \right] \\ &= \frac{P^2 - Q^2}{(P+, Q)^2} a + R^{-1}S\pi R = \frac{P^2 - Q^2}{P^2 + Q^2 - 2S.P.Q} a + R^{-1}S\pi R. \end{aligned}$$

If the forces are perpendicular $S.P.Q = 0$, and we have

$$\pi = \frac{P^2 - Q^2}{P^2 + Q^2} a + R^{-1}S\pi R.$$

In this case the distances along a from the central axis to P and Q are respectively $a \cdot \frac{2Q^2}{P^2 + Q^2}$ and $a \cdot \frac{2P^2}{P^2 + Q^2}$, so that these distances bear to each other the ratio of Q^2 to P^2 .

Ex. 4. Forces act at the angular points of a tetrahedron in directions respectively perpendicular to the opposite faces, and proportional to the areas of the faces in magnitude: show that the forces have the property considered in Art. 14.

Let a, b, c be three vector edges of the tetrahedron measured out from one vertex taken as the origin; then the forces may be represented by $V_a b, V_b c, V_c a$ and $V(c-a)(b-a)$. By eq. (32) we must have

$$V.\varepsilon\S(\rho S\varepsilon P) = 0, \quad \Sigma(\rho S\varepsilon P) = aS\varepsilon b, c + bS\varepsilon c, a + cS\varepsilon a, b = \varepsilon S_a b, c;$$

therefore the condition is satisfied.

(18). *Constrained Body*.—First let a rigid body have one fixed point, and let this point exert a force F against the body; then for equilibrium,

$$\Sigma(P) + R + F = 0, \quad (40)$$

and, if we take the origin at the fixed point,

$$G_0 = \Sigma(V, \rho, P) = 0. \quad (41)$$

Eq. (40) gives the pressure on the fixed point and (41) is the condition of equilibrium.

The system therefore reduces to a single resultant acting through this pt.

Second, let there be two fixed points in the body, and let the forces they exert on the body be respectively F_1 and F_2 . Since two points determine a right line, it is evident that all points in the right line joining the two fixed points will be incapable of motion, and the only condition of equilibrium is that the resultant moment about this line shall be zero. Let ϵ be a unit vector along the fixed line, and let the origin be taken on this line at distances a and b from the fixed points. Then the general equations (22) and (23) give

$$\Sigma(P) + F_1 + F_2 = 0, \quad (42)$$

$$V\Sigma(\rho, P) + aV\epsilon F_1 + bV\epsilon F_2 = 0. \quad (43)$$

Operate on (43) by $S\epsilon$, $\therefore$

$$S\epsilon\Sigma(\rho, P) = 0 = S\epsilon G_0. \quad (44)$$

Eq. (44) is the condition of equilibrium, for it makes the axis of the resultant couple perpendicular to ϵ , so that there can be no moment about ϵ . By elimination between (42) and (43) we find

$$\left. \begin{aligned} V\epsilon F_1 &= \frac{1}{a-b} \left[bV\epsilon\Sigma(P) - G_0 \right], & V\epsilon F_2 &= \frac{-1}{a-b} \left[aV\epsilon\Sigma(P) - G_0 \right]. \\ \text{But } F_1 &= \epsilon^{-1}S\epsilon F_1 + \epsilon^{-1}V\epsilon F_1 \text{ and } F_2 = \epsilon^{-1}S\epsilon F_2 + \epsilon^{-1}V\epsilon F_2, \end{aligned} \right\} \quad (44a)$$

hence the components of F_1 and F_2 along ϵ are indeterminate, as $S\epsilon F_1$ and $S\epsilon F_2$ are subject only to the condition derived from (42),

$$S\epsilon F_1 + S\epsilon F_2 + S\epsilon\Sigma(P) = 0. \quad (44b)$$

If the body is capable of sliding along the line ϵ as well as of turning around it, we must have the additional condition of equilibrium,

$$S\epsilon\Sigma(P) = 0, \quad (45)$$

which is equivalent to equation (3).

If *three* points of the body, not in one straight line, are fixed, it can have no motion either of rotation or translation, and the three vectors of the fixed points are indeterminate.

(19). *Body resting against a smooth plane.*—First let the body rest on one point, and call the force exerted by the plane against the body $N\epsilon$, ϵ being a unit vector perpendicular to the plane. Then if we take the origin at the point where the body rests on the plane, and take moments about

this point for equilibrium,

$$\Sigma(P) + N\varepsilon = 0, \quad (46)$$

$$G_0 = \Sigma(V, \rho, P) = 0. \quad (47)$$

As this gives $S, R, G = S\Sigma(P), G_0 = 0$, which by eq. (24) is the condition for a single resultant, it follows that the system reduces to a single force $R = -N\varepsilon$ perpendicular to the plane, and acting through the point of contact. The latter follows from equation (25).

Second, let the body rest on *two* points, and let $N_1\varepsilon$ and $N_2\varepsilon$ be the normal reactions of the plane against the body. Then

$$R + N_1\varepsilon + N_2\varepsilon = \Sigma(P) + \varepsilon(N_1 + N_2) = 0, \quad (48)$$

$\Sigma(V, \rho, P) + N_1 V, a_1 \varepsilon + N_2 V, a_2 \varepsilon = G_0 + V(N_1, a_1 + N_2, a_2)\varepsilon = 0$, (49) provided that moments are taken about the origin, and a_1 and a_2 are the vectors of the points of contact. We have at once from (48) $U, R = -\varepsilon$, and $T, R = R = \Sigma(P) = N_1 + N_2$. Operating on (49) by $S, R = -RSe$, we have

$$Se, G_0 = Se\Sigma(V, \rho, P) = 0, \quad (50)$$

which by (24) is the condition for a single resultant. Also operating on (49) by S, a_1 and S, a_2 we find

$$N_1 = \frac{S, a_2, G_0}{S, a_1, a_2 \varepsilon}, \quad N_2 = -\frac{S, a_1, G_0}{S, a_1, a_2 \varepsilon}. \quad (51)$$

The equation of the single resultant is by (25)

$$\pi = R^{-1}x\varepsilon - R^{-1}\varepsilon G_0,$$

that is R is applied at a distance from the origin, measured perpendicular to itself, of $-R^{-1}\varepsilon G_0$. From the equation just given $G_0 = R V\varepsilon\pi$; substitute this value in (49) and we have

$$\begin{aligned} V(-\pi R + N_1, a_1 + N_2, a_2)\varepsilon &= 0; \\ \therefore \pi &= \frac{N_1, a_1 + N_2, a_2}{R} = \frac{N_1, a_1 + N_2, a_2}{N_1 + N_2}, \end{aligned} \quad (52)$$

so that the resultant R is in the same line with the resultant of $N_1\varepsilon$ and $N_2\varepsilon$, as must evidently be the case.

Third, suppose the body to rest on *three* points. Then as before

$$\Sigma(P) + (N_1 + N_2 + N_3)\varepsilon = R + (N_1 + N_2 + N_3)\varepsilon = 0, \quad (53)$$

$$G_0 + V(N_1, a_1 + N_2, a_2 + N_3, a_3)\varepsilon = 0; \quad (54)$$

$\therefore U, R = -\varepsilon$, $T, R = R = N_1 + N_2 + N_3$, $Se, G_0 = Se\Sigma(V, \rho, P) = 0$, as before, so that the system reduces to a single resultant. The equation of this resultant will be precisely the same as in the last case. We shall also have a value of π like (52) with a term N_3, a_3 added in the numerator and a term N_3 in the denominator. The value of N_1 , found by elimination between (53) and (54), is

*Used here to represent ε .

$$N_1 = \frac{S(a_2 - a_3)G_0 + RS_{a_2, a_2}e}{S(a_1, a_2 + a_2, a_3 + a_3, a_1)e} \quad (55)$$

The values of N_2 and N_3 may be found from that of N_1 by cyclic permutation of subscripts.

If there are more than three points of contact the values of N_1, N_2 etc., become indeterminate, but we shall always have the condition for a single resultant, $Se, G_0 = 0$.

(20). *Example.*—The lid $ABCD$ of a cubical box, moveable about hinges at A and B , is held at a given angle α with the horizon by a horizontal string connecting C with a point vertically over A : find the pressure on each hinge.

We have, if the edge of the cube = 1,

$$\begin{aligned} \Sigma(P) &= -Wj - PU(i \cos \alpha + k) \\ &= -Wj - \frac{P}{\sqrt{1 + \cos^2 \alpha}} (i \cos \alpha + k), \end{aligned}$$

and $G_0 = V[-\frac{1}{2}W(i \cos \alpha + k)j - Pj \sin \alpha \cdot U(i \cos \alpha + k)]$



$$\begin{aligned} &= -\frac{1}{2}W(i \cos \alpha + k)j - \frac{P \sin \alpha}{\sqrt{1 + \cos^2 \alpha}} \cdot j(i \cos \alpha + k) \\ &= \left(\frac{P \sin \alpha}{\sqrt{1 + \cos^2 \alpha}} - \frac{W}{2} \right) (k \cos \alpha - i). \end{aligned}$$

The equation of condition (44) must be satisfied, therefore, substituting k for e , as the vector along the fixed line,

$$\begin{aligned} \left(\frac{P \sin \alpha}{\sqrt{1 + \cos^2 \alpha}} - \frac{W}{2} \right) \cos \alpha &= 0; \text{ whence} \\ P &= \frac{W\sqrt{1 + \cos^2 \alpha}}{2 \sin \alpha}. \end{aligned}$$

Making use of this value of P we have

$$\Sigma(P) = -Wj - \frac{W}{2 \sin \alpha} (i \cos \alpha + k), \text{ and } G_0 = 0.$$

Hence by (44a) and (44b) we find, if F_1 acts at A and F_2 at B ,

$$\begin{aligned} F_1 &= k^{-1}SkF_1 + kV \left[-Wk \left(j + \frac{1}{2}i \cot \alpha + \frac{1}{2 \sin \alpha} k \right) \right] \\ &= \frac{1}{2}Wi \cot \alpha + Wj - kSkF_1, \\ F_2 &= k^{-1}SkF_2 = -kSkF_2; \text{ with the relation} \\ SkF_1 + SkF_2 + \frac{1}{2}W \operatorname{cosec} \alpha &= 0. \end{aligned}$$

It is to be noted that the a and b of (43) are equal respectively to 0 and 1.

CRITICISMS.

BY ALEX. S. CHRISTIE, U. S. COAST AND GEODETIC SURVEY, WASH., D. C.

"THE tensor of a quaternion vector ρ , denoted by $T\rho$, is the length of the vector expressed in terms of the unit of length and without regard to direction, and hence is an abstract number, signless as those used by a child in counting, but called positive in contradistinction to the negatives of algebra. Multiplied into a vector of unit length it gives to it the length of ρ , and when the unit vector has the direction of ρ , or is $U\rho$ the result is ρ itself, viz., we have $T\rho U\rho = \rho$; and this operation performed upon the unit vector $U\rho$ to produce ρ may be conceived of as an act of *tension*, *extension* or *stretching*. But when ρ itself is operated upon by a scalar b to produce the *new* vector $b\rho$, or by the scalar c and the inversor ($-$) to produce the *new* vector $-c\rho$, the first by a stretching of ρ , the second by a stretching and an inversion, we must not suppose that in the one case $T\rho = b$, in the other $T\rho = c$, — while to write $T\rho = -c$ is to *include the very thing* which the characteristic T was devised to exclude, and to which that other symbol, U , has been appropriated, viz., the *versor* or *directional* element ($-$). b and c are factors of the tensors of the vectors *by them produced*, viz., $T(b\rho) = bT\rho$, $T(-c\rho) = T(c\rho) = cT\rho$. In fine, the tensor which produces ρ from a unit vector is the tensor of ρ , $T\rho$, in the technical sense in which Hamilton employs the term Tensor and the symbol T ; but the tensor which produces from ρ some *other* vector is *not* the tensor of ρ . In the one case ρ is the *result* of the stretching or tension, in the other the *subject*, and between these two cases it is necessary and easy to distinguish."

The above was sent to Prof. Wood immediately upon the publication of §9, p. 36 of the present volume of the ANALYST, and I insert it here because I deem the correction on p. 127 neither sufficiently general nor adequately emphatic.

At p. 69, line 5 we read: "the letter U may also be used to denote a unit vector; thus $U\beta$ is a unit vector parallel to vector β . As a versor it implies that β has been turned from some arbitrary direction into the given one. At p. 128 the latter sentence is expunged, because "it conflicts with the definition given by Hamilton, which is ' $U\beta$ is the versor of a right quotient.'"

Now, although there is nothing in Hamilton to *justify* the expunged statement, there is, on the other hand, nothing there with which it *conflicts*. The additional meaning thereby assigned to the symbol U is merely *worthless*. Hamilton calls a unit vector a *versor* from *analogy* and for *convenience*, since

we have $Uq = U\frac{\beta}{\alpha} = \frac{U\beta}{U\alpha}$; but he afterward shows that a vector, say β , may be usefully equated to the *right quotient* of which it is the *index*, and $U\beta$ to the corresponding *right radial*; furthermore, that every unit vector may be regarded as the axis of a *quadrantal rotation*, and that the vector may be considered as itself a *quaternion*, of which the tensor of the vector is the tensor, the *unit vector* $U\beta$, say, its *versor*. Hence U does not imply anything whatever, but plainly *expresses*, not that " β has been turned from some arbitrary direction into the given one", but that β itself has, as a quaternion, power to turn *other* vectors from *their* directions, this *turning* power being resident in $U\beta$. It will be seen that the distinction is that between the *actor* and the *thing acted upon*.

Turning now to Prof. Wood's treatment of the important symbols i, j, k , we find on p. 67 the equations $k = \frac{j}{i}$ (3), $k = \frac{-i}{j}$ (4), $-k = \frac{i}{j}$ (5), and thereafter is the remark: "A comparison of equations (3) and (5) shows that a reciprocal of the fraction *changes the sign* of the vector axis, instead of producing its reciprocal." Of course we have $\frac{i}{j} = -k = \frac{1}{k}$, which is the reciprocal of k , the "axis" of $j \div i$, and Prof. W. attends to the matter at top of p. 128; but he there restricts the necessity for the emendation to the case "When the direction of rotation of the fraction and its reciprocal are in opposite senses." It is matter of easy and direct perception, as well as of early definition, that the direction of rotation of a fraction is *always* opposite to that of its reciprocal. Again, on p. 68, the equation $kk = -1$ is obtained from (3) and (4) by the transformation

$$kk = \frac{j}{i} \cdot \frac{-i}{j} = \frac{-i}{i} = -1. \quad \text{Now by definition}$$

$$\frac{\beta}{\alpha} \cdot \alpha [= \beta \alpha^{-1} \alpha] = \beta, \frac{\gamma}{\beta} \cdot \frac{\beta}{\alpha} [= \gamma \beta^{-1} \beta \alpha^{-1} = \gamma \alpha^{-1}] = \frac{\gamma}{\alpha},$$

but not $\alpha \cdot \frac{\beta}{\alpha} [= \alpha \beta \alpha^{-1}] = \beta$, nor $\frac{\beta}{\alpha} \cdot \frac{\gamma}{\beta} [= \beta \alpha^{-1} \gamma \beta^{-1}] = \frac{\gamma}{\alpha}$; and hence

we should write $\frac{j}{i} \cdot \frac{-i}{j} [= -j i^{-1} i j^{-1} = -j j^{-1}] = -\frac{j}{j} = -1$, but

never $\frac{j}{i} \cdot \frac{-i}{j} = \frac{-i}{i}$. The j 's in Prof. W.'s transformation are *non-adjacent*—to combine them is to do away with the non-commutative principle of quaternion (or vector) multiplication, and this, we may say, is to do away with quaternions themselves. Had Prof. W. desired to pass through the form $\frac{-i}{i}$ he should have written $kk = \frac{-i}{j} \cdot \frac{j}{i} [= -i j^{-1} j i^{-1} = -i i^{-1}] = \frac{-i}{i}$

$= -1$, which is permissible. Owing to the *scalar* character of the product, which permits of cyclic permutation under the characteristic S , or more shortly, to the *complanarity* of the factor quaternions, the *result* arrived at is *correct*; but the *process* is not legitimate and is especially unjustifiable in a paper avowedly expository of the principles of this calculus.

On p. 123, Prof. W. says, "In division, the versor operating on the divisor line is conceived to turn it, positively, about the axis of the versor ($+i$) through an angle equal to that of the versor (θ) to coincide in direction with the dividend line. In multiplication the versor operating on the multiplier line [as a , eq. (27)] is conceived to turn it in a positive direction about the axis of the versor ($+i$) through an angle equal to the *supplement* of the angle of the versor [$\pi - (\pi - \theta) = \theta$], making it coincide in direction with the multiplicand line (as β)."

Prof. W. cites p. 85 of the *Lectures* for the above, but I can assure the readers of the *ANALYST* that Hamilton does not deserve the credit of it. In the "division"

$$\frac{\beta}{a} = \cos \theta + i \sin \theta,$$

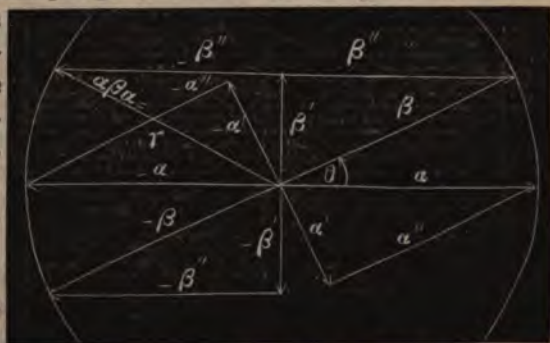
the versor $\cos \theta + i \sin \theta$, so far from "operating upon the divisor line" (a) in reality *does not operate upon any line whatever, does not act at all*, but simply *constitutes the conception* $\beta \div a$. So soon as the equivalency of the vector and the right part of the quaternion is established, we may change the "quotient" $\frac{\beta}{a}$ into the "product" $\beta \cdot \frac{1}{a} = -\beta a$, and then we have a *bonafide* "operation", viz., β and the inversor ($-$) operate on a ; and for all such products of complanar unit vectors a multitude of simple constructions may be given.

As a specimen, let the figure be a diametral section of the unit sphere by the plane of the paper. Adopting left hand rotation as positive, imagine the unit vector i drawn perpendicular to this section and in front. Resolve a into a' and a'' , the former perpendicular, the latter parallel to β , &c.

Then we have by inspection,

$$\frac{\beta}{a} = -(\beta a) = -(\beta' + \beta'') a$$

$$= -(\beta' a) - (\beta'' a) = -[(-i) \sin \theta] + (a^2 \cos \theta) = i \sin \theta + \cos \theta,$$



$$\frac{\beta}{a} = -(\beta a) = -\beta(a' + a'') = -(\beta a') - (\beta a'') = -(-i) \sin \theta - (\beta^2 \cos \theta) = i \sin \theta + \cos \theta,$$

$$\frac{\beta}{a} = (-\beta)a = (-\beta)(a' + a'') = (-\beta)a' + (-\beta)a'' = i \sin \theta - \beta^2 \cos \theta = i \sin \theta + \cos \theta,$$

&c., &c., &c.,

$$a\beta = (a' + a'')\beta = a'\beta + a''\beta = i \sin \theta + \beta^2 \cos \theta = i \sin \theta - \cos \theta,$$

$$a\beta = a(\beta' + \beta'') = a\beta' + a\beta'' = i \sin \theta + a^2 \cos \theta = i \sin \theta - \cos \theta.$$

$$\frac{\beta}{a} \cdot a = (i \sin \theta + \cos \theta)a = i a \sin \theta + a \cos \theta = \beta' + \beta'' = \beta.$$

$$a\beta \cdot a = (i \sin \theta - \cos \theta)a = i a \sin \theta + (-a) \cos \theta = \beta' + (-\beta'') = \gamma,$$

and so on interminably.

It is easily seen that the product of an *even* number of complanar vectors is a *quaternion* in the same plane, that of an *odd* number, a *vector* complanar with the factors, and that for every such *odd* product we have $Sa\beta\gamma \dots \omega = 0$, special cases of which are the constantly recurring formulæ $Sa = 0$, $Sa\beta\gamma = 0$. But these matters must be left to expositors of the method.

To recapitulate results: It is seen that in the *division* $\beta \div a$ the versor $\cos \theta + i \sin \theta$ does not operate at all; in the equivalent *multiplication* $\beta \cdot \frac{1}{a}$, β and the inversor $(-)$ operate upon a to produce the versor $\cos \theta + i \sin \theta$. Let this be compared with the first part of Prof. W's statement.

In the multiplication $a\beta$, the versor $-\cos \theta + i \sin \theta$, so far from operating upon the multiplier line a , is the *result* of a operating upon β —the multiplier line upon the multiplicand line, as it should be.

It is possible that Prof. W. had in mind that

$$\frac{\beta}{a} \cdot a = (\cos \theta + i \sin \theta)a = \beta,$$

which is matter of definition; but this is quite a different affair from the division $\beta \div a$; and, on the other hand, we do *not* have

$$a\beta \cdot a = (-\cos \theta + i \sin \theta)a = \beta.$$

In fact this versor $-\cos \theta + i \sin \theta$, when it comes to operate upon a (as a *multiplicand*, of course) turns the latter, as shown by the last of the above equations, through an angle $\pi - \theta$ into coincidence with γ , what Hamilton would call the *reflection* of β with respect to the line β' .

EXPERIMENTAL DETERMINATION OF THE CONSTANT IN THE FORMULA FOR PROBABLE ERROR.

BY R. J. ADCOCK, ROSEVILLE, ILL.

THE law of errors is expressed by eq. (4), ANALYST, page 51, Vol. VI, viz.;

$$\frac{\tan^{-1}cx}{\tan^{-1}cl} = \frac{z}{n}, \quad (4)$$

where z is the number of errors not greater than x , and n the number not greater than l , $S(d_1^2)$ = the sum of the squares of the errors, and

$$c = \sqrt{\frac{n}{S(d_1^2)}}.$$

Making $z \div n = \frac{1}{2}$, gives $x = \sqrt{[S(d_1^2) \div n]} \tan \frac{1}{2} \tan^{-1} cl$. It is required to find the numerical value of $\tan \frac{1}{2} \tan^{-1} cl$.

The following 80 numerical values of a quantity were observed by me:

34, 30, 40, 38, 36, 33, 32, 33, 38, 36, 32, 34, 32, 33, 32, 34, 34, 33, 32, 33, 33, 33, 40, 33, 34, 32, 44, 41, 40, 32, 32, 41, 34, 31, 31, 40, 33, 40, 34, 33, 38, 38, 40, 34, 38, 35, 40, 35, 39, 38, 40, 32, 36, 34, 35, 34, 36, 34, 39, 38, 37, 35, 36, 38, 34, 35, 32, 34, 36, 35, 32, 34, 35, 32, 32, 36, 38, 36, 38, 36.

The true value of the observed quantity as determined by the method of Least Squares, which in this case coincides with the mean, is 35.6. Subtracting 35.6 from each of the above numbers gives the errors, of which the greatest is 8.4 = l , and the sum of their squares is $729.96 = S(d_1^2)$, $\frac{8.4}{\sqrt{729.96}} = c^2 = .109602$, $c = .33106$, $cl = 2.7809$, $\tan^{-1} cl = \text{arc of } 70^\circ 13' 17''$, $\tan \frac{1}{2} \tan^{-1} cl = 35^\circ 06' 38'' = .703$. In another case of 120 observed values of another quantity the calculated value was $36\frac{1}{5}$, the greatest observed error $5\frac{4}{5}$, $S(d_1^2)$ was found to be $408\frac{4}{5}$, and the resulting value of $\tan \frac{1}{2} \tan^{-1} cl = .709$. I therefore conclude, from these two examples and others that I have examined that the true value of $\tan \frac{1}{2} \tan^{-1} cl$ is $.707 = \frac{1}{2}\sqrt{2}$, and not .6745 as heretofore given.

Corollary. Since after a sufficient number of observations the value of l will cease to vary sensibly, $\sqrt{[S(d_1^2) \div n]}$ will have its limit also. Therefore I conclude that after this limit is reached, or nearly so, increasing the number of observations does not increase the accuracy of the result calculated from them. Prof. Peirce arrived at this conclusion in the Coast Survey Report for 1854.

NEW DEMONSTRATION OF THE BINOMIAL THEOREM.

BY JOHN MACNIE, A. M., EASTON, CONN.

1. Let the expansion of $(1+z)^n$ be

$$(1+z)^n = c_0 + c_1 z + c_2 z^2 + c_3 z^3 + \dots + c_r z^r, \quad [1]$$

where n may have any value, and $c_0, c_1, \dots, c_r$, are coefficients independent of z .

Multiplying both members of [1] by $(1+z)$ we obtain

$$(1+z)^{n+1} = c_0 + (c_1 + c_0)z + (c_2 + c_1)z^2 + (c_3 + c_2)z^3 + \dots + (c_r + c_{r-1})z^r,$$

in which the $(r+1)$ th coefficient is the sum of the $(r+1)$ th and r th of [1].

By proceeding in the same manner, we obtain in succession,

$$(1+z)^{n+2} = c_0 + (c_1 + 2c_0)z + (c_2 + 2c_1 + c_0)z^2 + (c_3 + 2c_2 + c_1)z^3 + \dots + (c_r + 2c_{r-1} + c_{r-2})z^r,$$

$$(1+z)^{n+3} = c_0 + (c_1 + 3c_0)z + (c_2 + 3c_1 + 3c_0)z^2 + (c_3 + 3c_2 + 3c_1 + c_0)z^3 + \dots + (c_r + 3c_{r-1} + 3c_{r-2} + 3c_{r-3})z^r$$

$$(1+z)^{n+4} = c_0 + (c_1 + 4c_0)z + (c_2 + 4c_1 + 6c_0)z^2 + (c_3 + 4c_2 + 6c_1 + 4c_0)z^3 + (c_4 + 4c_3 + 6c_2 + 4c_1 + c_0)z^4 + \dots + (c_r + 4c_{r-1} + 6c_{r-2} + 4c_{r-3} + c_{r-4})z^r.$$

By noting the law according to which these coefficients are formed, we easily prove by induction that the $(r+1)$ th term of $(1+z)^{n+m}$ is

$$\left[c_r + mc_{r-1} + \frac{m(m-1)}{2} c_{r-2} + \dots + mc_{r-m+1} + c_{r-m} \right] z^r,$$

and $(1+z)^{n+m} = c_0 + (c_1 + mc_0)z + \left[c_2 + mc_1 + \frac{m(m-1)}{2} \right] z^2 + \dots \quad [2]$

If now in [1] we suppose $n = 0$, then since $(1+z)^0 = 1+0\dots$, we have $c_0 = 1, c_1 \dots c_r$, each $= 0$, and we obtain the expansion of $(1+z)^m$ either from [2], or by substituting in succession, $0, 1, 2, \dots, m$ for r in the general term; noting that every term in the coeff. of that expression vanishes, except that for which the subscript of c becomes 0 for assumed value of r . Thus we obtain

$$(1+z)^m = 1 + mz + \frac{m(m-1)}{1.2} z^2 + \frac{m(m-1)(m-2)}{1.2.3} z^3 + \dots + mz^{m-1} + mz^m,$$

m being any positive integer.

2. As by the aid of [2], the expansion of $(1+z)^n$ being given, we can find that of $(1+z)^{n+m}$, so, conversely, the expansion of $(1+z)^{n+m}$ being given, say

$$(1+z)^{n+m} = 1 + C_1 z + C_2 z^2 + C_3 z^3 + \dots + C_r z^r, \quad [4]$$

we can find in succession $c_1, c_2, c_3, \dots, c_r$, the coefficients of $(1+z)^n$, by means of the series of equations:

$$c_0 = 1; c_1 = C_1 - m; c_2 = C_2 - mc_1 - \frac{m(m-1)}{1.2};$$

$$c_3 = C_3 - mc_2 - \frac{m(m-1)}{1.2}c_1 - \frac{m(m-1)(m-2)}{1.2.3}; \text{ \&c.,}$$

and thus obtain

$$(1+z)^n = 1 + (C_1 - m)z + \left(C_2 - mc_1 - \frac{m(m-1)}{1.2}\right)z^2 + \left(C_3 - mc_2 - \frac{m(m-1)}{1.2}c_1 - \frac{m(m-1)(m-2)}{1.2.3}\right)z^3. \quad [5]$$

Since, by hypothesis, n may have any value, let $n = -m$, then [4] becomes $(1+z)^0 = 1+0+\dots$, that is, $C_1 \dots C_r$ each $= 0$; and substit'ng these values in [5], we obtain, after some simplifications:

$$(1+z)^{-m} = 1 + (-m)z + \frac{-m(-m-1)}{1.2}z^2 + \frac{-m(-m-1)(-m-2)}{1.2.3}z^3 + \dots$$

where the exponent is any negative integer.

3. Again, since $(1+z)^{nm} = (1+z)^{n+(m-1)n}$, we can, if the expansion of $(1+z)^n$ is given, obtain that of $(1+z)^{nm}$ by substituting $(m-1)n$ for m in [2]; and, conversely, if the expansion of $(1+z)^{nm}$ be given, we can obtain that of $(1+z)^n$ by substituting $(m-1)n$ for m in [5].

Let $n = s \div m$, s and m being integers, then $(1+z)^{nm} = (1+z)^s$, and we have, by [3],

$$C_1 = s, C_2 = \frac{s(s-1)}{1.2} \dots \text{ Substituting, in [5], these values for } C_1 \dots C_r, \text{ and } (m-1)(s \div m) \text{ for } m, \text{ and simplifying, we obtain}$$

$$(1+z)^{\frac{s}{m}} = 1 + \frac{s}{m}z + \frac{(s \div m)[(s \div m)-1]}{1.2}z^2 + \frac{(s \div m)[(s \div m)-1][(s \div m)-2]}{1.2.3}z^3 + \dots$$

Hence, since s may be pos. or neg., we find, whatever n may be,

$$(1+z)^n = 1 + nz + \frac{n(n-1)}{1.2}z^2 + \frac{n(n-1)(n-2)}{1.2.3}z^3 + \dots$$

and thence, by putting $y \div x$ for z , and multiplying by x^n ,

$$(x+y)^n = x^n + nx^{n-1}y + \dots$$

SOLUTION OF PROB. 315 BY CHAS. H. KUMMELL. (SEE P. 166.)—The total distance of A is $D_a = 1+2+3 = 6^{in}$. It is required to find the most probable similar quantity for B .

The most probable position of B 's bull's eye is evidently the center of gravity of his shots, viz., that point for which

$$D_1^2 + D_2^2 + D_3^2 = (x-0)^2 + (y-0)^2 + (x-1)^2 + (y-1)^2 + (x-2)^2 + (y-3)^2 = \text{min'm.}$$

$$\therefore \begin{aligned} (x-0)+(x-1)+(x-2) &= 0; \quad x = 1. \\ (y-0)+(y-1)+(y-3) &= 0; \quad y = \frac{4}{3}. \end{aligned}$$

We have then

$$\begin{aligned} D_1 &= \sqrt{1 + \frac{16}{9}} = 1.667, \\ D_2 &= \sqrt{0 + \frac{16}{9}} = 0.333, \\ D_3 &= \sqrt{1 + \frac{16}{9}} = 1.944. \end{aligned}$$

The total distance from the most probable center would be then

$$D_s = 3^{1.944}, < D_a.$$

It would not be fair however to compare D_s , which is only probable, with D_a , which is true.

The rule agreed upon for comparing the skill makes evidently an abstraction of the direction of the shot from the center; it only considers the dist. The question arises: what is (with this abstraction) the most probable distance of the center of gravity from the true center?

We have for the mean error ϵ of the center of gravity being the true center

$$\epsilon = \sqrt{\frac{[D^2]}{3 \times 2}} = \frac{1}{3}\sqrt{10} = 1.054 \text{ in.}$$

We have then the probability of the true center being in the circumference of a circle of radius r about the center of gravity

$$\theta(r) = \frac{dr}{\epsilon_0 \sqrt{2\pi}} e^{-\frac{r^2}{2\epsilon_0^2}};$$

therefore the most probable distance r_0 of the center of gravity from the true center is,

$$r_0 = \frac{\int_0^\infty r \frac{dr}{\epsilon_0 \sqrt{2\pi}} e^{-\frac{r^2}{2\epsilon_0^2}}}{\int_0^\infty \frac{dr}{\epsilon_0 \sqrt{2\pi}} e^{-\frac{r^2}{2\epsilon_0^2}}} = \epsilon_0 \sqrt{\frac{2}{\pi}} = 0.841^{11}.$$

The most probable distance of each of the three shots from the true center is then $D'_1 = \sqrt{(r_0^2 + D_1^2)} = 1.867 \text{ in.}$, $D'_2 = \sqrt{(r_0^2 + D_2^2)} = 0.905 \text{ in.}$, $D'_3 = \sqrt{(r_0^2 + D_3^2)} = 2.118 \text{ in.}$; hence $D'_s = D'_1 + D'_2 + D'_3 = 4.890$, $< D_a$. B was therefore the most probable winner.

A more correct mode of comparison of their skill would be, however, by means of the measures of precision. We have the mean error for A ;

$$\epsilon_a = \sqrt{\left(\frac{1^2 + 2^2 + 3^2}{3}\right)} = \sqrt{\frac{14}{3}}; \therefore h_a = \frac{1}{\epsilon_a \sqrt{2}} = \frac{1}{\frac{1}{3}\sqrt{2}} = 0.327; \text{ for } B,$$

$$\epsilon_b = \sqrt{\left(\frac{[D^2]}{3-1}\right)} = \sqrt{\frac{10}{3}}; \therefore h_b = \frac{1}{\epsilon_b \sqrt{2}} = \frac{1}{\frac{1}{3}\sqrt{2}} = 0.387.$$

However, the ratio of skill agrees satisfactorily from the two methods for,

$$D_a \div D_s = \frac{4.890}{3.944} = 1.23, \quad \text{and } h_b \div h_a = \frac{0.387}{0.327} = 1.19.$$

SOLUTION OF PROB. 313 (SEE P. 164) BY PROF. D. V. WOOD. — There being no escape of heat due to compression, the law of pressure would be expressed by

$$pv^k = \text{constant} = p'v'^k, \quad (1)$$

where p' = initial pressure of the atmosphere = 15lbs, nearly, v' = vol. of the cylinder = al , if a is the area of the base, p = the pressure within the cylinder when the weight has descended a distance x , $a(l-x)$ = the volume when the pressure is p , $k = 1.408$; then, from (1),

$$p = \frac{p'l^k}{(l-x)^k}, \quad \text{and} \quad \frac{wd^2x}{gdt^2} = w + p'a - \frac{ap'l^k}{(l-x)^k}.$$

Integrating, observing that for $x = 0$, $v = 0$,

$$v^2 = \left(1 + \frac{p'a}{w}\right) 2gx + \frac{2ap'g}{w(k-1)} \left(\frac{l[l-x]^{k-1} - l^k}{[l-x]^{k-1}} \right).$$

At the end of the downward movement $v = 0$; therefore

$$x(w+pa)(l-x)^{k-1} + \frac{ap'l}{k-1}(l-x)^{k-1} = \frac{ap'l^{k+1}}{k-1},$$

from which x may be found by trial after numerical values have been substituted for the known quantities.

SOLUTIONS OF PROBLEMS IN NUMBER FIVE.

SOLUTIONS of problems in No. 5, have been received as follows:

From Prof. W. P. Casey, 320, 321, 323; Prof. J. H. Kershner, 324 and Miscellaneous prob. No. 1 (see p. 149); Octavian L. Mathiot, 320; Prof. D. J. Mc Adam, 323; Prof. E. B. Seitz, 324; Prof. M. C. Woodward, 322; Prof. D. V. Wood, 322; H. Heaton, 322.

320. "The transverse and conjugate axes of an ellipse being given, to find the diameter of the circular base, and the altitude, of the right cone, and where to pass a plane so as to produce the given ellipse."

SOLUTION BY OCTAVIAN L. MATHIOT, BALTIMORE, MD.

Construction. — Draw $AB = 2a$, the given transverse diameter. At P , the centre of AB , erect a perpendicular, and with A as a centre and radius AB , describe a vertical arc in which suppose a point D to be taken such that if BD be produced to meet the perpendicular from P in C , and the triangle PBC be revolved about PC as an axis, a cone will be described a section of which, having AD for transverse axis, will have the given conjugate axis.

From D to A pass a plane whose cutting edge shall be at right angles with AB and parallel to the base of the cone, producing the ellipse $AFDE$, the transverse axis of which is $AD = AB$. Through K , the centre of AD , pass a plane at right angles to PC and parallel to the base of the cone, producing the circle $LEMF$. Join E and F , the points where the circle intersects the ellipse, EF will be the given conjugate axis $= 2b$, say. Through K , the middle point of EF , draw LM a diameter of the circle $LEMF$, intersecting PC in R . EF is a chord of the circle $LEMF$.



$$\therefore LK = EK^2 \div KM = b^2 \div KM. \quad (1)$$

Through D , draw DG perpendicular to, and intersecting PC in I , and draw DS parallel to PC .

Calculation.—Since the triangles ALK and AGD are similar, and $AD = 2AK = 2a$, $GD = 2LK$ and $ID = LK$; $\therefore LR = KH = \frac{1}{2}(LK + KM) = (EK^2 + KM^2) \div 2KM$, by substituting for LK from (1), or, because $KM = KD = a$,

$$KH = (a^2 + b^2) \div 2a. \quad (2)$$

We have $HD = \sqrt{(KD^2 - KH^2)} = \sqrt{[(3a^4 - 2a^2b^2 - b^4) \div 4a^2]}$, and $HM = KH - LK = (a^2 - b^2) \div 2a$, and thence, by similar triangles we find $PC = a \sqrt{[(3a^4 - 2a^2b^2 - b^4) \div (a^2 - b^2)]}$, the altitude of the cone.

Again we have $SP = ID = LK = b^2 \div a$, $\therefore AS = (a^2 + b^2) \div a$, and consequently $SD = \sqrt{(AD^2 - AS^2)} = \sqrt{[4a^4 - (a^2 + 2ab + b^2) \div a^2]}$, the altitude of the point D .

321. "From any point B of the circle $x^2 + y^2 = a^2$, BR is drawn perpendicular to the straight line $x = b$. Find the locus of P , the symmetrical point of R with respect to the tang't at B and its evolute."

SOLUTION BY PROF. W. P. CASEY, SAN FRANCISCO, CAL.

Let $\theta = \angle BCR$, and let p, h be coordinates of P . Then $2y \sin \theta = 2RO = RP$, and $2y \sin^2 \theta = RP \sin \theta = h$; $\therefore 4y^2 \sin^4 \theta = h^2$. Also $PR \cos \theta = RN$ or $2y \sin \theta \times \cos \theta = RN$; $\therefore 2y \sin \theta \cos \theta + x = p$ and $(2y \sin \theta \cos \theta + x)^2 = p^2$. Therefore $4y^2 \sin^4 \theta + 4y^2 \sin^2 \theta \cos^2 \theta + 4xy \sin \theta \cos \theta + x^2 = p^2 + h^2$, or $4y^2 \sin^2 \theta + 4xy \sin \theta \cos \theta + x^2 = p^2 + h^2$, and by assigning numerical values to θ we obtain the locus of P .



322. "The end of a prism is an isosceles triangle, altitude a and base b . The end of this prism is welded to the vertical face of a block of elastic material, the base of the triangular end of the prism being parallel with the horizontal face of the block and projecting a given height h above the block. A given force F_1 is applied at the projecting extremity the direction of which is normal to the vertical face of the block and which is in equilibrium with the forces F_2 and F_3 , of tension and pressure, above and below the neutral axis. Find the position of the neutral axis."

SOLUTION BY PROF. M. C. WOODWARD, WASHINGTON UNIV., ST. LOUIS, MO.

Since the prism is in equilibrium, we have

$$F_1 + F_2 + F_3 = 0, \quad (1)$$

$$M_1 + M_2 + M_3 = 0, \quad (2)$$

in which M is the moment of F about any axis.

Let x = the distance of the required neutral axis from the vertex of the triangular base of the prism. Let r be the distance of any point from the neutral axis; then, supposing the prism to be rigid, we have, for the intensity of the force acting on the prism, $p = -nr$. This is negative, for when r is positive the stress is like F_2 , opposite to F_1 .

$$F_2 = -\frac{bn}{a} \int_0^{a-x-h} (x+r)rdr,$$

$$F_3 = -\frac{bn}{a} \int_{-x}^0 (x+r)rdr.$$

Letting $s = a-h$, integrating, putting in limits and reducing we have

$$F_2 = -\frac{bn}{6a} (x^3 - 3xs^2 + 2s^3),$$

$$F_3 = \frac{bn}{6a} x^3.$$

Substituting in (1) we have

$$F_1 = \frac{bn}{6a} s^2(2s - 3x). \quad (3)$$

Taking the neutral axis as the axis for moments we have

$$M_1 = +F_1(a-x),$$

$$M_2 = -\frac{bn}{a} \int_0^{a-x-h} (x+r)r^2dr,$$

$$M_3 = -\frac{bn}{a} \int_{-x}^0 (x+r)r^2dr.$$

Integrating and reducing, we find

$$M_2 = -\frac{bn}{12a}(3s^4 - 8s^3x + 6s^2x^2 - x^4).$$

$$M_3 = -\frac{bn}{12a}x^4.$$

Substituting in (2) we have, after reducing,

$$F_1(a-x) = \frac{bn}{12a}(3s^4 - 8s^3x + 6s^2x^2). \quad (4)$$

Combining (3) and (4) we readily get

$$x = \frac{s}{2} \cdot \frac{4a - 3s}{3a - 2s}.$$

Cor. I. If h is infinite and s finite, we shall have M_1 finite, $F_1 = 0$ (in order that F_2 and F_3 may be finite), and $x = \frac{2}{3}a$, or we have the neutral axis of a triangular beam exposed only to a bending moment. In this case F_1 is the sum of the two forces of a couple.

Cor. II. If $h = 0$, we have $a = s$, and $x = \frac{1}{2}s$.

Cor. III. If $h = -\frac{1}{3}s$, we have $3a - 2s = 0$, $x = \infty$, and the stress is uniform.

Scholium. Regarding the triangular prism as a sector of a solid cylinder, it is easily seen that the general case $h > 0$, is illustrated (approximately) by the tension upon a circular-headed bolt; and the neutral axis becomes a *neutral circle*.

When $h = 0$, we have (still approximately) the action of a nut on a bolt.

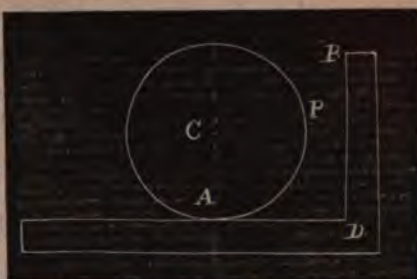
When $h = -\frac{1}{3}s$ we have a state of stress very desirable to attain, but very difficult to reach in practice.

[We cannot perceive that the foregoing elegant solution of problem 322 illustrates, even approximately, the tension upon a circular-headed bolt; for the forces acting on the head of the bolt are only *two*, instead of three as in the problem. The head of the bolt being rigid, as the force F_1 is applied at all points of its circumference, there can be no rotation, as is supposed in the problem (322), and therefore no neutral axis. Hence, although we may suppose the bolt elastic, we have no representative of F_3 , or rather F_2 and F_3 constitute but one force in the bolt, as they act in the same direction, their *sum* being the equivalent of F_1 , instead of their difference as in problem 322. That there can be no neutral axis, or neutral circle, in this case, is evident from the fact that its existence would require a depression in the center which is incompatible with the assumed rigidity of the head of the bolt.—Ed.]

323. "Describe the spiral of Archimedes by continuous motion."

SOLUTION BY PROF. D. J. MC ADAM, WASHINGTON, PA.

Lay a circle, cut from a board or paste board, upon the paper, as *C* in the figure. Lay a square flat upon the paper and make it press against the circumference of the circle. Put a pencil against the arm of the square as at *P*, at a distance from the tang't *AD*, of the radius *AC* of the circle.



Roll the square upon the circle and keep the pencil pressed against the p't *P*. The pencil will describe a spiral of Archimedes.

That the curve is the spiral of Archimedes is evident, for there is a constant relation between the angle of rotation and the radius vector *CP*. Also the curve cuts the radial lines from *C* at distances from *C* which increase uniformly by the circumference of the fixed circle. This principle is referred to by Prof. Rankine in his "Machinery and Millwork", page 54.

324. "Find the envelope of the straight line

$$x \cos \varphi + y \sin \varphi = a (\cos n\varphi)^{\frac{1}{n}}.$$

SOLUTION BY PROF. E. B. SEITZ, KIRKSVILLE, MO.

Differentiating the given equation with respect to φ , we have

$$x \sin \varphi - y \cos \varphi = a \sin n\varphi (\cos n\varphi)^{\frac{1-n}{n}}. \quad (2)$$

Let $x = r \cos \theta$ and $y = r \sin \theta$, then (1) and (2) become

$$r \cos (\varphi - \theta) = a (\cos n\varphi)^{\frac{1}{n}} \quad (3), \quad r \sin (\varphi - \theta) = a \sin n\varphi (\cos n\varphi)^{\frac{1-n}{n}}. \quad (4)$$

The square root of the sum of the squares of (3) and (4) is

$$r = a (\cos n\varphi)^{\frac{1-n}{n}}. \quad (1)$$

Dividing (4) by (3) we have $\tan (\varphi - \theta) = \tan n\varphi$; hence $\varphi - \theta = n\varphi$, or $\varphi = \theta \div (1 - n)$. Substituting this value of φ in (5), we have

$$\cos \frac{n}{n-1} \theta = \left(\frac{a}{r} \right)^{\frac{n}{n-1}},$$

the polar equation of the envelope. When $n = 2$ we have $r^2 \cos 2\theta = a^2$, the equation of the equilateral hyperbola.

$n = -1,$	$r \cos^{\frac{1}{2}} \theta = a,$	equation of the parabola.
$n = \frac{1}{2},$	$r = a \cos \theta,$	" " " circle.
$n = \frac{2}{3},$	$r = a^2 \cos 2\theta,$	" " " lemniscate.

SOLUTION OF MISCEL. PROB. (1) (SEE P. 149), BY PROF. KERSHNER.

$$\begin{aligned}(1+x)^n &= 1 + C_1x + C_2x^2 + \dots + C_2x^{n-2} + C_1x^{n-1} + x^n, \\ (1+x)^n &= x^n + C_1x^{n-1} + C_2x^{n-2} + \dots + C_2x^2 + C_1x + 1; \\ \therefore (1+x)^{2n} &= x^n + C_1x^{n+1} + C_1x^{n-1} + C_2x^{n+2} + C_1x^n + C_2x^{n-2} + \&c. \\ \text{The coefficient of } x^n \text{ in } (1+x)^{2n} &\text{ is}\end{aligned}$$

$$\frac{1.3.5 \dots (2n-1)}{1.3.5 \dots n} \times 2^n,$$

by the binomial expansion, and in the product above, $1 + C_1^2 + C_2^2 + \&c.$;

$$\therefore 1 + n^2 + \left\{ \frac{n(n-1)}{1.2} \right\}^2 + \&c. = \frac{1.3.5 \dots (2n-1)}{1.3.5 \dots n} \times 2^n.$$

QUERY BY PROF. WM. WOOLSEY JOHNSON, ANNAPOLIS, MD.—“It is stated in Brande and Cox’ Dictionary of Sc. and Lit., with respect to Fermat’s ‘last theorem’, viz.: that $x^n + y^n = z^n$ is insoluble in integers except when $n=2$, that no complete demonstration has yet been given. Yet Barlow in his Theory of Numbers gives what professes to be a demonstration of the impossibility of the equivalent equation $x^n - y^n = z^n$, p. 164.

What is the fallacy in Barlow’s demonstration and by whom was it first exposed?”

ANSWER BY PROF. JOHN E. DAVIES, MADISON, WISCONSIN.

By reference to the famous “Report on the Theory of Numbers” by Prof. H. J. Stephen Smith of Oxford, in the British Association for the Advancement of Science Reports, 1859–62; in the Report for 1860 will be found a *resume* of Kummer’s work on Fermat’s Theorems. Kummer has shown its impossibility for all values of the exponent up to 100 and seems to have considered his proof general as the title of his paper is “Allgemeiner Beweis des Fermat’schen Satzes dass die Gleichung $x^\lambda + y^\lambda = z^\lambda$ unlösbar ist für all diejenigen Potenz-Exponenten λ welche ungeraden Primzahlen sind, und in der zählern der ersten $\frac{1}{2}(\lambda-3)$ Bernouille’schen Zahlen als Factoren nicht vorkommen.” The excepted primes, numbers like 37, 59, 67, &c., which divide the numerators of some one of the 1st $\frac{1}{2}(\lambda-3)$ fractions of Bernouilli were afterward treated by Kummer, and all those satisfying certain conditions were also found to be included in the theorem.

Concerning Barlow’s proof, Prof. Smith says—“The proof in Barlow’s Theory of Numbers, pp 160–169, is erroneous, as it reposes (see p. 168) on an elementary proposition (Cor. 2, p. 20) which is untrue.” I think it very likely that the error *had* been noticed before, but I do not know where.

PROBLEMS.

325. *By Prof. F. P. Matz, King's Mountain, N. C.*—Given the altitude and radius of the circumscribed and inscrib'd circles of a plane scalene triangle; to find the three sides.

326. *By Prof. J. H. Kershner.*—Draw a line bisecting a given triangle so that the part lying within shall be, 1st, a minimum, 2nd, a maximum.

327. *By Request.*—The poles of the radical axis of two circles taken with respect to each circle, and the two centers of similitude of the circles, are four harmonic points. (Ex. 8, p. 367, Chauvanet's Modern Geom.)

328. *By Prof. D. J. Mc Adam.*—A body is projected from the top of a tower 100 ft high at an angle of elevation of 45° , with a velocity of 60 ft per second. Find the distance from the point at which it first strikes the horizontal plane to the second point at which it strikes the plane. The modulus of elasticity being $\frac{1}{2}$, and the resistance of the atmosphere neglected.

329. *By E. J. Edmunds, Professeur de Francais d'Anglais et de Mathématiques, 11, rue Toullier, Paris.*—Three points, A, B, C , being given, to find a point M , whose distance from A, B and C , shall be a minimum.

330. *By Prof. E. B. Seitz, Kirksville, Mo.*—Two points are taken at random within a circle on opposite sides of a given diameter, and a third point is taken at random in the circumference; find the average area of the triangle formed by joining the points.

331. *By W. E. Heal, Marion, Ind.*—Show that the reciprocal of the Hessian of the reciprocal of a given curve passes through the points of contact of multiple tangents, and points of inflexion of the given curve.

ANNOUNCEMENT OF VOL. VIII.—As the number of persons, in this country, who are able and willing to devote time and money to the cultivation of Mathematics as a Science, is comparatively small, and as the ANALYST depends entirely upon its subscribers for the pecuniary aid necessary for its production, and for voluntary contributions to fill its pages, it was scarcely to be expected that it would survive to the age it has already attained.

It has been our intention from its birth, as we have heretofore stated to our subscribers, to continue the ANALYST as long as our health will permit and the interest in it, manifested by its readers, continues unabated. We are, therefore pleased to be able to say that, so far as we can judge at present, the ANALYST will survive yet several years; and as we trust it has been of some service in promoting the cultivation of the Science of mathemat-

ics, which is the key to all other sciences, we hope the interest and support of our subscribers will continue unabated.

The Nos. of Vol. VIII shall appear promptly as they are due, No. 1, about the 1st of Jan., 1881.

J. E. HENDRICKS.

PUBLICATIONS RECEIVED.

Address before the American Association for the Advancement of Science, Section A. By PROF. ASAPH HALL. Boston meeting, Aug. 25, 1880.

Geology of Wisconsin. Vols. II and III. 8vo. 1877 and 1879.

Each Vol. contains nearly 800 pages, is finely illustrated with many engravings, and is accompanied by an Atlas of 14 Maps, 24 by 30 inches, each. This is a magnificent work and is creditable to the State, as well as to the parties who made the surveys.

Ray's New Higher Arithmetic. 408 pp. 12mo. Van Antwerp, Bragg and Co. Cincinnati. 1880.

This is a very neat Revision of a very popular book. And the fact that the Revision has been in charge of Prof. J. M. Greenwood, is a guaranty that the work is brought up to the present demands of the science.

The American Journal of Mathematics, Vol. III, No. 1.

This Number contains two papers of special interest: 1. *Regular Figures in n-dimensional Space*, by W. J. Stringham, and, 2. *On the Algebra of Logic*, by C. S. Peirce.

The Mathematical Visitor, No. 4. Erie, Pa. Artemas Martin, A. M., Editor and Publisher. Semi-annual. \$1.00 per annum.

Versuch einer mathematischen Theorie zur Erklärung des Lichtwechsels der veränderlichen Sterne. Von Hugo Gylden. Helsingfors. 1880. 63 pp. 4to.

The author attempts to find a mathematical expression for the brightness of a variable star. His theory is founded on certain notions of Zollner modified or generalized. He considers a variable star as a glowing body of globular form having a motion of rotation; but the points of whose surface emit light of very different degrees of intensity, on account of the presence of scoria or similar matter. These scoria-areas, although probably slowly variable in size and position on the surface, may, he thinks, for the comparatively short time which observation at present covers, be considered as fixed. The moments of inertia are regarded as having any values whatever, and the instantaneous axis of rotation may have any direction with respect to the principal axes. The brightness comes out as a periodic function dependent on two arguments increasing proportionally to the time, the ratio of whose periods is a function of the differences of the moments of inertia. The author has not compared his theory with the data of observation, for the reason that sufficiently extended series of observations of the intensity of the light of the variable stars have not been published in the astronomical journals. The mere times of maxima and minima will not, he says, afford a crucial test of his theory.

G. W. H.

ERRATA.

On page 150, at the beginning of the first of the two general formulæ, for " $\frac{1}{2}$ " read $\frac{3}{2}$.

" " 184, dele " r " in the Figure,

N. L.
a. a. s.





